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Different χ^2 Forms for Correlated Data and Their Properties*

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Abstract The properties of four factorized chi-square forms, which are used in minimization of correlated data, are studied, including their biasness and unbiasedness. The simplified R-value measurement are quoted to test the conclusion quantitatively.

Key words χ^2 form, biasness, unbiasedness, R-value measurement, correlated data

1 Introduction

There are two often used methods, the covariance matrix and the scale factor method, to deal with the correlated data, and the equivalence between them was discussed by D'Agostini in Ref.[1], where two typical experiment cases, the offset and the normalization cases, have been studied in detail mainly for two measurements. For the normalization case[2)], the equivalent conclusion has been expanded to multi-measurements[2,3].

In previous study[3], two points are worthy of notice: first, it is easy to acquire analytical results by using the factorized χ^2 form, which avoids complex calculations of inverse matrix; second, the estimates of parameters obtained from both the matrix and the factor approach deviate from the expected average value, and the deviation may be considerably striking, if the measurement points are quite many, or the uncertainty of the scale factor is rather large.

In this article, the study is devoted to factorized χ^2 forms. Besides the two depicted in Ref.[1], the other two forms come also into our sight. The properties of and the relations between different χ^2 forms are the main topics of the following sections. In addition, attention is paid to biasness and unbiasedness of minimization estimates. At last, some simplified experimental results are adopted to confirm the theoretical conclusions.

2 Four χ^2 forms

In the experimental data analysis, a scale factor f, by which all data points are multiplied, is introduced in the expression of χ^2 to take into account the normalization uncertainty[4,5]:

$$\chi^2_B = \sum_{i=1}^{n} \frac{(fx_i - k)^2}{(\sigma_i)^2} + \frac{(1-f)^2}{\sigma_f^2}, \tag{1}$$

where σ_f is the error of the factor f. Another similar form, where the individual errors are also scaled, is

$$\chi^2_A = \sum_{i=1}^{n} \frac{(fx_i - k)^2}{(f\sigma_i)^2} + \frac{(1-f)^2}{\sigma_f^2}. \tag{2}$$

Here subscripts A and B follow D'Agostini's notation. In above two equations, x_i's denote n statistically independent observations, k a physical quantity, which is expected as a constant. In minimization process k is a fitting parameter. However, in some actual analyses, the normalized factor is usually combined with fitting parameters instead of acting on observations. For example, in $\psi(2S)$ scan experiment, χ^2 can be expressed as[6]

$$\chi^2 = \sum_{i=1}^{n} \frac{(N_i - N_i^0)^2}{\sigma_i^2}, \tag{3}$$

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2) For the offset case, the equivalent proof can be found in Appendix of this article.

where N_i is the experimentally observed event number, N_i^0 corresponding theoretical expectation, which could be written in detail as

$$N_i^0 = L_i \cdot \sigma_i(\boldsymbol{\eta}) \cdot \varepsilon,$$

where L_i indicates luminosity; σ_i theoretical cross section, which is a function of parameter vector $\boldsymbol{\eta}$; ε the efficiency for a certain process, which could be figured out by Monte Carlo simulation. Because of simulation imperfection, the error of ε could not be neglected, and sometimes could be rather large. Under such situation, ε could be treated as a kind of scale factor whose variation would affect all data points in the same way. If ε's error is denoted as Δ, taking the correlation coming from ε into account, Eq. (3) becomes

$$\chi^2 = \sum_{i=1}^{n} \frac{(N_i - n_i^0)^2}{\sigma_i^2} + \frac{(\varepsilon - \varepsilon^0)^2}{\Delta^2}, \qquad (4)$$

with

$$= L_i \cdot \sigma_i(\boldsymbol{\eta}) \cdot \varepsilon^0$$

Here ε^0 is a fitting parameter. If define

$$f \equiv \frac{\varepsilon^0}{\varepsilon}, \text{ and } \sigma_f \equiv \frac{\Delta}{\varepsilon},$$

then

$$n_i^0 = L_i \cdot \sigma_i(\boldsymbol{\eta}) \cdot (\varepsilon \cdot f) = f \cdot N_i^0,$$

and

$$\chi^2 = \sum_{i=1}^{n} \frac{(N_i - fN_i^0)^2}{\sigma_i^2} + \frac{(1-f)^2}{\sigma_f^2}.$$

