$AdS_3\times S^3$弦的可积性*

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摘要　从Rahmfeld和Rajaraman构造的在$AdS_3\times S^3$背景中具有κ对称的GS弦的作用量出发, 推导出$AdS_3\times S^3$弦的运动方程, 然后利用连续的扭曲对偶变换构造了带有自由参数λ的平联络, 利用这些平联络可进一步得到无穷多守恒量, 说明此系统是可积的.

关键词　κ对称性　Green-Schwarz超弦　扭曲对偶变换　$AdS_3\times S^3$

1　引言

为了研究AdS/CFT对应[1—3], AdS背景中的弦理论受到广泛关注. 1998年Metsaev和Tseytlin(MT)提出超陪集方法[4], 构造了$AdS_5\times S^5$背景中Green-Schwarz(GS)超弦作用量, 这个作用量具有局域的κ对称性. 2003年Bena, Polchinski和Roiban在MT的基础上, 进一步构造了此模型无穷多非定域的守恒荷[5], 说明世界叶上的σ模型是完全可积的. 最近Polyakov[6]和侯伯宇等人[7, 8]分别利用连续的Hodge对偶变换, 在群上给出了此模型的无穷多守恒流. 这两种方法是等价的, 它们都说明在$AdS_5\times S^5$背景中, 具有κ对称性的GS超弦是可积的, 这使得可积场论的方法可以用来研究AdS_5弦.

在MT之后, Rahmfeld和Rajaraman利用超陪集方法构造了$AdS_3\times S^3$背景中GS超弦作用量[9]. 最近, 陈斌等人进一步研究此模型, 利用Bena, Polchinski和Roiban的方法, 在陪集上构造了无穷多平流[10]. 文中将用连续的Hodge对偶变换, 在群上构造带自由参数的平联络. 从Rahmfeld和Rajaraman给出的$AdS_3\times S^3$背景中GS超弦作用量出发, 推导其运动方程, 然后构造平联络, 利用世界叶上Maurer-Cartan 1-形式的Hodge对偶以及运动方程, 详细证明带自由参数的新联络满足平曲率条件, 由此可以进一步得到无穷多守恒量, 说明$AdS_3\times S^3$弦是可积的.

2　$AdS_3\times S^3$背景中GS弦的作用量

下面重温$AdS_3\times S^3$背景中GS弦的作用量. 对于这种情况, 超陪集流形是$\mathscr{G}=\dfrac{SU(1,1|2)_{\rm L}\times SU(1,1|2)_{\rm R}}{SO(1,2)\times SO(3)}$, 它的玻色部分是$\dfrac{(SU(1,1)\times SU(2))^2}{SO(1,2)\times SO(3)}\simeq\dfrac{SO(2,2)}{SO(1,2)}\times\dfrac{SO(4)}{SO(3)}\simeq AdS_3\times S^3$. 作为构造作用量的第一步, 先来研究$SU(1,1|2)^2$超代数.

2.1　$SU(1,1|2)^2$超代数

$SU(1,1|2)^2$超代数的生成元如下, $T_A=(P_a, P_{a'}; J_{ab}, J_{a'b'}|Q_{\alpha\alpha' I})$, 其中不带撇的是$AdS_3$的, 带撇的是$S^3$的, $Q_{\alpha\alpha' I}$是费米生成元. 这里$a=0,1,2$, $a'=3,4,5$, $I=1,2,\alpha,\alpha'=1,2$.

