Bayesian credible interval construction for Poisson statistics*

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Abstract The construction of the Bayesian credible (confidence) interval for a Poisson observable including both the signal and background with and without systematic uncertainties is presented. Introducing the conditional probability satisfying the requirement of the background not larger than the observed events to construct the Bayesian credible interval is also discussed. A Fortran routine, BPOCI, has been developed to implement the calculation.

Key words Bayesian credible interval, systematic uncertainties, Poisson distribution

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1 Introduction

The commonly accepted way to report errors on results in high energy physics experiments is to present confidence intervals for the parameters to be determined based on observed data. Very often, observables in an experiment are Poisson variables including both the signal and background processes, therefore, the construction of confidence interval for parameter of signal in a Poisson process is of great importance. For such a question, Conrad et al.[1] reviewed the methods of confidence belt construction in the frame of frequentist statistics, and developed a FORTRAN program, POLE[2], to calculate the confidence intervals for a maximum of observed events of 100 and a maximum signal expectation of 50. The ordering schemes for frequentist construction supported are the Neyman method[3], likelihood ratio ordering[4] and improved likelihood ratio ordering[5]. The systematic uncertainties in both the signal and background efficiencies as well as systematic uncertainty of background expectation have been taken into account in the confidence belt construction by assuming a probability density function (pdf) which parameterizes our knowledge on the uncertainties and integrating over this pdf. This method, combining classical and Bayesian elements, is referred to as semi-Bayesian approach.

In the frame of Bayesian statistics[6], Narsky[7, 8] depicted the estimation of upper limits for Poisson statistic with the known background expectation. Roe and Woodroofe proposed using a Bayes procedure with uniform prior to determine credible intervals without inclusion of systematic uncertainties of signal efficiency and background expectation[9]. Treatment of background uncertainty is discussed with the flat prior for simplified cases of background expectation distributions in Refs. [10,11]. Inclusion of systematic uncertainties of signal efficiency and background expectation in the upper limit calculation via Bayesian approach has been recently discussed by ZHU[12].

In this paper, we describe the construction of Bayesian credible (confidence) interval for a Poisson observable including both signal and background with and without systematic uncertainties. Introducing the conditional probability satisfying the requirement of the background not larger than observed events to construct Bayesian credible interval is also discussed. A Fortran program, BPOCI, has been developed to implement the calculation[13].

2 Bayesian credible interval

Throughout this paper we assume that in the signal window, where the signal events (if exist) shall

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reside, the number of signal events is a Poisson variable with unknown expectation s to be inferred, and the number of background events is a Poisson variable with known expectation b, the conditional pdf of observing n total events given s is represented by a Poisson probability

$$p(n|s)_b = \mathrm{e}^{-(s+b)}\frac{(s+b)^n}{n!} \ . \tag{1}$$

In the Bayesian statistics, any statistical inference on parameters is based on posterior pdf. The posterior pdf in our question can be expressed as

$$h(s|n) = \frac{q(n|s)_b\pi(s)}{\displaystyle\int_0^\infty q(n|s)_b\pi(s)\mathrm{d}s} \ , \tag{2}$$

where $q(n|s)_b$ is the conditional probability of observing n total events given s signal events, and $\pi(s)$ is the non-informative prior

$$\pi(s) \propto \frac{1}{(s+b)^m} \ , \quad s \geqslant 0, \quad b \geqslant 0 \quad 0 \leqslant m \leqslant 1, \tag{3}$$

where m=0 corresponds to Bayes prior, m=0.5 to $1/\sqrt{s+b}$ prior, and m=1 to $1/(s+b)$ prior. The statistical bases on these three priors are referred to Refs. [14—17]. It should be kept in mind that different m values will give different answers for the credible interval. The expected coverage and length of Bayesian intervals constructed with these three priors and of frequentist intervals with Neyman construction[3] and unified approach[4] can be found in Ref. [8]. It has been shown that the $1/\sqrt{s+b}$ prior is the most versatile choice among the Bayesian methods, and it provides a reasonable mean coverage for the credible interval for Poisson observable.

