Global Bipartite Entanglement in the Three-Qubit Heisenberg XXX Spin Chain with Impurity*

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Abstract We study the global bipartite entanglement of the three-qubit Heisenberg XXX spin chain with impurity. Through calculating the negativities $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$, we show that the critical temperature T_c above which the entanglement vanishes increases with the increase of the impurity parameterJ_1. For a given T, the corresponding critical impurity parameterJ_{1c} below which the entanglement vanishes increases with the increase of the magnetic field B, and by adjusting J_1 and B one can control the values of $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$. The maximum value of $\mathcal{N}_{12\text{-}3}$ decreases from 0.5 to 0.3727 as the temperature rises, but the one of $\mathcal{N}_{1\text{-}23}$ keeps the constant value of about 0.4714.

Key words Heisenberg XXX chain, global bipartite entanglement, impurity

1 Introduction

Quantum entanglement, which has no classical analog, is one of the most notable features of quantum mechanics. It provides a new perspective for the analysis of correlations and transitions in many-body quantum systems[1, 2]. And more importantly, it also provides a key resource in realizing quantum information, such as quantum teleportation[3, 4], quantum key distribution[5], super-dense coding[6], quantum cloning[7] et al. In particular, as an important entanglement resource in the field of condensed-matter-physics, the Heisenberg spin system has also been used to construct a quantum computer and quantum dots[8]. Consequently, it has been extensively studied in recent years[9—14].

However, as far as we know, most discussions mentioned above only focused on the calculation and analysis of the concurrence C, which measures the entanglement between any and only two qubits. Recently, N. Canosa and R. Rossignoli[15] discussed the global bipartite entanglement in the XXZ chains with another measurement named negativity[16] associated with bipartitions of the whole system into two subsystems. Nagativity is a measure of the degree of violation of the criterion of positive partial transpose(PPT) in entangled states and is sufficient just for two-qubit or qubit+qutrit systerm. Moreover, impurity and magnetic filed play an important role in the 1D quantum system[17, 18], so it is meaningful to investigate the global bipartite entanglement of the Heisenberg chain with impurity.

In this paper, we consider the global bipartite entanglement in the three-qubit Heisenberg XXX chain with impurity in the presence of a uniform magnetic field B. Our paper is organized as follows. In Sec. 2 we give the analytical solution of the model. In Sec. 3, we analyze the ground-state entanglement, obtain the exact expressions of the negativities $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$ at finite temperature without magnetic field B, and

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also we give the numerical solution of thermal entanglement. Finally, in Sec. 4, we give a conclusion of this paper.

2 Solution of the model

We assume the impurity spin is located at the first site and impose the periodic boundary condition, then the corresponding Hamiltonian can be written as

$$H = J_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_1)+J\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3+B\sum_{i=1}^{3}\sigma_i^z \ , \quad (1)$$

where J_1 denotes the coupling between the normal site and the impurity site and J denotes that between the normal sites. In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the eigenvalues of the Hamiltonian (1) are analytically obtained as

$$\begin{aligned} E_{1,2} &= -3J\pm B, \quad E_{3,4}=J-4J_1\pm B, \\ E_{5,6} &= 2J_1+J\pm B, \ E_{7,8}=2J_1+J\pm 3B, \end{aligned} \quad (2)$$

with the corresponding eigenstates

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|010\rangle-|001\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|110\rangle-|101\rangle), \\ |\psi_3\rangle &= \frac{1}{\sqrt{6}}(|001\rangle+|010\rangle-2|100\rangle), \\ |\psi_4\rangle &= \frac{1}{\sqrt{6}}(2|011\rangle-|101\rangle-|110\rangle), \\ |\psi_5\rangle &= \frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle), \\ |\psi_6\rangle &= \frac{1}{\sqrt{3}}(|011\rangle+|101\rangle+|110\rangle), \\ |\psi_7\rangle &= |000\rangle, \\ |\psi_8\rangle &= |111\rangle. \end{aligned} \quad (3)$$