6维的Dirac矩阵表示为

$$\Gamma^a=\gamma^a\otimes\boldsymbol{I}\otimes\sigma^1,\tag{1}$$

$$\Gamma^{a'}=\boldsymbol{I}\otimes\gamma^{a'}\otimes\sigma^2,\tag{2}$$

其中$\gamma^0={\rm i}\sigma^3$, $\gamma^{1,2}=\sigma^{1,2}$, $\gamma^{a'}=\sigma^{a'-2}$, 并且共轭超荷定义为

$$\bar{Q}^I=(Q^I)^\dagger\gamma^0.\tag{3}$$

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由此 $SU(1,1|2)^2$ 超代数表述如下,

$$\begin{aligned}\{Q_I,\bar{Q}_J\} &= 2\delta_{IJ}\left(\mathrm{i}P_a\gamma^a - P_{a'}\gamma^{a'}\right)+\\ &\quad \epsilon_{IJ}\left(J_{ab}\gamma^{ab}-J_{a'b'}\gamma^{a'b'}\right),\end{aligned}$$

$$\begin{aligned}&[P_a,Q_I]=-\frac{\mathrm{i}}{2}\epsilon_{IJ}\gamma_aQ_J, \quad [P_{a'},Q_I]=\frac{1}{2}\epsilon_{IJ}\gamma_{a'}Q_J,\\ &[J_{ab},Q_I]=-\frac{1}{2}\gamma_{ab}Q_I, \quad [J_{a'b'},Q_I]=-\frac{1}{2}\gamma_{a'b'}Q_I,\\ &\left[P_a,\bar{Q}_I\right]=\frac{\mathrm{i}}{2}\epsilon_{IJ}\bar{Q}_J\gamma_a, \quad \left[P_{a'},\bar{Q}_I\right]=-\frac{1}{2}\epsilon_{IJ}\bar{Q}_J\gamma_{a'},\\ &\left[J_{ab},\bar{Q}_I\right]=\frac{1}{2}\bar{Q}_I\gamma_{ab}, \quad \left[J_{a'b'},\bar{Q}_I\right]=\frac{1}{2}\bar{Q}_I\gamma_{a'b'},\end{aligned} \tag{4}$$

$$\begin{aligned}[M_{AB},M_{CD}] &= \eta_{BC}M_{AD}+\eta_{AD}M_{BC}-\\ &\quad \eta_{AC}M_{BD}-\eta_{BD}M_{AC},\end{aligned} \tag{5}$$

$$\begin{aligned}[M_{A'B'},M_{C'D'}] &= \delta_{B'C'}M_{A'D'}+\delta_{A'D'}M_{B'C'}-\\ &\quad \delta_{A'C'}M_{B'D'}-\delta_{B'D'}M_{AC}.\end{aligned} \tag{6}$$

这里定义

$$P_a=M_{0a},\ J_{ab}=M_{ab},\ P_{a'}=M_{0'a'},\ J_{a'b'}=M_{a'b'}, \tag{7}$$

并且 $\eta=(-++-)$.

$SU(1,1|2)^2$ 代数的左不变的Cartan 1-形式

$$L^A=dX^ML^A_M, \quad X^M=(x,\theta) \tag{8}$$

定义为

$$\begin{aligned}G^{-1}dG=L^AT_A &\equiv L^aP_a+L^{a'}P_{a'}+\frac{1}{2}L^{ab}J_{ab}+\\ &\frac{1}{2}L^{a'b'}J_{a'b'}+\frac{1}{2}\left(\bar{L}^IQ^I+\bar{Q}^IL^I\right),\end{aligned} \tag{9}$$

这里 $G=G(x,\theta)$ 是群 $SU(1,1|2)^2$ 中的陪集代表元. 将群 $SU(1,1|2)^2$ 的结构方程

$$d\left(G^{-1}dG\right)=-G^{-1}dG\wedge G^{-1}dG, \tag{10}$$

按群的生成元展开, 得到如下Maurer-Cartan方程:

$$dL^a = -L^b\wedge L^{ba}-\frac{\mathrm{i}}{2}\bar{L}^I\gamma^a\wedge L^I, \tag{11}$$

$$dL^{a'} = -L^{b'}\wedge L^{b'a'}+\frac{1}{2}\bar{L}^I\gamma^{a'}\wedge L^I, \tag{12}$$

$$dL^{ab} = -L^a\wedge L^b-L^{ac}\wedge L^{cb}+\epsilon_{IJ}\bar{L}^I\gamma^{ab}L^J, \tag{13}$$