It is easy to build the central interval $[S_\mathrm{L}, S_\mathrm{U}]$ for s at a given Bayesian posterior credible level $CL = 1-\alpha$ by posterior pdf $h(s|n)$

$$\int_0^{S_\mathrm{L}} h(s|n)\mathrm{d}s = \frac{\alpha}{2} = \int_{S_\mathrm{U}}^\infty h(s|n)\mathrm{d}s \ , \tag{4}$$

and upper limit S_UP

$$1-\alpha = \int_0^{S_\mathrm{UP}} h(s|n)\mathrm{d}s \ . \tag{5}$$

However, the central interval and upper limit determined in such a way suffers from the so-called "flip-flopping" policy, namely, to report a central interval or an upper limit is artificially decided by the experimenter's choice, which is similar to the Neyman construction[3] and certainly undesirable. To avoid this drawback, it is better to construct the highest posterior density (HPD) credible interval R expressed as

$$1-\alpha = \int_R h(s|n)\mathrm{d}s \ , \tag{6}$$

where for any $s_1 \in R$ and $s_2 \notin R$, the following inequality holds

$$h(s_1|n) \geqslant h(s_2|n) \ . \tag{7}$$

The HPD interval is the optimum and shortest interval at a given credible level in the Bayes framework. It automatically provides a two-sided interval or an upper limit, decided by the observed data itself, which is the same as in the unified approach[4], superior over the Neyman construction.

2.1 Bayesian interval without inclusion of systematic uncertainties

In the case that the systematic uncertainties of the signal efficiency and background expectation can be neglected, the signal expectation s is an unknown constant and the background expectation b is a known value. In this case, $q(n|s)_b$ in Eq. (2) is simply equal to $p(n|s)_b$ in Eq. (1), and the posterior pdf is then given by

$$h(s|n) = \frac{(s+b)^{n-m}\mathrm{e}^{-(s+b)}}{\Gamma(n-m+1,b)} \ , \tag{8}$$

where

$$\Gamma(x,b) = \int_b^\infty s^{x-1}\mathrm{e}^{-s}\mathrm{d}s \ , \quad x>0, \ b>0 \tag{9}$$

is an incomplete gamma function. Solving the eqation of

$$\frac{\mathrm{d}h(s|n)}{\mathrm{d}s} = 0$$

gives the unique solution of $s_m = n-(b+m)$, where $s_m$is the maximum of $h(s|n)$, which can be known from the behaviors of $p(n|s)_b$ and $\pi(s)$. While the $h(s|n)$ is a monotonic decreasing function of s for $n \leqslant b+m$, it is a function of s with a single maximum for $n > b+m$. The bounds of the HPD interval R, $[S_\mathrm{L}, S_\mathrm{U}]$, at a given credible level $CL = 1-\alpha$ can be acquired by solving Eq. (6) numerically with posterior pdf Eq. (8) from the measured values of n and b.

2.2 Bayesian interval with inclusion of systematic uncertainties

Now we take into account the systematic uncertainties. In this case, both the signal expectation and background expectation are not the constants, but the variables; they have respective distributions.

If only the uncertainty of background expectation is present, and the distribution of the background expectation is represented by a pdf $f_{b'}(b,\sigma_b)$ with the mean b and standard deviation σ_b, the conditional pdf expressed by Eq. (1) now is modified to

$$q(n|s)_b = \int_0^\infty p(n|s)_{b'}\cdot f_{b'}(b,\sigma_b)\mathrm{d}b' \ , \tag{10}$$

where $p(n|s)_{b'}$ has the same expression in Eq. (1) with b replaced by b'.