Rewrite N_i as x_i, and only consider the constant fitting problem (so N_i^0 becomes k), then

$$\sum_{i=1}^{n} \frac{(x_i - fk)^2}{(\sigma_i)^2} + \frac{(1-f)^2}{\sigma_f^2}, \qquad (5)$$

which is similar to χ^2_B except the factor f acts on the fitting parameter k rather than on the observations x_i. Corresponding to χ^2_A, there is a similar chi-square form:

$$\chi^2_\beta = \sum_{i=1}^{n} \frac{(x_i - fk)^2}{(f\sigma_i)^2} + \frac{(1-f)^2}{\sigma_f^2}. \qquad (6)$$

Here subscripts α and β are chosen in order to display the duality of estimates and covariances between different chi-square forms. This point would be seen clearly in the next section. Thus, there are totally four χ^2 forms as shown in (5) and (6).

Properties of χ^2 forms

In this section, the parameter estimates and covariances of four χ^2 forms are to be worked out. Firstly, for χ^2_A, from

$$\begin{cases} \dfrac{\partial \chi^2_A}{\partial k} = 0 \\ \dfrac{\partial \chi^2_A}{\partial f} = 0 \end{cases}$$

it can be obtained

$$\begin{cases} \hat{k}_A = f\bar{x}\,|_{f=\hat{f}_A} \\ \hat{f}_A = 1 \end{cases}, \quad \text{that is} \quad \begin{cases} \hat{k}_A = \bar{x} \\ \hat{f}_A = 1 \end{cases},$$

where $\bar{x}$ is the weighted average defined as

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} \Big/ \sum_{i=1}^{n} \frac{1}{\sigma_i^2}.$$

The inverse of covariance matrix is

$$= \frac{1}{2} \begin{pmatrix} \dfrac{\partial^2 \chi^2_A}{\partial k \partial k} & \dfrac{\partial^2 \chi^2_A}{\partial f \partial k} \\ \dfrac{\partial^2 \chi^2_A}{\partial k \partial f} & \dfrac{\partial^2 \chi^2_A}{\partial f \partial f} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} \dfrac{1}{\sigma_i^2} & -\sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} \\ -\sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} & \dfrac{1}{\sigma_f^2} + \bar{x} \cdot \sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} \end{pmatrix},$$

therefore

$$\frac{1}{D_{V_A^{-1}}} \cdot \begin{pmatrix} \dfrac{1}{\sigma_f^2} + \bar{x} \cdot \sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} & \sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} \\ \sum_{i=1}^{n} \dfrac{x_i}{\sigma_i^2} & \sum_{i=1}^{n} \dfrac{1}{\sigma_i^2} \end{pmatrix},$$

where

$$D_{V_A^{-1}} = |\boldsymbol{V}_A^{-1}| = \frac{1}{\sigma_f^2} \cdot \sum_{i=1}^{n} \frac{1}{\sigma_i^2}.$$

From $\boldsymbol{V}_A$, the variance of $\hat{k}_A$ can be acquired,

$$\sigma^2_{\hat{k}_A} = \sigma_x^2 + (\sigma_f \bar{x})^2$$

where σ_x^2 is defined as

$$\frac{1}{\sigma_x^2} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}, \text{ or } \sigma_x^2 = 1 \Big/ \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right).$$

(8) indicates the use of χ^2_A always gives the result of $\hat{k}_A = \bar{x}$. The reason can be seen from another form

$$= \sum_{i=1}^{n} \frac{(x_i - k/f)^2}{(\sigma_i)^2} + \frac{(1-f)^2}{\sigma_f^2}$$

In minimization process, for any f, the estimated $\hat{k}_A$ is determined by the summation terms of the above equation, at the same time, the last term constrains f to 1. The only influence of f is on $\sigma^2_{\hat{k}_A}$, which turns out to be equal to quadric combination of the weighted average error σ_s with $\sigma_f \bar{x}$, the normalization uncertainty on the average. This result corresponds to such a case, when the normalization factor in the definition of χ^2 is not included, the overall uncertainty is added in quadrature at the end[1)].