Let ρ be the density matrix, then the global bipartite entanglement between the subsystem of $\{1\}$ and $\{2,3\}$ and the one between that of $\{3\}$ and $\{1,2\}$ can be measured by means of the negativity[15, 19, 20]

$$\begin{aligned} \mathcal{N}_{1\text{-}23} &= (||\rho^{t_{1\text{-}23}}||-1)/2 \quad \text{or} \quad \textstyle\sum_i |(\mu_{1\text{-}23})_i|, \\ \mathcal{N}_{12\text{-}3} &= (||\rho^{t_{12\text{-}3}}||-1)/2 \quad \text{or} \quad \textstyle\sum_i |(\mu_{12\text{-}3})_i|, \end{aligned} \quad (4)$$

where $\rho^{t_{1\text{-}23}}(\rho^{t_{12\text{-}3}})$ denotes the partial transpose (PT) of ρ, and $||\rho^{t_{1\text{-}23}}||(||\rho^{t_{12\text{-}3}}||)$ denotes the trace norm of $\rho^{t_{1\text{-}23}}(\rho^{t_{12\text{-}3}})$. $(\mu_{1\text{-}23})_i$ $((\mu_{12\text{-}3})_i)$ is the negative eigenvalues of $\rho^{t_{1\text{-}23}}(\rho^{t_{12\text{-}3}})$ and 1-23 (12-3) denotes the bipartition of the system.

In the next section we will give our results, where we only consider the case of $J>0$ and $B\geqslant 0$ because it is easy to find from Eq. (2) that the entanglement is invariant under the substitution $B\to -B$.

3 Results and discussion

3.1 Bipartite entanglement of the ground states

To see the dependence of the global bipartite entanglement on the impurity parameter J_1 and the magnetic field B at the ground states, we give our calculation results in Table 1. First we consider the case of B=0. Obviously, the impurity enhances the entanglement when $J_1 > J$. But when $J_1 < J$, it decreases $\mathcal{N}_{1\text{-}23}$ to zero, but may increase $\mathcal{N}_{12\text{-}3}$ to its maximum value 0.5 (when $J > J_1 > -2J$) or may decrease it to zero (when $J_1 > -2J$). For a given J_1, the influences of B on $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$ are similar: it may enhance bipartite entanglement when $B < B_c$, suppress entanglement at $B = B_c$, but destroy the entanglement completely when $B > B_c$, where B_c is a critical value which is associated with J_1. So we can control bipartite entanglement via adjusting J_1 and B.

Table 1. The dependence of the negativities $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$ on J_1 and B at ground states.

J_1	$B\geqslant 0$	$\mathcal{N}_{1\text{-}23}$	$\mathcal{N}_{12\text{-}3}$
$J_1 > J$	B=0	0.3333	0.2676
	$0<B<3J_1$	0.4714	0.3727
	$B=3J_1$	0.0936	0.0618
	$B>3J_1$	0	0
$J_1 = J$	B=0	0.1667	0.1667
	$0<B<3J$	0.2357	0.2357
	$B=3J$	0.0624	0.0624
	$B>3J$	0	0
$-2J<J_1<J$	$B<J_1+2J$	0	0.5
	$B=J_1+2J$	0	0.1036
	$B>J_1+2J$	0	0
$J_1\leqslant -2J$	$B\geqslant 0$	0	0

3.2 Thermal entanglement

After analyzing the global bipartite entanglement in the ground states at zero absolute temperature, we turn to the more realistic case of thermal entanglement. For a system with temperature T at thermal equilibrium, the density matrix can be written as

$$\rho(T) = Z^{-1}\exp(-H/T), \quad (5)$$

where the partition function $Z = \mathrm{Tr}[\exp(-H/T)]$ and the Boltzmann's constant is set to 1 for simplicity.