Next we consider both the uncertainties of the signal efficiency and background expectation exist, and assume they are independent of each other. The distribution of the relative signal efficiency ε (with respect to the mean of the signal detection efficiency η) is expressed by a pdf $f_\varepsilon(1,\sigma_\varepsilon)$ with the mean 1 and standard deviation σ_ε. The conditional pdf described by Eq. (1) is then further modified to

$$q(n|s)_b=\int_0^\infty\int_0^\infty p(n|s\varepsilon)_{b'}f_{b'}(b,\sigma_b)f_\varepsilon(1,\sigma_\varepsilon)\mathrm{d}b'\mathrm{d}\varepsilon \ , \quad (11)$$

where $p(n|s\varepsilon)_{b'}$ represents that b is replaced by b', and s by $s\varepsilon$ in Eq. (1). One notices that the lower limits of integrals in Eqs. (10), (11) are all zero, which are the possible minimum value of any efficiencies and number of background events. One can determine the corresponding posterior pdf $h(s|n)$ according to Eq. (2)[18], and then the Bayesian interval on s at any given credible level with inclusion of systematic uncertainties in terms of Eq. (6).

2.3 Connection to the "Conditioning"

In the unified approach, there is a background dependence of the confidence interval for a Poison observable in the case of fewer events observed than expected background. Roe and Woodroofe[5] propose a solution to this problem by using such an argument that, given an observation n, the background b can not be larger than n in any case. Replacing the usual Poisson probability by the conditional probability satisfying this requirement in confidence interval construction is called "Conditioning". The conditional probability of observing n total events with background b not larger then n is

$$g(n|s)_b=\frac{p(n|s)_b}{\sum_{i=0}^{n}p(i)_b} \ , \quad p(i)_b=\frac{1}{i!}b^i\mathrm{e}^{-b} \ . \quad (12)$$

As pointed out by authors of Ref. [5] that in the case the systematic uncertainties of the signal efficiency and background expectation can be neglected, $g(n|s)_b$ is a density in s, and, is the posterior distribution that is obtained when s is given an (improper) uniform prior distribution over the range $0\leqslant s<\infty$. Using $g(n|s)_b$ as the posterior function $h(s|n)$ in Eq. (6) leads to a Bayesian credible interval with flat prior (m=0). Therefore, if the systematic uncertainties are taken into account in the Bayesian approach, $q(n|s)_b$ with conditioning can be expressed by Eqs. (10), (11) with $p(n|s)_b$ replaced by $g(n|s)_b$. Thus, the Bayesian interval with conditioning can be built by solving Eq. (6) with this $q(n|s)_b$. Notice the factors irrelevant to s in the numerator and denominator of $q(s|n)$ are cancelled out, the consequences of introducing the conditioning are quite amusing: only the intervals with inclusion of the systematic uncertainty of background expectation (and also the systematic uncertainty of signal efficiency simultaneously) differ from those without conditioning, while the intervals kept unchanged in other cases. Besides, introducing the conditioning produces shorter Bayesian intervals.

3 BPOCI: An algorithm for calculating Bayesian interval

We have developed an algorithm for calculating Bayesian interval for the Poisson observable at a given credible level with or without inclusion of systematic uncertainties in background (bkgd) expectation and signal efficiency. It has been implemented as a FORTRAN program, BPOCI (Bayesian POissonian Credible Interval)[13].

To run BPOCI, following valuables are required to input:

IC, II, ID, IBK, IE, N, B, SIGBK, SIGE, ETA, CL, AM.

Their meanings are listed in Table 1.

Table 1. Input variables of BPOCI and their meanings.

notation	meaning
IC	flag for conditioning: 1—no conditioning, 2—conditioning.
II	flag for type of interval: 1—HPD, 2—central, 3—upper limit.
ID	flag for systematic uncertainties. (see below)
IBK	flag for selecting the distribution of bkgd expectation. (see below)
IE	flag for selecting the distribution of signal detection efficiency. (see below)
N	number of total events observed in signal window, n.
B	predicted bkgd expectation in signal window, b.
SIGBK	standard deviation of the distribution for relative bkgd expectation, σ_b/b.
SIGE	standard deviation of the distribution for relative signal efficiency, σ_ε.
ETA	predicted signal detection efficiency, η.
CL	credible level, $CL=1-\alpha$.
AM	prior selection. $AM=m$, prior is $1/(s+b)^m$, $0\leqslant m\leqslant 1$.