For the other three χ^2 forms, repeat the similar process, the needed results could be acquired[2)], which are summarized in Table 1.

Table 1. Results of four χ^2 forms.

Method	Parameter estimate	Scale factor	Estimate variance
χ^2_A	$\hat{k}_A = \hat{f}\cdot\bar{x}$	$\hat{f} = 1$	$\sigma^2_{\hat{k}_A} = \sigma_s^2 + (\sigma_f \bar{x})^2$
	$\hat{k}_B = \hat{f}\cdot x$	$\hat{f} = \frac{1}{1+\Sigma_{\sigma_f}}$	$\sigma^2_{\hat{k}_B} = \hat{f}\cdot(\sigma_s^2 + \sigma_f^2 \overline{x^2})$
	$\hat{k}_\alpha = \frac{1}{\hat{f}}\cdot\bar{x}$	$\hat{f} = 1$	$\sigma^2_{\hat{k}_\alpha} = \sigma_s^2 + (\sigma_f \bar{x})^2$
	$\hat{k}_\beta = \frac{1}{\hat{f}}\cdot\bar{x}$	$\hat{f} \in \{f^3(f-1) = \Sigma_{\sigma_f} \& f \neq 0\}$	$\sigma^2_{\hat{k}_\beta} = \sigma_s^2 + \frac{(\sigma_f\cdot\bar{x})^2}{\hat{f}^4 + 3\Sigma_{\sigma_f}}$

Note: Σ_{σ_f} is defined as $\Sigma_{\sigma_f} \equiv \sigma_f^2 \cdot \sum_{i=1}^{n} \frac{x_i(x_i - x)}{\sigma_i^2}$, with the expectation $\langle \Sigma_{\sigma_f} \rangle = \sigma_f^2 \cdot (n-1)$.

It can be seen that the scale factor $\hat{f}$ of χ^2_β should be solved from the quartic equation

$$f^3(f-1) = \Sigma_{\sigma_f}.$$

Four roots of the above equation are

$$= \frac{1}{4} - \frac{1}{2}S \mp \frac{1}{2}\cdot\left[\frac{3}{4} - S^2 - \frac{1}{4S}\right]^{1/2},$$

$$= \frac{1}{4} + \frac{1}{2}S \mp \frac{1}{2}\cdot\left[\frac{3}{4} - S^2 + \frac{1}{4S}\right]^{1/2},$$

where

$$S = \sqrt{\frac{1}{4} - \frac{4\cdot A}{T} + B\cdot T},$$

$$T = \sqrt[3]{\sqrt{3}\sqrt{27 + 256\Sigma_{\sigma_f}} - 9},$$

$$A = \sqrt[3]{\frac{2\Sigma^2_{\sigma_f}}{3}},$$

$$B = \sqrt[3]{\frac{\Sigma_{\sigma_f}}{18}}.$$

For the limit $\sigma_f = 0$, which corresponds to the case that there is no correlation between different points, and the scale factor should be identically equal to 1, then $\Sigma_{\sigma_f} = 0$, so it is easy to derive $f_{1,2,3} = 0$ and $f_4 = 1$. Therefore f_4 is the reasonable physical solution.

According to the results listed in Table 1, the minimization of χ^2_B and χ^2_β will give the biased parameter estimate k, which is not equal to the weighted average $\bar{x}$. To see the deviation in χ^2_β method, the scale factor $1/\hat{f}$ as the function of σ_f with different number of experiment

