High Energy Physics and Nuclear Physics

Volume 21, Number 4

q-Analog Boson Inverse Operators and Their Application

Wei Lianfu, Wang Shunjin, and Jie Quanlin

(Institute of Modern Physics, Southwest Jiaotong University, Chengdu, China)

The inverse of the q-analog boson creation and annihilation operators is introduced. By virtue of the properties of a_q^{-1} and a_q^{+-1}, the q-analog deformation form of the photon-added coherent states is constructed and its completeness relation is discussed.

Key words: q-analog boson operators, inverse operators, photon-added coherent states, completeness relation.

1. INTRODUCTION

It is well known that boson operators a and a^+, which form a Heisenberg algebra, play an important role in group theory and many branches of physics, e.g., quantum mechanics, quantum optics, condensed matter theory, etc.

Various physical states of a radiation field can be described by the eigenstates of boson operators or their combinations. For example, the coherent state is the eigenstate of a and the eigenstates of a^+a are nothing but the Fock states. Recently, the boson inverse operators a^{-1} and a^{+-1} introduced by Metha *et al.* [1] have been used to develop the nonunitary transformation theory in quantum mechanics [2]. On the other hand, the q-analog boson operators a_q and a_q^+, which generate a q-analog Heisenberg algebra [3]:

$$a_q a_q^+ - q a_q^+ a_q = q^{-N_q} , \tag{1.1}$$

$$[N_q, a_q] = -a_q \quad [N_q, a_q^+] = a_q^+ \tag{1.2}$$

and their related problems have been paid much more attention. The boson realization of many

Received on December 6, 1996. Supported by the National Natural Science Foundation of China.

quantum algebras, e.g., $SU_q(2)$ [3] and $SU_q(1, 1)$ [4], etc., has been studied through use of the q-analog boson operators. Furthermore, the eigenstates of q-analog boson operators or some of their combinations have been used to describe various q-analog quantum states: the q-coherent state [3], the q-charged coherent state [5], and q-even and odd coherent states [6], etc. Do the q-analog boson inverse operators exist also? In this paper, we first introduce the inverse of q-analog boson inverse operators a_q^- and a_q^{+-1} by their action on the q-deformed number states. By virtue of the properties of a_q^- and a_q^{++1}, we shall construct the q-analog form of the photon-added coherent states and discuss its completeness relation.

2. INVERSE OF q-ANALOG BOSON CREATION AND ANNIHILATION OPERATORS

It seems that we may always define the inverse of q-analog boson operators a_q^- and a_q^{+-1} by the following conditions: $a_q a_q^{-1} = 1$, $a_q^{-1} a_q = 1$; and $a_q + a_q^{+-1} = 1$, $a_q^{+-1} + a_q^+ = 1$. However, this is not true. Since q-analog boson operators a_q and a_q^+ are singular, they do not possess any inverse in a strict sense. We now introduce the generalized inverse of q-analog boson operators by their action on the representation space of q-analog Heisenberg algebra, i.e., the q-analog Fock space: $\{|n\rangle_q;\ |n\rangle_q = \frac{(a_q^+)^n}{\sqrt{[n]!}}|0\rangle_q,\ (n=0,1,2,\ldots)\}$. With the help of the following relations [3]:

$$a_q |n\rangle_q = \sqrt{[n]}\ |n-1\rangle_q, \quad a_q^+|n\rangle_q = \sqrt{[n+1]}\ |n+1\rangle_q, \tag{2}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and q is the deformation parameter, the generalized inverse of q-analog boson operators can be defined by

$$a_q^{-1}|n\rangle_q = \frac{1}{\sqrt{[n+1]}}\ |n+1\rangle_q, \quad a_q^{+-1}|n\rangle_q = (1-\delta_{n,0})\frac{1}{\sqrt{[n+1]}}\ |n+1\rangle_q \tag{3}$$