Flag ID can take four values with the following assignment:

1 — without considering any systematic uncertainties,

2 — incorporating systematic uncertainty of bkgd expectation,

3 — incorporating systematic uncertainty of signal efficiency,

4 — incorporating systematic uncertainties of bkgd expectation and signal efficiency simultaneously,

and they are assumed to be independent of each other.

For the distribution of relative signal efficiency (signal efficiency divided by η) or relative background expectation (background expectation divided by b), three types of functions with the mean 1 and standard deviation σ are supported: Gaussian, Log-Gaussian and flat distributions, which correspond to IE (IBK) equal to 1, 2 and 3, respectively.

The program will automatically generate an output file "BPOCI.out", which gives the bounds of the required interval at a given credible level. Simultaneously, it creates a HBOOK file "bpoci.hbk", which can be looked at by using PAW and to generate a corresponding "bpoci.eps" file, drawing the posterior density $h(s|n)$ as a function of s with both linear and log scales. Fig. 1 shows the posterior density for n=8, b=2 with Gaussian systematic uncertainty σ_b/b=0.3 using uniform prior, the HPD interval at CL=0.9 is [2.07, 11.77]. Tables 2—4 list Bayesian HPD intervals for signal expectation s without inclusion of systematic uncertainties using uniform prior at the most commonly used 68.27%, 90% and 95% CL.

The inclusion of systematic uncertainties leads to widening the credible interval. The relations of the interval's length versus the type of priors, the type of pdfs for uncertainties and the uncertainty's size, have similar tendencies as those of Bayesian upper limit, which are referred to Ref. [12].

Table 2. Bayesian credible intervals for signal expectation s without incorporating systematic uncertainties using flat prior. n and b represents the total events observed and background events, respectively.