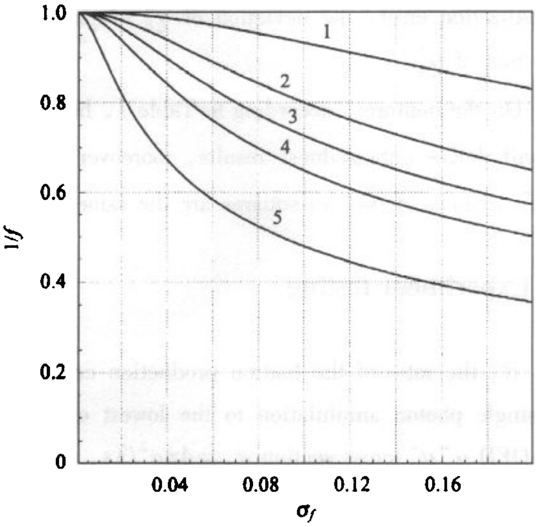

Fig.1. $1/\hat{f}$ variation for χ^2_β. The numbers on curve, 1,2,3 4 and 5, indicate different numbers of experiment points, which is 10, 50, 100, 200 and 1000, respectively.

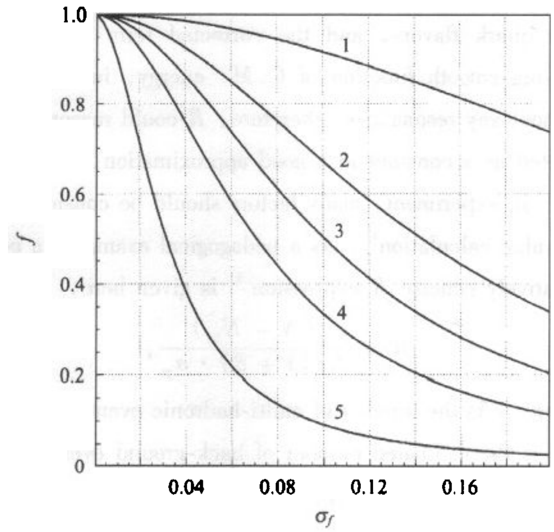

Fig.2. $\hat{f}$ variation for χ^2_B. The numbers on curve, 1,2,3, 4 and 5, indicate different numbers of experiment points, which is 10, 50, 100, 200 and 1000, respectively.

1) In fact, one special matrix approach can be used to obtain the same unbiased estimate, see appendix.

2) For χ^2_B, the detail of calculation could be found in Ref.[3].

points n, has been drawn in Fig.1. Correspondingly, the variation of factor $\hat{f}$ from χ^2_B is also drawn in Fig.2. Here the expectation of Σ_{σ_f}, namely $\langle \Sigma_{\sigma_f} \rangle = \sigma_f \cdot (n-1)$ is used to calculate scale factor f. Comparing Fig.1 and 2, it can be seen, both χ^2_B and χ^2_β will lead to biasness of minimization results. However, the degree of deviation from the weighted average is different. As to the same normalization error, the deviation of χ^2_β is much smaller than that of χ^2_B.

On the contrary, according to Table 1, both χ^2_A and χ^2_α will derive unbiasedness results, moreover, the estimate variances of two chi-squares are the same exactly.

4 Experiment testing

R, the ratio of the hadron production cross section via single photon annihilation to the lowest order point-like QED $\mu^+\mu^-$ cross section $\sigma_{pt} = 4\pi\alpha^2/3s$, is a fundamental quantity in e^+e^- interaction. It is calculated in the naive quark-parton model as $R = 3\sum_q Q_q^2$, where Q_q is the quark electric charge, and the summation runs over all the produced flavors. Taking the lowest order QCD correction and the electro-weak effect into consideration, R value would be larger than the naive value (10/3) for four quark flavors, and the corrected term is a slowly varying smooth function of C.M. energy, in the region without any resonances, therefore, R could reasonably be treated as a constant in a good approximation.

In experiment, many factors should be considered in R value calculation[1]. As a pedagogical example, a comparatively concise R expression[5] is given here,

$$R = \frac{(N - N_{bg})}{L\varepsilon(1+\delta)\cdot\sigma_{pt}},$$

where N is the number of multi-hadronic events detected, N_{bg} is the estimated number of back-ground events, L is the integrated luminosity, $\varepsilon(1+\delta)$ is the acceptance for the multi-hadronic events with radiative effect included and $(1+\delta)$ is the radiative correction factor due to higher order QED processes up to order α^3. Table 2 lists thirty eight experiment R-values[9]. From a study of data taken at different times at the same C.M. energy, the estimated systematic point-to-point errors are given as ± 3%. For the R value used here, the systematic uncertainty in the detection efficiency (± 8%), the luminosity measurement (± 6%), the event selection procedure (± 2%), and the background substraction (± 3%) yielded an common systematic error of ± 10%, which should be considered as normalization error. Now these thirty eight R-values will be used to test foregoing conclusions. For minimization, the MINUIT packages, one of useful CERN package in high energy physics[10], is utilized. In the χ^2 construction, the following substitute is adopted:

$$x_i \to R^i_{exp.},\ \sigma_i \to \Delta R^i_{exp.},\ \text{and}\ k \to R.$$

Table 2. Values for R[9]. The errors quoted are point-to-point systematic errors.

E_{cm}/GeV	R value	ΔR	E_{cm}/GeV	R value	ΔR
5.60	4.08	0.32	6.60	4.50	0.17
5.70	4.09	0.16	6.65	4.25	0.16
5.75	4.12	0.20	6.70	4.63	0.15
5.80	4.13	0.16	6.75	4.38	0.15
5.85	4.13	0.19	6.80	4.44	0.16
5.90	4.09	0.14	6.85	4.50	0.13
5.95	4.17	0.16	6.90	4.41	0.15
6.00	4.17	0.09	6.95	4.23	0.17
6.05	4.16	0.18	7.00	4.10	0.12
6.10	4.04	0.15	7.05	4.31	0.09
6.15	4.34	0.16	7.10	4.32	0.14
6.20	4.05	0.08	7.15	4.29	0.11
6.25	3.96	0.14	7.20	4.27	0.11
6.30	4.27	0.14	7.25	4.39	0.11
6.35	4.47	0.17	7.30	4.29	0.11
6.40	4.31	0.13	7.35	4.33	0.09
6.45	4.23	0.14	7.40	4.46	0.08
6.50	4.40	0.15	7.45	4.51	0.14
6.55	4.66	0.16	7.50	4.18	0.59

For example, the standard chi-square form should be written as

$$\chi^2 = \sum_i \frac{(R^i_{exp.} - R)^2}{(\Delta R^i_{exp.})^2}$$

First, the data listed in Table 2 are considered as uncorrelated (namely, the common sys-tematic error of 10% is not taken into account), and Formula (11) used in the fitting. Minimization gives $R = 4.284 \pm 0.021$

1) The R value measurement at BESII has been described in Refs.[7,8], where the detailed calculation about experiment R value could be found.

(fitting error only) with $\chi^2/n_{dof} = 55.93/37$, see Fig.3. In fact, as to this simple example, R and corresponding χ^2 could be computed directly from Formulae (9) and (11), which give the same results as those from the fitting.

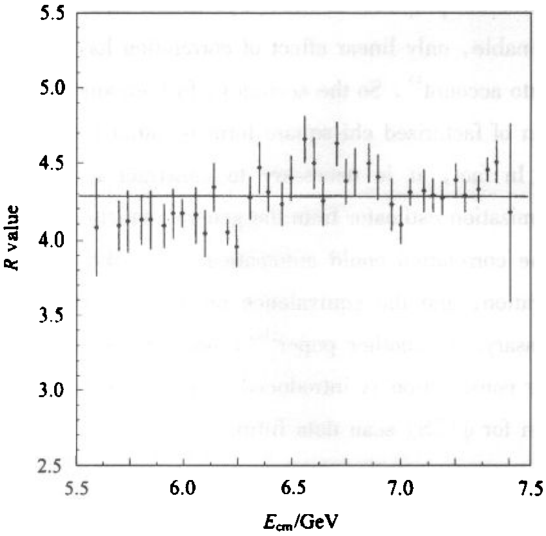

Fig.3. Fitted R value without any correlation (solid line indicates fitting result).

Next, in order to consider the correlation, a scale factor f is introduced, whose uncertainty ranges from 1 % to 20 %, centering around the normalization error 10 %. The fitted values and their errors (fitting errors only) are shown in Fig.4. The top straight line denotes the fitted R value for χ^2_A and χ^2_α forms, which is 4.284. The other two lines correspond to the theoretical curves of χ^2_B and χ^2_β, respectively. Unlike Fig.1 and 2, the value of Σ_{σ_f} instead of $\langle \Sigma_{\sigma_f} \rangle$ is used in curve drawing.