$$dL^{a'b'} = L^{a'}\wedge L^{b'}-L^{a'c'}\wedge L^{c'b'}-\epsilon_{IJ}\bar{L}^I\gamma^{a'b'}L^J, \tag{14}$$

$$\begin{aligned}dL^I &= -\frac{\mathrm{i}}{2}\epsilon_{IJ}L^a\wedge\gamma_aL^J+\frac{1}{2}\epsilon_{IJ}L^{a'}\wedge\gamma_{a'}L^J-\\ &\quad \frac{1}{4}L^{ab}\wedge\gamma_{ab}L^I-\frac{1}{4}L^{a'b'}\wedge\gamma_{a'b'}L^I,\end{aligned} \tag{15}$$

$$\begin{aligned}d\bar{L}^I &= \frac{\mathrm{i}}{2}\epsilon_{IJ}\bar{L}^J\gamma_a\wedge L^a-\frac{1}{2}\epsilon_{IJ}\bar{L}^J\gamma_{a'}\wedge L^{a'}-\\ &\quad \frac{1}{4}\bar{L}^I\gamma_{ab}\wedge L^{ab}-\frac{1}{4}\bar{L}^I\gamma_{a'b'}\wedge L^{a'b'}.\end{aligned} \tag{16}$$

2.2 GS弦的作用量和运动方程

在 $AdS_3\times S^3$ 背景中运动的超弦采用Green-Schwarz描述形式, 它的作用量表述如下[9],

$$S = S_0+S_1, \tag{17}$$

$$S_0 = -\frac{1}{2}\int_{\partial M_3}\mathrm{d}^2\sigma\sqrt{g}g^{ij}(L^a_iL^a_j+L^{a'}_iL^{a'}_j), \tag{18}$$

$$\begin{aligned}S_1 &= \mathrm{i}\int_{M_3}\mathscr{L}_{WZ}=\\ &\frac{\mathrm{i}}{4}\int_{M_3}S^{IJ}\big((L^a\wedge\bar{L}^I\gamma^a\wedge L^J+\\ &\mathrm{i}L^{a'}\wedge\bar{L}^I\gamma^{a'}\wedge L^J)+\mathrm{c.c.}\big),\end{aligned} \tag{19}$$

其中 $\sqrt{g}=\sqrt{-\det g_{ij}}$, $g_{ij}g^{jk}=\delta_{ik}$, $i,j=0,1$, $S^{IJ}=\mathrm{diag}\{1,-1\}$, 并且 g_{ij} 是世界叶上的度规.

为了推导运动方程, 先求出Cartan 1-形式的变分. 将下面的等式

$$\delta\left(G^{-1}dG\right)=[G^{-1}dG,G^{-1}\delta G]+d\left(G^{-1}\delta G\right), \tag{20}$$

按李代数的生成元分解, 就得到了Cartan 1-形式的变分:

$$\begin{aligned}\delta L^a &= d\delta x^a+L^{ab}\delta x^b+L^b\delta x^{ba}+\\ &\frac{\mathrm{i}}{2}(\bar{L}^I\gamma^a\delta\theta^I-\delta\bar{\theta}^I\gamma^aL^I),\end{aligned} \tag{21}$$

$$\begin{aligned}\delta L^{a'} &= d\delta x^{a'}+L^{a'b'}\delta x^{b'}+L^{b'}\delta x^{b'a'}-\\ &\frac{1}{2}(\bar{L}^I\gamma^{a'}\delta\theta^I-\delta\bar{\theta}^I\gamma^{a'}L^I),\end{aligned} \tag{22}$$