68.27% CL

$n\backslash b$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
0	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15
1	0.27,2.50	0.00,1.99	0.00,1.79	0.00,1.66	0.00,1.57	0.00,1.51	0.00,1.46	0.00,1.42	0.00,1.39	0.00,1.35
2	0.87,3.85	0.38,3.30	0.00,2.66	0.00,2.37	0.00,2.16	0.00,2.01	0.00,1.89	0.00,1.80	0.00,1.72	0.00,1.61
3	1.56,5.14	1.06,4.64	0.58,4.08	0.16,3.42	0.00,2.95	0.00,2.68	0.00,2.47	0.00,2.30	0.00,2.16	0.00,1.96
4	2.29,6.40	1.79,5.90	1.30,5.39	0.83,4.83	0.39,4.20	0.00,3.52	0.00,3.20	0.00,2.94	0.00,2.73	0.00,2.40
5	3.06,7.63	2.56,7.13	2.06,6.63	1.57,6.11	1.10,5.56	0.65,4.96	0.24,4.30	0.00,3.73	0.00,3.43	0.00,2.96
6	3.85,8.83	3.35,8.34	2.85,7.83	2.35,7.33	1.86,6.82	1.38,6.27	0.93,5.69	0.51,5.06	0.12,4.38	0.00,3.64
7	4.65,10.03	4.15,9.53	3.65,9.03	3.15,8.53	2.66,8.02	2.16,7.51	1.69,6.97	1.23,6.40	0.80,5.79	0.01,4.45
8	5.47,11.21	4.97,10.71	4.47,10.21	3.97,9.71	3.47,9.21	2.98,8.70	2.48,8.19	2.00,7.66	1.54,7.10	0.68,5.88
9	6.30,12.37	5.80,11.88	5.30,11.38	4.80,10.88	4.30,10.38	3.80,9.87	3.30,9.37	2.81,8.86	2.33,8.33	1.41,7.21
10	7.14,13.54	6.64,13.04	6.14,12.54	5.64,12.04	5.14,11.54	4.64,11.04	4.14,10.53	3.64,10.03	3.15,9.52	2.19,8.46
11	7.99,14.69	7.49,14.19	6.99,13.69	6.49,13.19	5.99,12.69	5.49,12.19	4.99,11.69	4.49,11.19	3.99,10.69	3.01,9.66
12	8.84,15.83	8.34,15.33	7.84,14.83	7.34,14.33	6.84,13.83	6.34,13.33	5.84,12.84	5.34,12.33	4.84,11.83	3.85,10.82
13	9.70,16.98	9.20,16.47	8.70,15.97	8.20,15.47	7.70,14.97	7.20,14.47	6.70,13.97	6.20,13.47	5.70,12.97	4.70,11.97
14	10.56,18.11	10.06,17.61	9.56,17.11	9.07,16.61	8.57,16.11	8.06,15.61	7.57,15.11	7.07,14.61	6.56,14.11	5.57,13.11
15	11.43,19.24	10.94,18.74	10.43,18.24	9.93,17.74	9.44,17.24	8.94,16.74	8.43,16.24	7.93,15.74	7.43,15.24	6.43,14.24
16	12.31,20.37	11.81,19.86	11.31,19.36	10.81,18.87	10.31,18.36	9.81,17.87	9.31,17.37	8.81,16.86	8.31,16.37	7.31,15.37
17	13.18,21.49	12.68,20.99	12.18,20.49	11.69,19.99	11.19,19.49	10.68,18.98	10.18,18.49	9.69,17.99	9.19,17.49	8.19,16.49
18	14.07,22.61	13.57,22.10	13.07,21.60	12.57,21.11	12.07,20.61	11.57,20.11	11.07,19.60	10.57,19.11	10.07,18.61	9.07,17.61
19	14.95,23.72	14.45,23.22	13.95,22.72	13.45,22.22	12.95,21.72	12.45,21.22	11.95,20.72	11.45,20.22	10.95,19.72	9.95,18.72
20	15.84,24.83	15.34,24.33	14.84,23.83	14.34,23.33	13.84,22.83	13.34,22.34	12.84,21.83	12.34,21.33	11.84,20.83	10.84,19.84