We notice that a_q^{-1} behaves as a creation operator while a_q^{+-1} behaves as an annihilation operator. It is easily shown that a_q^{-1} is the right inverse of a_q:

$$a_q a_q^{-1} = 1, \quad a_q^{-1} a_q = 1 - |0\rangle_{qq}\langle 0| \neq 1, \tag{4}$$

while a_q^{+-1} is the left inverse of a_q^+, i.e.,

$$a_q^{+-1} a_q^+ = 1, \quad a_q^+ a_q^{+-1} = 1 - |0\rangle_{qq}\langle 0| \neq 1, \tag{5}$$

Here $|0\rangle_q$ is the q-analog vacuum state defined by $a_q|0\rangle_q = 0$. From Eqs. (3) and (4), one has

$$a_q a_q^+ a_q^{+-1} = a_q, \quad a_q^{-1} a_q a_q^+ = a_q^+, \tag{6.1}$$

$$[a_q, a_q^{-1}] = [a_q^{+-1}, a_q^+] = |0\rangle_{qq}\langle 0|\ , \tag{6.2}$$

and

$$a_q^m a_q^{-m} = a_q^{+-m} a_q^{+m} = 1, \quad a_q^{-m} a_q^m = a_q^{+m} a_q^{+-m} = 1 - \sum_{i=0}^{m-1} |i\rangle_{qq}\langle i|\ , \tag{7}$$

where $m > 1$ is an arbitrary integer. It is worth pointing out that q-coherent state $|\alpha\rangle_q$ [3]

$$|\alpha\rangle_q = A(\alpha)\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}\ |n\rangle_q\ , \tag{8}$$

is the eigenstate of a_q

$$a_q|\alpha\rangle_q = a|a\rangle_q \ , \tag{9}$$

but it is not the eigenstate of a_q^-, i.e., $a_q^{-1}|\alpha\rangle_q \neq a^{-1}|\alpha\rangle_q$. In Eq. (8), $[n]! = [n][n-1]!$, $[0] = 1$, and $A(\alpha) = (\exp(|\alpha|^2))^{-1/2}$, where $(\exp|\alpha|^2) = \sum_{n=0}\frac{(|\alpha|^2)^n}{[n]!}$ is the q-analog exponential function. In fact, there exists no right eigenstate of the operator a_q^-. Using Eq. (3), we have

$$a_q^{-1}|\alpha\rangle_q = a^{-1}(|\alpha\rangle_q - A(\alpha)|0\rangle_q) \ . \tag{10}$$

Similarly, we can prove that there exists no right eigenstate of the operator a_q^{+-1} also except for the vacuum $|0\rangle_q$ with zero eigenvalue.

3. q-DEFORMATION OF PHOTON-ADDED COHERENT STATES AND ITS COMPLETENESS

So far, researchers have not understood the physical meaning of q-deformation clearly. In the last decade, some attention has been paid to the problem by studying the non-linear effect in quantum optics and using the q-analog Heisenberg algebra to describe the radiation field. In 1992, Agarwal *et al.* [7] first introduced the photon-added coherent state, which was generated by the repeated application of boson creation operator a^+ on the coherent state $|\alpha\rangle$. Although this state still has not been realized experimentally, it is interesting as a theory because of its non-classical features. In the following, we shall investigate the q-analog of such a state.

Consider the following quantum state

$$|\alpha,m\rangle_q = C\alpha_q^{+m}|\alpha\rangle_q \ , \tag{11}$$

where m is the positive integer and C is a normalization constant. Using Eqs. (7) and (9), we readily find that the state $|\alpha, m\rangle_q$ is a right eigenstate of $a_q^{+m}\, a_q\, a_q^{+-m}$ with eigenvalue α, i.e.,

$$a_q^{+m}a_qa_q^{+-m}|\alpha,m\rangle_q = \alpha\,|\alpha,m\rangle_q \ . \tag{12}$$