$$\begin{aligned}\delta L^I &= d\delta\theta^I-\frac{\mathrm{i}}{2}\epsilon_{IJ}\left(\delta x^a\gamma^a+\mathrm{i}\delta x^{a'}\gamma^{a'}\right)L^J+\\ &\frac{\mathrm{i}}{2}\epsilon_{IJ}\left(L^a\gamma^a+\mathrm{i}L^{a'}\gamma^{a'}\right)\delta\theta^J-\\ &\frac{1}{4}\left(\delta x^{ab}\gamma^{ab}+\delta x^{a'b'}\gamma^{a'b'}\right)L^I+\\ &\frac{1}{4}\left(L^{ab}\gamma^{ab}+L^{a'b'}\gamma^{a'b'}\right)\delta\theta^I,\end{aligned} \tag{23}$$

$$\begin{aligned}\delta\bar{L}^I &= d\delta\bar{\theta}^I-\frac{\mathrm{i}}{2}\epsilon_{IJ}\bar{L}^J\left(\delta x^a\gamma^a+\mathrm{i}\delta x^{a'}\gamma^{a'}\right)+\\ &\frac{\mathrm{i}}{2}\epsilon_{IJ}\delta\bar{\theta}^J\left(\gamma^aL^a+\mathrm{i}\gamma^{a'}L^{a'}\right)+\\ &\frac{1}{4}\bar{L}^I\left(\delta x^{ab}\gamma^{ab}+\delta x^{a'b'}\gamma^{a'b'}\right)-\\ &\frac{1}{4}\delta\bar{\theta}^I\left(\gamma^{ab}L^{ab}+\gamma^{a'b'}L^{a'b'}\right).\end{aligned} \tag{24}$$

作用量中WZ项 $\mathscr{L}_{\mathrm{WZ}}$ 仍然是闭的3-形式, 它的变分如下,

$$\delta\mathscr{L}_{WZ}=d\Lambda, \tag{25}$$

$$\begin{aligned}\Lambda &= \frac{1}{2}S^{IJ}(\bar{L}^I\gamma^a\wedge L^J\delta x^a-\bar{L}^I\gamma^a\wedge L^a\delta\theta^J-\\ &\delta\bar{\theta}^I\gamma^aL^a\wedge L^J+\mathrm{i}\bar{L}^I\gamma^{a'}\wedge L^J\delta x^{a'}-\\ &\mathrm{i}\bar{L}^I\gamma^{a'}\wedge L^{a'}\delta\theta^J-\mathrm{i}\delta\bar{\theta}^I\gamma^{a'}L^{a'}\wedge L^J).\end{aligned} \tag{26}$$

现在对作用量(17)做变分, 就可以得到如下的运动方程,

$$\partial_i(\sqrt{g}g^{ij}L_j^a)+\sqrt{g}g^{ij}L_i^{ab}L_j^b+\frac{\mathrm{i}}{2}\epsilon^{ij}S^{IJ}\bar{L}_i^I\gamma^a L_j^J=0, \tag{27}$$

$$\partial_i(\sqrt{g}g^{ij}L_j^{a'})+\sqrt{g}g^{ij}L_i^{a'b'}L_j^{b'}-\frac{1}{2}\epsilon^{ij}S^{IJ}\bar{L}_i^I\gamma^{a'}L_j^J=0, \tag{28}$$

$$(L_i^a\gamma^a+\mathrm{i}L_i^{a'}\gamma^{a'})(\sqrt{g}g^{ij}\delta^{IJ}-\epsilon^{ij}S^{IJ})L_j^J=0, \tag{29}$$

$$\bar{L}_i^I(\gamma^a L_j^a+\mathrm{i}\gamma^{a'}L_j^{a'})(\sqrt{g}g^{ij}\delta^{IJ}+\epsilon^{ij}S^{IJ})=0. \tag{30}$$

作用量(17)式对度规g_{ij}做变分就给出了标准的Virasoro约束,

$$L_i^aL_j^a+L_i^{a'}L_j^{a'}=\frac{1}{2}g_{ij}g^{kl}\left(L_k^aL_l^a+L_k^{a'}L_l^{a'}\right). \tag{31}$$

从以上做变分的过程, 可以很容易验证作用量(17)式具有κ对称性.