On the basis of simplified R-value measurement results, it can be concluded that the actual fitting results, shown in Fig.4, quantitatively agree with theoretical expectations, listed in Table 1:

1. The fitted R value in χ^2_A and χ^2_α forms consists with each other and with the weighted average. The deviation between χ^2_B and χ^2_β forms to the weighted average increase with the enhancement of σ_f.

2. The fitted uncertainties of R in χ^2_A and χ^2_α increase with the velocity proportional to the size of σ^2_f, but for χ^2_B and χ^2_β, the increasing velocity is relatively slow, especially in χ^2_β case[1)]

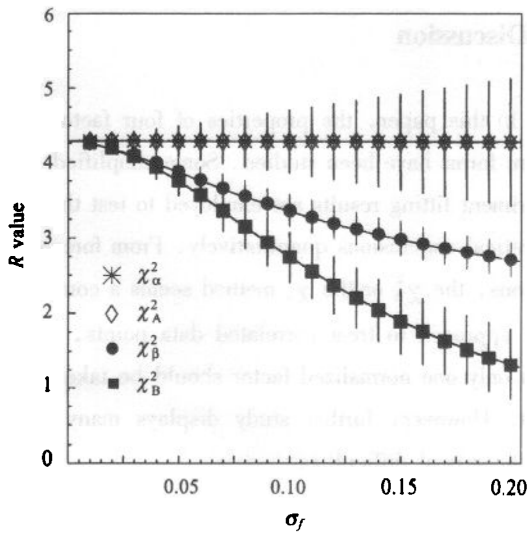

Fig.4. Fitted R value with correlation (solid lines indicate theoretical expectation).

By virtue of the concrete example, χ^2_B and χ^2_β are unfavorable from the minimization point of view. The reasons are two-fold. First, χ^2_B and χ^2_β produce biased fitting value and the bias might be rather severe when the common error σ_f is considerably large. Concretely speaking, for the 10 % common systematic error in our quoted R-value measurement, the fitted R equal to 2.747 for χ^2_B and 3.369 for χ^2_β, respectively. The deviation from the weighted average $\bar{R}$ is up to 36 % and 21 % correspondingly. Under such condition, the correction from factor f must be taken into account conscientiously. Second, χ^2_B and χ^2_β tend to underestimate the fitted error which is hard to correct. Comparing with χ^2_A or χ^2_α, the enhancement of fitting error is too slow with the increasing of common error σ_f for χ^2_B, while for χ^2_β, the fitting error almost keep the same for the variation of σ_f. But from the intuition of experiment physics, the common σ_f should be added directly into the final fitting error, just as that for χ^2_A or χ^2_α.

1) So far as uncertainty is concerned, the effect from σ^2_s could be neglected when σ_i is small and n is large. This point could be seen clearly for a simple case that all σ_i's are approximately equal, i.e. $\sigma_i \approx \sigma_p$. Under such condition, according to the definition of σ^2_s, it could be obtained that $\sigma^2_s \approx \sigma^2_p/n$. As to the example quoted in this paper, $\sigma_p \sim 3\%$, then $\sigma_s \approx 3\%/\sqrt{38} = 0.49\%$ which is small enough to be neglected.

5 Discussion

In this paper, the properties of four factorized chi-square forms have been studied. Some simplified R-value experiment fitting results are employed to test the relevant theoretical conclusions quantitatively. From foregoing discussions, the χ^2_A or the χ^2_a method seems a comparatively ideal approach to treat correlated data points, especially when only one normalized factor should be taken into account. However, further study displays many technical and theoretical difficulties.

First, the trick used in the equivalence proof for single factor and single parameter (i.e. constant) fitting case would lose its magic, when multi-factor and multi-parameter must be taken into consideration. Therefore, for minimization results from chi-square which contains many factors, it is hard to know if the matrix chi-square form could be used as a cross check for fitted results. This point makes it more difficulty to ensure that results from factorized chi-square fitting are reliable.