We notice that the q-analog Heisenberg algebra is reduced to the usual Heisenberg algebra as $q \to 1$. Therefore, $|\alpha, m\rangle_q$ is nothing but the q-analog of the photon-added coherent state defined in Ref. [7]. We now discuss the completeness of such a q-analog photon-added coherent state. Substituting Eq. (8) into Eq. (11), we have

$$|\alpha,m\rangle_q = CA(\alpha)\sum_{n=0}^{\infty}\frac{\alpha^n\sqrt{[n+m]!}}{[n]!}\,|n+m\rangle_q \ . \tag{13}$$

It is observed that the number states $\{|n\rangle_q, n = 0, 1, 2, \cdots, m-1\}$ are absent from this family of states. Equation (12) implies that these number states are the eigenstates of $a_q^{+m}\, a_q\, a_q^{+-m}$ with a common eigenvalue $\alpha = 0$. Thus, we obtain $(m+1)$-fold degeneracy for this eigenvalue. This means that the family of state $|\alpha, m\rangle_q$ does not form a complete set. However, they, along with the number states $\{|n\rangle_q, n = 0, 1, 2, \cdots, m-1\}$, do form a complete set. The corresponding completeness relations can be obtained through the identity relation involving the q-coherent state $|\alpha\rangle_q$ [8]

$$\int|\alpha\rangle_{q\,q}\langle\alpha|\mathrm{d}\mu(\alpha) = 1 \tag{14}$$

where $d\mu(\alpha) = \frac{1}{2\pi} \exp_q(|\alpha|^2)\exp_q(-|\alpha|^2)d_q|\alpha|^2 d\theta$. The integral over $d\theta$ is a normal integration but the integration over $d_q|\alpha|^2$ is the q-integration. We multiply this relation by a_q^{+m} from the left and a_q^m from the right, and then obtain

$$\int C^{-2}|\alpha,m\rangle_{qq}\langle\alpha,m|d\mu(\alpha)_q = a_q^{+m}a_q^m \ . \tag{15}$$

where Eq. (11) was used. Multiplying Eq. (15) by $a_q^{-m}\ a_q^{+-m}$ from the right and using Eq. (7), we have

$$\int C^{-2}|\alpha, m\rangle_{qq}\langle\alpha, m|_q a_q^{-m}a_q^{+-m}d\mu(\alpha) + \sum_{i=0}^{m-1}|i\rangle_{qq}\langle i| = 1 \ . \tag{16}$$

Equation (16) is essentially the completeness relation of state $|\alpha, m\rangle_q$.

Similarly to a_q^+, the operator a_q^- behaves as a creation operator. Therefore, another family of q-analog photon-added coherent states can be generated by the repeated application of a_q^- on the q-coherent state $|\alpha\rangle_q$:

$$|\alpha, m\rangle_q = Da_q^{-m}|\alpha\rangle_q \ , \tag{17}$$

where D is a normalization constant. In an analogous way, we can also obtain the completeness relation of these states (17). It is worth pointing out that though the q-coherent state can form a representation space of the q-analog Heisenberg algebra by stating the q-analog photon-added coherent states together with the number states $\{|n\rangle_q,\ n = 0, 1, 2, \cdots, m-1\}$ can form a representation space of the q-analog Heisenberg algebra.

It is interesting to point out that another typical quantum state, i.e., the q-analog photon-depleted coherent state, can be generated by the repeated application of a_q^{+-1} on the q-coherent state $|\alpha\rangle_q$:

$$|\alpha, -m\rangle_q = Ga_q^{+-m}|\alpha\rangle_q \ , \tag{18}$$

where G is a normalization constant. Since a_q^+-1 behaves as an annihilation operator, the state $|\alpha, -m\rangle_q$ indicates that it is a state in which m photons have been depleted from the q-coherent state $|\alpha\rangle_q$. It is easily seen that $|\alpha, -m\rangle_q$ is nothing but the eigenstate of operator a_q^{+-m}, a_q^{+m}, i.e.,

$$a_q^{+-m}a_q a_q^{+m}|\alpha, -m\rangle_q = \alpha|\alpha, -m\rangle_q \ , \tag{19}$$