First we consider the analytical solutions of $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$ at B=0. From Eq. (4), one can obtain the negativities $\mathcal{N}_{1\text{-}23}$ and $\mathcal{N}_{12\text{-}3}$

$$\mathcal{N}_{1\text{-}23} = \begin{cases} 0 & J_1 \leqslant (\ln 2)T/3, \\ 4(2c-b)/3Z & J_1 > (\ln 2)T/3, \end{cases} \tag{6}$$

$$\mathcal{N}_{12\text{-}3} = \begin{cases} 0 & \mu \leqslant 0, \\ 2\mu & \mu > 0, \end{cases} \tag{7}$$

where

$$a = \exp(3J/T)/2,$$

$$b = \exp[(4J_1 - J)/T]/6,$$

$$c = \exp[(-2J_1 - J)/T]/3,$$

$$\mu = (\sqrt{13b^2 - 10bc + 4c^2 - 2ac + a^2 - 2ab} - 2b - 2c)/Z. \tag{8}$$

Next we consider the numerical results where we have set J to 1 for simplicity. Fig. 1 gives the critical temperature T_c above which the entanglement vanishes as a function of J_1. It is clear that $T_{c1\text{-}23}$ linearly increases with the increase of J_1, which can also get from Eq. (6). But $T_{c12\text{-}3}$ initially increases then almost keeps unchanged with the increase of J_1 when $-2J < J_1 < J$ and increases with the increase of J_1 when $J_1 > J$.

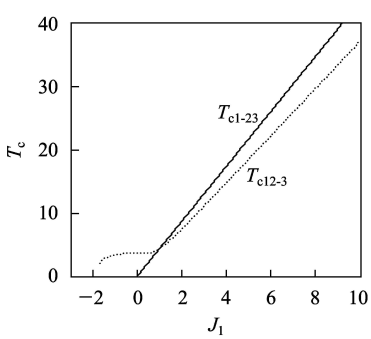

Fig. 1. The critical temperature plotted as a function of $J_1(B$=0) for bipartitions: 1-23 and 12-3. The parameter J=1.

Figure 1 also shows there exists a corresponding critical impurity parameter J_{1c} below which the bipartite entanglement vanishes for a given T and this critical value increases with the increase of T. These results show that the impurity can be used as a switch to turn on or turn off the bipartite entanglement.

Then we consider the negativity as a function of J_1 at different temperatures T. Fig. 2(a) shows that when the temperature is low enough, $\mathcal{N}_{12\text{-}3}$ initially increases to a upper limited value $\mathcal{N}_1$ at J_1=0, then decreases to a lower limited value $\mathcal{N}_2$ and finally increases to a asymptotic value $\mathcal{N}_a$ with the increase of J_1. The values $\mathcal{N}_1$ and $\mathcal{N}_2$ decrease to zero gradually as the temperature rises, but the asymptotic value is independent of the temperature (see T=0.5, 1.5, 2.5, 3.5, 4.5 in Fig. 2(a)). This means that for a given T, one can control $\mathcal{N}_{12\text{-}3}$ to obtain its maximum value via adjusting J_1, and this maximum value decreases from 0.5 to 0.2676 and then almost keeps unchanged as the temperature rises. From Fig. 2(b), one can observe that at any fixed temperature, $\mathcal{N}_{1\text{-}23}$ increases with the increase of J_1 and reaches its maximum value when J_1 is large enough. Generally, the higher the temperature is, the smaller the $\mathcal{N}_{1\text{-}23}$ is for a given J_1 (see T=1.5, 2.5, 3.5, 4.5 in Fig. 2(b)). But when T is low enough (see T=0.5 and 1.5 in Fig. 2(b)), a higher temperature may correspond to a larger $\mathcal{N}_{1\text{-}23}$ if J_1 is within a special value. Comparing Fig. 2(a) with Fig. 2(b), the variations of $\mathcal{N}_{12\text{-}3}$ and $\mathcal{N}_{1\text{-}23}$ with the increase of J_1 have significant differences when T is low enough, and these differences narrow down gradually with the rise of the temperature.