3 $AdS_3\times S^3$弦的平联络

3.1 Cartan形式L^a和$L^{a'}$的Hodge对偶

下面介绍构造$AdS_3\times S^3$弦的平联络, 首先引入Maurer-Cartan 1-形式L^a和$L^{a'}$在世界叶上的Hodge对偶. 令

$$\sqrt{g}g^{ij}L_i^a\equiv\epsilon^{jk}\,{}^*L_k^a, \tag{32}$$

即

$$\sqrt{g}g^{i1}L_i^a\equiv\epsilon^{12*}L_2^a={}^*L_2^a, \tag{33}$$

$$\sqrt{g}g^{i2}L_i^a\equiv\epsilon^{21*}L_1^a=-{}^*L_1^a. \tag{34}$$

将上式带入运动方程(27)中, 得

$$\epsilon^{jk}\partial_j^*L_k^a+L_j^{ab}\epsilon^{jk*}L_k^b+\frac{\mathrm{i}}{2}S^{IJ}\epsilon^{jk}\bar{L}_j^I\gamma^aL_k^J=0, \tag{35}$$

两边同乘$d\sigma^j\wedge d\sigma^k$得到如下方程

$$d^*L^a+L^{ab}\wedge{}^*L^b+\frac{\mathrm{i}}{2}S^{IJ}\bar{L}^I\gamma^aL^J=0. \tag{36}$$

采用相同的办法, 将(33), (34)两式带入剩余的运动方程中, 得到

$$d^*L^{a'}+L^{a'b'}\wedge{}^*L^{b'}-\frac{1}{2}S^{IJ}\bar{L}^I\gamma^{a'}L^J=0, \tag{37}$$

$$\begin{aligned}&\delta^{IJ}\left({}^*L^a\gamma^a+\mathrm{i}^*L^{a'}\gamma^{a'}\right)\wedge L^J+\\&S^{IJ}\left(L^a\gamma^a+\mathrm{i}L^{a'}\gamma^{a'}\right)\wedge L^J=0,\end{aligned} \tag{38}$$

$$\begin{aligned}&\bar{L}^I\wedge\left(\gamma^{a*}L^a+\mathrm{i}\gamma^{a'*}L^{a'}\right)\delta^{IJ}+\\&\bar{L}^I\wedge\left(\gamma^aL^a+\mathrm{i}\gamma^{a'}L^{a'}\right)S^{IJ}=0.\end{aligned} \tag{39}$$

以上4个方程就是运动方程的Hodge对偶, 它们在构造平联络的过程中非常重要.

3.2 连续的扭曲对偶变换

下面引入$SU(1,1|2)^2$超代数上连续的扭曲对偶变换[6—8]:

$$L^a(\lambda)=\frac{1}{2}(\lambda^2+\lambda^{-2})L^a+\frac{1}{2}(\lambda^2-\lambda^{-2})\,{}^*L^a, \tag{40}$$

$$L^{a'}(\lambda)=\frac{1}{2}(\lambda^2+\lambda^{-2})L^{a'}+\frac{1}{2}(\lambda^2-\lambda^{-2})\,{}^*L^{a'}, \tag{41}$$

$$L^{ab}(\lambda)=L^{ab},\quad L^{a'b'}(\lambda)=L^{a'b'}, \tag{42}$$

$$L^1(\lambda)=\lambda L^1,\quad L^2(\lambda)=\lambda^{-1}L^2, \tag{43}$$

$$\bar{L}^1(\lambda)=\lambda\bar{L}^1,\quad \bar{L}^2(\lambda)=\lambda^{-1}\bar{L}^2. \tag{44}$$