We will show that the family of $|\alpha, -m\rangle_q$ can form a complete set like that of the q-coherent state $|\alpha\rangle_q$. The corresponding completeness relation of $|\alpha, -m\rangle_q$ reads as

$$\int G^{-2}|\alpha, -m\rangle_{qq}\langle\alpha, -m|\ a_q^m a_q^{+m}d\mu(\alpha) = 1, \tag{20}$$

This is in contrast to that of the family of the q-analog photon-added coherent state $|\alpha, m\rangle_q$ as above. Using Eq. (20), any quantum state $|\psi\rangle$ can be expressed in terms of the state $|\alpha, -m\rangle_q$ as

$$|\psi\rangle = \int G^{-2}{}_q\langle\alpha, -m|a_q^m a_q^{+m}|\psi\rangle|\alpha, -m\rangle_q d\mu\,(d) \ . \tag{21}$$

4. CONCLUSION

The inverse of q-analog boson creation and annihilation operators is first introduced by their action on the q-deformed number states in this paper. It is shown that a_q^- behaves as a creation operator, while a_q^{+-1} behaves an annihilation operator. We also find that a_q^- is the right inverse of a_q, while a_q^{+-1} is the left inverse of a_q^+. By virtue of the generalized inverse of q-analog boson operators, we have constructed the q-analog photon-added coherent state by the repeated application of a_q^+ or a_q^- on the q-coherent state; and the q-analog photon-depleted coherent state by the repeated application of a_q^{+-1} on the q-coherent state. It is shown that the q-coherent state $|\alpha\rangle q$ is not the eigenstate of operator a_q^-. The operator $a_q^{+\mathrm{m}}$ and $a_q^{+-\mathrm{m}}$ ($m \geq 1$) acting on the q-coherent state will result in the new q-analog quantum states $|\alpha,\ m\rangle_q$ and $|\alpha,\ -m\rangle_q$, respectively. The former is the eigenstate of the operator $a_q^{+m}\, a_q^{+-m}$, while the latter is the eigenstate of the operator $a_q^{+-m}\, a_q^{+m}$. We have also shown that the family of q-analog photon-deleted coherent states $|\alpha,\ -m\rangle_q$ can form a complete set. However, the family of q-analog photon-added coherent states $|\alpha,\ m\rangle_q$, along with the number states $\{|n\rangle_q,\ n = 0,\ 1,\ 2,\ \ldots,\ m - 1\}$, do form a complete set. Finally, it is worth pointing out that our q-analog photon-added coherent state is reduced to the photon-added coherent state defined in Ref. [7] as $q = 1$. When $q \neq 1$, we can study the nonlinear effects of the q-analog photon-added coherent state by the numerical method.

REFERENCES

[1] C.L. Metha, A.K. Roy, and G.M. Saxena, *Phys. Rev.*, **A46**(1992), p. 1565.
[2] Zhang Tang, *Phys. Rev.*, **A52**(1995), p. 3448.
[3] L.C. Biedenharn, *J. Phys.*, **A22**(1989), L873.
[4] D. Fairlie, J. Nutyts, and C. Zachos, *Phys. Lett.*, **B202**(1988), p. 320.
[5] H.Y. Fan and C.P. Sun, *Commun. Theor. Phys.*, **17**(1992), p. 243.
[6] L.F. Wei, *Act. Phys. Sin.* (in Chinese), **42**(1993), p. 757.
[7] G.S. Agarwal and K. Tara, *Phys. Rev.*, **A43**(1991), p. 492.
[8] R.W. Gray and C.A. Nelson, *J. Phys.*, **A23**(1990), L945.