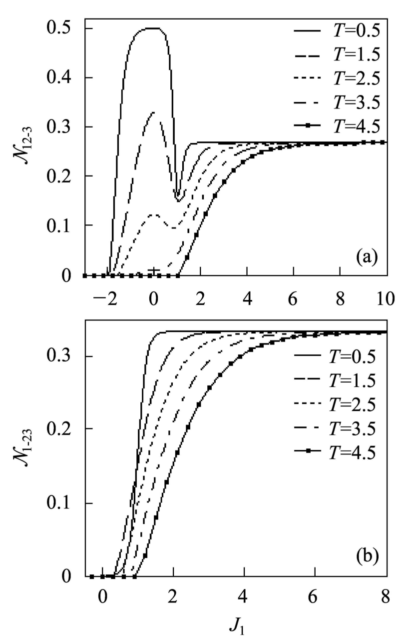

Fig. 2. The negativities $\mathcal{N}_{12\text{-}3}$ (a) and $\mathcal{N}_{1\text{-}23}$ (b) as a function of the impurity parameter J_1. The parameter J=1.

Now we consider the global bipartite entanglement in the presence of a uniform magnetic filed B. Fig. 3 gives the plots of $\mathscr{N}_{12\text{-}3}$ and $\mathscr{N}_{1\text{-}23}$ as a function of J_1 and B for T=0.5 and T=10. It is clear that for a particular fixed B, the behaviors of $\mathscr{N}_{12\text{-}3}$ and $\mathscr{N}_{1\text{-}23}$ are similar to that of the B=0 case.

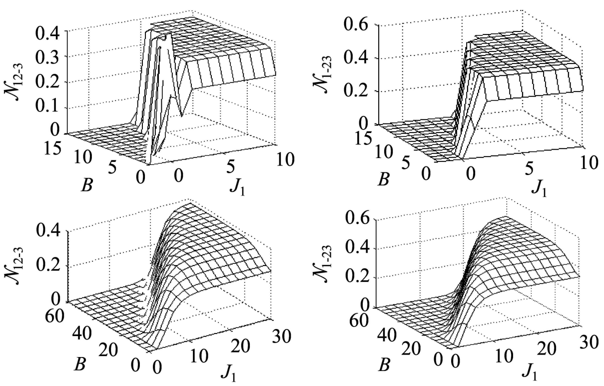

Fig. 3. The negativities $\mathscr{N}_{1\text{-}23}$ and $\mathscr{N}_{12\text{-}3}$ as a function of J_1 and B for T=0.5 (the upper two figures) and 10 (the lower two figures). The parameter J=1.

When the temperature is fixed, the influences of the magnetic filed B on $\mathscr{N}_{12\text{-}3}$ and $\mathscr{N}_{1\text{-}23}$ are similar for a given J_1. When J_1 is within a threshold value, the negativity decreases to zero with the increase of B. When J_1 exceeds this value, the negativity first increases and then decreases to zero with the increase of B. These results show that one can control $\mathscr{N}_{12\text{-}3}$ and $\mathscr{N}_{1\text{-}23}$ to their corresponding maximum values via adjusting J_1 and B. Moreover, the maximum value is independent of the temperature with the exception of $\mathscr{N}_{12\text{-}3}$ at low temperature. From Fig. 3, one also can observe that for a given temperature, the corresponding critical impurity parameter J_{1c} keeps nearly unchanged when B is within a special value and increases with the increase of B when B exceeds this critical value. Furthermore, we give the plot of J_{1c} as a function of B for T=0.5 and T=10 in Fig. 4 for visualization.

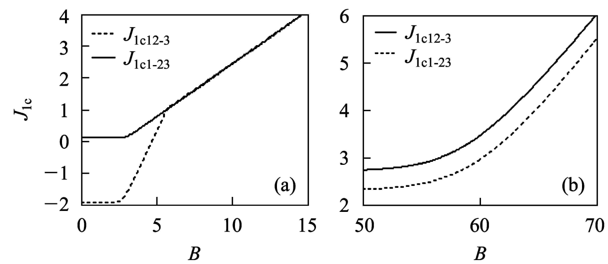

Fig. 4. The critical impurity parameter J_{1c} as a function of B for a given temperature: (a) T=0.5; (b) T=10.