3.3 构造平联络

上面通过利用连续的扭曲对偶变换得到了新的Cartan形式, 这些Cartan形式都带有自由的参数λ. 下面将证明这些新的Cartan形式都满足Maurer-Cartan方程(11)—(16), 从而构造带有自由的参数λ的平联络. 首先, 来验证$L^a(\lambda)$满足的方程,

$$\begin{aligned}dL^a(\lambda)=&\frac{1}{2}(\lambda^2+\lambda^{-2})dL^a+\frac{1}{2}(\lambda^2-\lambda^{-2})d^*L^a=\\&-\left(\frac{1}{2}(\lambda^2+\lambda^{-2})L^b+\frac{1}{2}(\lambda^2-\lambda^{-2})\,{}^*L^b\right)\wedge\\&L^{ba}-\frac{\mathrm{i}}{2}\lambda^2\bar{L}^1\gamma^aL^1-\frac{\mathrm{i}}{2}\lambda^{-2}\bar{L}^2\gamma^aL^2=\\&-L^b(\lambda)\wedge L^{ba}(\lambda)-\frac{\mathrm{i}}{2}\bar{L}^I(\lambda)\gamma^a\wedge L^I(\lambda).\end{aligned} \tag{45}$$

其次, 验证$L^{ab}(\lambda)$满足的方程.

$$\begin{aligned}dL^{ab}(\lambda)=&dL^{ab}=-L^a\wedge L^b-L^{ac}\wedge L^{cb}+\\&\epsilon_{IJ}\bar{L}^I\gamma^{ab}\wedge L^J=-L^a(\lambda)\wedge L^b(\lambda)-\\&L^{ac}(\lambda)\wedge L^{cb}(\lambda)+\epsilon_{IJ}\bar{L}^I(\lambda)\gamma^{ab}\wedge L^J(\lambda),\end{aligned} \tag{46}$$

这里后两项$\left(\epsilon_{IJ}\bar{L}^I\gamma^{ab}\wedge L^J=\epsilon_{IJ}\bar{L}^I\gamma^{ab}\wedge L^J\right)$和$(L^{ac}\wedge L^{cb}=L^{ac}(\lambda)\wedge L^{cb}(\lambda))$是直截了当的. 对于第一项, 利用Cartan形式的性质

$$A\wedge{}^*B=-{}^*A\wedge B \tag{47}$$

$${}^*A\wedge{}^*B=-A\wedge B. \tag{48}$$

可得

$$-L^a\wedge L^b=-L^a(\lambda)\wedge L^b(\lambda). \tag{49}$$

同理可得

$$dL^{a'}(\lambda)=-L^{b'}(\lambda)\wedge L^{b'a'}(\lambda)+\frac{1}{2}\bar{L}^I(\lambda)\gamma^{a'}\wedge L^I(\lambda), \tag{50}$$

$$dL^{a'b'}(\lambda) = L^{a'}(\lambda)\wedge L^{b'}(\lambda) - L^{a'c'}(\lambda)\wedge L^{c'b'}(\lambda) - \epsilon_{IJ}\bar{L}^{I}(\lambda)\gamma^{a'b'}\wedge L^{J}(\lambda). \tag{51}$$

下面证明最后两个Maurer-Cartan方程. 因为费米的Cartan 1-形式L^I和$\bar{L}^I$的分量变换规则不同, 所以以下的证明过程中将采用分量式. 方程(38)写成分量式变成

$$\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'} + {}^*L^a\gamma^a + \mathrm{i}{}^*L^{a'}\gamma^{a'}\right)\wedge L^1 = 0 \ \ (J=1), \tag{52}$$

$$\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'} - {}^*L^a\gamma^a - \mathrm{i}\ {}^*L^{a'}\gamma^{a'}\right)\wedge L^2 = 0 \ \ (J=2). \tag{53}$$

要验证第5个Maurer-Cartan方程, 先考虑方程右边的前两项.

$$-\frac{\mathrm{i}}{2}\epsilon_{IJ}L^a(\lambda)\wedge\gamma_a L^J(\lambda) + \frac{1}{2}\epsilon_{IJ}L^{a'}(\lambda)\wedge\gamma_{a'}L^J(\lambda) = -\frac{\mathrm{i}}{2}\epsilon_{IJ}\left[\left(\frac{\lambda^2}{2}\right)\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'} + {}^*L^a\gamma^a + \mathrm{i}\ {}^*L^{a'}\gamma^{a'}\right)\wedge L^J(\lambda) + \left(\frac{\lambda^{-2}}{2}\right)\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'} - {}^*L^a\gamma^a - \mathrm{i}{}^*L^{a'}\gamma^{a'}\right)\wedge L^J(\lambda)\right], \tag{54}$$