Second, although the scheme of covariance matrix construction proposed in Ref.[1], seems fairly crafty and reasonable, only linear effect of correlation has been taken into account1). So the accuracy, furthermore the application of factorized chi-square form is rather limited.

In fact, it is necessary to construct a reasonable minimization estimator from the general statistic principle, so the correlation could automatically be taken into consideration, and the equivalence proof would become unnecessary. In another paper[11], one method about estimator construction is introduced, together with its application for $\psi(2S)$ scan data fitting.

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Appendix

Matrix χ^2 form

Let us consider a certain offset case where all experiment results $x_i \pm \sigma_i$ $(i = 1, 2, \cdots, n)$ are affected by one calibration constant c whose error is σ_c and the best estimate value of c is 0 ($\hat{c} = 0$), then the error matrix could be constructed2) as follows:

$$V_c = \begin{pmatrix} \sigma_1^2 + \sigma_c^2 & \sigma_c^2 & \cdots & \sigma_c^2 \\ \sigma_c^2 & \sigma_2^2 + \sigma_c^2 & \cdots & \sigma_c^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_c^2 & \sigma_c^2 & \cdots & \sigma_n^2 + \sigma_c^2 \end{pmatrix}. \quad \text{(A1)}$$

Correspondingly, the χ^2 reads

$$\chi^2_c = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - k)\cdot(V_c^{-1})_{ij}\cdot(x_j - k), \quad \text{(A2)}$$

where subscript "c" indicates the effect from offest constant c. The first and second derivative of χ^2_c lead to the minimization estimate and variance:

$$\hat{k}_c = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i V_{ij}^{*}}{\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}^{*}} \quad \left(\text{from } \frac{\partial \chi^2_c}{\partial k} = 0\right), \quad \text{(A3)}$$

1) In Ref.[1], the equivalence between the factorized and the matrix chi-square form has been proved strictly. However, in matrix form, the covariance matrix is constructed according to the standard formalism of error propagation where only first derivatives are considered. But for a general and non-linear function, the first derivative is usually not accurate enough as approximation.

2) The details of correlated matrix construction can be found in Ref.[1], where two frequently happed cases, the offset and the normalization cases, have been studied.

$$\sigma_{\hat{k}_c}^2 = \frac{2}{\frac{\partial^2 \chi_c^2}{\partial k \partial k}} = \frac{DV_c}{\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}^{c*}}, \quad (A4)$$

where V_{ij}^{c*} indicates the element of the adjoint matrix of V_c. Using the following two formulae

$$D_{V_c} = \left\{1 + \sigma_c \cdot \sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right\}\cdot\left[\prod_{i=1}^{n}\sigma_i^2\right], \quad (A5)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}^{c*} = \left\{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\right\}\cdot\left[\prod_{i=1}^{n}\sigma_i^2\right], \quad (A6)$$

it can be worked out,

$$\hat{k}_c = \bar{x}, \quad (A7)$$

$$\sigma_{\hat{k}_c}^2 = \sigma_{\bar{x}}^2 + \sigma_c^2. \quad (A8)$$

Comparing Eq. (A8) with (8), together with Eq. (A8) with (10), it can be see if let $\sigma_c = \sigma_f\hat{k}(\hat{k} = x)$, then χ_A and χ_c would give the exactly same minimization estimate and variance. In another word, if it is assumed that normalization uncertainty produces a "global" correlation with constant covariance $(\sigma_f\hat{k})^2$, instead of a "local" correlation with variant covariance $(\sigma_f^2 x_i x_j)$, then the normalization uncertainty could transform into a special form of offset uncertainty. So the matrix method (i.e. χ_c^2 method) can also used to figure out the unbiased estimate.

关联实验数据的4种 χ^2 形式及其性质*

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摘要　研究了关联实验数据的四种因子化形式的 χ^2 表达式及其极小化性质,同时讨论了极小化估计值的有偏性与无偏性. 利用简化的R值测量定量地检验了相关的理论结果.

关键词　χ^2 形式　有偏性　无偏性　R值测量　关联实验数据

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更　正

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