4 Conclusion

In this paper, we investigate the global bipartite entanglement in the three-qubit Heisenberg XXX spin chain with impurity. We give the exact results of $\mathscr{N}_{12\text{-}3}$ and $\mathscr{N}_{1\text{-}23}$ at zero absolute temperature and numerical solutions at finite temperature. We show the critical temperature T_c above which the entanglement vanishes increases with the increase of the impurity parameter J_1. For a given T, the corresponding critical J_{1c} increases with the increase of the magnetic filed B. The impurity can be used as a switch to turn on or turn off the bipartite entanglement, and a proper B can enhance the bipartite entanglement. Moreover, for a given T, one can control bipartite entanglement via adjusting J_1 and B, and furthermore, the maximum value of $\mathscr{N}_{12\text{-}3}$ decreases from 0.5 to 0.3727 as the temperature rises, but the one of $\mathscr{N}_{1\text{-}23}$ keeps the constant value of about 0.4714.

References

1 Verstraete F, Martin-Delgado M A, Cirac J I. Phys. Rev. Lett., 2004, **92**: 087201

2 Vidal G, Latorre J I, Rico E et al. Phys. Rev. Lett., 2003, **90**: 227902

3 Bennett C H, Brassard G, Crepeau C et al. Phys. Rev. Lett., 1993, **70**: 1895—1899

4 YE Y, LIU T Q et al. J. Phys., 2005, **A38**: 3235—3243

5 Ekert A K. Phys. Rev. Lett., 1991, **67**(6): 661—663

6 Bennett C H. Phys. Rev. Lett., 1992, **69**: 2881—2884

7 Dagmar Bruß, DiVincenzo D P, Ekert A et al. Phys. Rev., 1998, **A57**: 2368—2378

8 Loss D, DiVincenzo D P. Phys. Rev., 1998, **A57**: 120—126

9 Nielsen M A. 1998 Ph.D. Thesis, University of New Mexico

10 WANG X G. Phys. Rev., 2002, **A66**: 034302

11 WANG X G. Phys. Rev., 2001, **A64**: 012313

12 WANG X G. Phys. Lett., 2001, **A281**: 101

13 WANG X G, FU H, Solomon A I. J. Phys., 2001, **A34**: 11307

14 HU M L, TIAN D P. HEP & NP, 2006, **30**(11): 1132—1136

15 Canosa N, Rossignoli R. Phys. Rev., 2006, **A73**: 022346

16 Zyczkowski K, Horodecki P, Samperm A. Phys. Rev., 1998, **A58**: 883—892

17 XI X Q, HAO S R, CHEN W X et al. Phys. Lett., 2002, **A297**: 291—299
18 Arnesen M C, Bose S, Vedral V. Phys. Rev. Lett., 2001, **87**(1): 017901
19 WANG X G, WANG Z D. Phys. Rev., 2006, **A73**: 064302
20 Peres A. Phys. Rev. Lett., 1996, **77**: 1413

含杂质三量子位Heisenberg XXX链的全局两体纠缠*

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摘要 研究了含杂质三量子位Heisenberg XXX链的全局两体纠缠, 通过计算$\mathcal{N}_{12\text{-}3}$和$\mathcal{N}_{1\text{-}23}$, 发现两体纠缠存在的临界温度$T_c$随杂质参数$J_1$的增加而升高. 给定温度$T$, 相应的纠缠存在的临界杂质参数$J_{1c}$随磁场的增加而增加, 而且可以通过调节杂质参数$J_1$和磁场$B$来控制$\mathcal{N}_{12\text{-}3}$和$\mathcal{N}_{1\text{-}23}$的取值. 此外, 随着温度的增加, $\mathcal{N}_{12\text{-}3}$的最大值将由0.5减小到0.3727, 而$\mathcal{N}_{1\text{-}23}$的最大值保持0.4714不变.

关键词 Heisenberg XXX链 全局两体纠缠 杂质

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