将方程(52)和方程(53)带入上式, 有

$$-\frac{\mathrm{i}}{2}\epsilon_{IJ}L^a(\lambda)\wedge\gamma_a L^J(\lambda) + \frac{1}{2}\epsilon_{IJ}L^{a'}(\lambda)\wedge\gamma_{a'}L^J(\lambda) = \begin{cases} -\frac{\mathrm{i}}{2}\lambda\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'}\right)\wedge L^2 & (I=1) \\ \frac{\mathrm{i}}{2}\lambda^{-1}\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'}\right)\wedge L^1 & (I=2)\end{cases}, \tag{55}$$

因此, 得到

$$dL^1(\lambda) = \lambda dL^1 = -\lambda\frac{\mathrm{i}}{2}\left(L^a\gamma^a + \mathrm{i}L^{a'}\gamma^{a'}\right)\wedge L^2 - \lambda\frac{1}{4}L^{ab}\wedge\gamma_{ab}L^1 - \lambda\frac{1}{4}L^{a'b'}\wedge\gamma_{a'b'}L^1 = -\frac{\mathrm{i}}{2}L^a(\lambda)\wedge\gamma_a L^2(\lambda) + \frac{1}{2}L^{a'}(\lambda)\wedge\gamma_{a'}L^2(\lambda) - \frac{1}{4}L^{ab}(\lambda)\wedge\gamma_{ab}L^1(\lambda) - \frac{1}{4}L^{a'b'}(\lambda)\wedge\gamma_{a'b'}L^1(\lambda) \tag{56}$$

和

$$dL^2(\lambda) = \frac{\mathrm{i}}{2}L^a(\lambda)\wedge\gamma_a L^1(\lambda) - \frac{1}{2}L^{a'}(\lambda)\wedge\gamma_{a'}L^1(\lambda) - \frac{1}{4}L^{ab}(\lambda)\wedge\gamma_{ab}L^2(\lambda) - \frac{1}{4}L^{a'b'}(\lambda)\wedge\gamma_{a'b'}L^2(\lambda). \tag{57}$$

合并上面两式, 即

$$dL^I(\lambda) = -\frac{\mathrm{i}}{2}\epsilon_{IJ}L^a(\lambda)\wedge\gamma_a L^J(\lambda) + \frac{1}{2}\epsilon_{IJ}L^{a'}(\lambda)\wedge\gamma_{a'}L^J(\lambda) - \frac{1}{4}L^{ab}(\lambda)\wedge\gamma_{ab}L^I(\lambda) - \frac{1}{4}L^{a'b'}(\lambda)\wedge\gamma_{a'b'}L^I(\lambda). \tag{58}$$

同理, 可以证明

$$d\bar{L}^I(\lambda) = \frac{\mathrm{i}}{2}\epsilon_{IJ}\bar{L}^J(\lambda)\gamma_a\wedge L^a(\lambda) - \frac{1}{2}\epsilon_{IJ}\bar{L}^J(\lambda)\gamma_{a'}\wedge L^{a'}(\lambda) - \frac{1}{4}\bar{L}^I(\lambda)\gamma_{ab}\wedge L^{ab}(\lambda) - \frac{1}{4}\bar{L}^I(\lambda)\gamma_{a'b'}\wedge L^{a'b'}(\lambda). \tag{59}$$

综上, 证明了这些新的Cartan 1-形式都满足Maurer-Cartan方程(11)—(16). 下面定义带有自由参数λ的平流,

$$\mathscr{J}(\lambda) \equiv L^a(\lambda)P_a + L^{a'}(\lambda)P_{a'} + \frac{1}{2}L^{ab}(\lambda)J_{ab} + \frac{1}{2}L^{a'b'}(\lambda)J_{a'b'} + \frac{1}{2}\left(\bar{Q}^I L_I(\lambda) + \bar{L}^I(\lambda)Q_I\right), \tag{60}$$

它满足平联络条件

$$d\mathscr{J}(\lambda) + \mathscr{J}(\lambda)\wedge\mathscr{J}(\lambda) = 0. \tag{61}$$

有了带有自由参数λ的平联络, 就可以很容易构造出无穷多的守恒量.

4 结论

从$AdS_3\times S^3$背景中具有局域的κ对称性的GS超弦作用量出发, 我们构造了带自由参数的非定域守恒流. 由这些守恒流可以通过标准的方法得到一系列的非定域守恒荷. 这说明世界叶上的2维σ模型至少在经典的水平上是完全可积的. 构造守恒荷首先需要一族流$\mathscr{J}(\lambda)$光滑的依赖于谱参数λ, 并且满足平流条件: $d\mathscr{J}(\lambda) + \mathscr{J}(\lambda)\wedge\mathscr{J}(\lambda) = 0$. 因此可以构造monodromy矩阵

$$T_{(\lambda)}(t) = P\exp\int_{(-\infty,t)}^{(+\infty,t)}\mathscr{J}(\lambda),$$

因为$\mathscr{J}(\lambda)$满足平流条件, 所以monodromy矩阵$T_{(\lambda)}(t)$不随时间改变. 因而将$T_{(\lambda)}$按谱参数λ的幂次展开, 展开系数就给出了一系列守恒荷. 对于周期边界条件(闭弦), 有

$$\mathrm{Tr}\,T_{(\lambda)}(t) = \mathrm{Tr}\left(P\exp\oint\mathscr{J}_\sigma(\lambda,t,\sigma)\,\mathrm{d}\sigma\right),$$

它仍然是守恒的. 总之, 对于以上两种边界条件, 均可构造出带自由参数λ的守恒量, 这说明此系统是可积的.

参考文献(References)

1 Maldacena J M. Adv. Theor. Math. Phys., 1998, **2**: 231

2 Gubser S S, Klebanov I R, Polyakov A M. Phys. Lett., 1998, **B428**: 105

3 Witten E. Adv. Theor. Math. Phys., 1998, **2**: 253

4 Metsaev R R, Tseytlin A A. Nucl. Phys., 1998, **B533**: 109

5 Bena I, Polchinski J, Roiban R. Phys. Rev., 2004, **D69**: 046002

6 Polyakov A M. Mod. Phys. Lett., 2004, **A19**: 1649

7 HOU Bo-Yu, PENG Dan-Tao, XIONG Chuan-Hua et al. The Affine Hidden Symmetry and Integrability of Type IIB Superstring in $AdS_5 \times S^5$. arXiv:hep-th/0406239

8 XIONG Chuan-Hua. Acta Physica Sinica, 2005, **54**: 0047 (in Chinese)
(熊传华. 物理学报, 2005, **54**: 0047)

9 Rahmfeld J, Rajaraman A. Phys. Rev., 1999, **D60**: 064014

10 CHEN Bin, HE Ya-Li, ZHANG Peng et al. Phys. Rev., 2005, **D71**: 086007

Integrability of $AdS_3 \times S^3$ Superstring*

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Abstract　From the κ symmetric action of IIB string in $AdS_3 \times S^3$ background given by Rahmfeld and Rajaraman, we derive the equations of motion. Then using the twisted dual transformation which was introduced by Hou et al. We construct the flat currents, and hence conserved non-local charge with one free parameter, for the Green-Schwarz superstring in $AdS_3 \times S^3$. Thus we show the $AdS_3 \times S^3$ string is integrable.

Key words　κ-symmetry, Green-Schwarz superstring, twisted daul transformation, $AdS_3 \times S^3$

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