$n\backslash b$	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	14.0	15.0
0	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15	0.00,1.15
1	0.00,1.32	0.00,1.30	0.00,1.28	0.00,1.27	0.00,1.26	0.00,1.25	0.00,1.24	0.00,1.23	0.00,1.23	0.00,1.22
2	0.00,1.54	0.00,1.48	0.00,1.44	0.00,1.41	0.00,1.38	0.00,1.36	0.00,1.34	0.00,1.33	0.00,1.32	0.00,1.30
3	0.00,1.82	0.00,1.72	0.00,1.64	0.00,1.58	0.00,1.53	0.00,1.50	0.00,1.47	0.00,1.44	0.00,1.42	0.00,1.40
4	0.00,2.17	0.00,2.00	0.00,1.88	0.00,1.79	0.00,1.71	0.00,1.65	0.00,1.61	0.00,1.57	0.00,1.53	0.00,1.50
5	0.00,2.61	0.00,2.36	0.00,2.18	0.00,2.04	0.00,1.93	0.00,1.84	0.00,1.77	0.00,1.71	0.00,1.67	0.00,1.62
6	0.00,3.16	0.00,2.81	0.00,2.54	0.00,2.34	0.00,2.19	0.00,2.06	0.00,1.97	0.00,1.89	0.00,1.82	0.00,1.76
7	0.00,3.83	0.00,3.35	0.00,2.99	0.00,2.71	0.00,2.50	0.00,2.33	0.00,2.19	0.00,2.09	0.00,2.00	0.00,1.92
8	0.00,4.61	0.00,4.01	0.00,3.53	0.00,3.16	0.00,2.88	0.00,2.65	0.00,2.47	0.00,2.32	0.00,2.20	0.00,2.10
9	0.57,5.96	0.00,4.77	0.00,4.18	0.00,3.70	0.00,3.33	0.00,3.03	0.00,2.79	0.00,2.60	0.00,2.45	0.00,2.32
10	1.29,7.31	0.48,6.04	0.00,4.92	0.00,4.34	0.00,3.86	0.00,3.48	0.00,3.18	0.00,2.93	0.00,2.73	0.00,2.56
11	2.06,8.58	1.19,7.40	0.39,6.10	0.00,5.06	0.00,4.49	0.00,4.01	0.00,3.63	0.00,3.32	0.00,3.06	0.00,2.85
12	2.88,9.78	1.95,8.68	1.09,7.48	0.31,6.16	0.00,5.20	0.00,4.63	0.00,4.16	0.00,3.77	0.00,3.45	0.00,3.19
13	3.71,10.95	2.75,9.90	1.84,8.77	1.00,7.55	0.23,6.22	0.00,5.34	0.00,4.77	0.00,4.30	0.00,3.91	0.00,3.58
14	4.57,12.10	3.59,11.07	2.64,10.00	1.74,8.86	0.92,7.62	0.16,6.28	0.00,5.47	0.00,4.91	0.00,4.44	0.00,4.04
15	5.44,13.24	4.44,12.22	3.47,11.19	2.53,10.10	1.65,8.94	0.84,7.68	0.10,6.34	0.00,5.59	0.00,5.04	0.00,4.57
16	6.31,14.36	5.31,13.36	4.33,12.34	3.36,11.29	2.43,10.19	1.56,9.01	0.76,7.74	0.03,6.39	0.00,5.72	0.00,5.17
17	7.18,15.49	6.18,14.48	5.20,13.48	4.21,12.45	3.26,11.39	2.34,10.27	1.48,9.08	0.70,7.80	0.00,6.47	0.00,5.84
18	8.07,16.61	7.07,15.61	6.07,14.60	5.08,13.59	4.10,12.56	3.16,11.48	2.25,10.35	1.41,9.14	0.63,7.85	0.00,6.58
19	8.95,17.72	7.95,16.72	6.95,15.72	5.96,14.71	4.97,13.69	4.00,12.65	3.06,11.57	2.17,10.43	1.34,9.20	0.57,7.91
20	9.84,18.84	8.84,17.83	7.84,16.83	6.84,15.83	5.85,14.82	4.86,13.80	3.90,12.75	2.97,11.66	2.09,10.50	1.27,9.26

Table 3. 90% CL.

$n\backslash b$	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30
1	0.08,3.95	0.00,3.50	0.00,3.27	0.00,3.11	0.00,2.99	0.00,2.91	0.00,2.84
2	0.45,5.45	0.00,4.82	0.00,4.43	0.00,4.11	0.00,3.87	0.00,3.67	0.00,3.52
3	0.95,6.91	0.45,6.40	0.00,5.70	0.00,5.28	0.00,4.92	0.00,4.62	0.00,4.36
4	1.52,8.33	1.01,7.84	0.53,7.29	0.09,6.60	0.00,6.08	0.00,5.69	0.00,5.34
5	2.14,9.70	1.63,9.21	1.14,8.71	0.65,8.15	0.22,7.48	0.00,6.85	0.00,6.44
6	2.79,11.05	2.29,10.55	1.79,10.05	1.29,9.54	0.81,8.98	0.37,8.34	0.00,7.60
7	3.47,12.36	2.97,11.86	2.47,11.37	1.97,10.86	1.48,10.35	1.00,9.80	0.55,9.18
8	4.18,13.66	3.67,13.16	3.17,12.66	2.67,12.16	2.17,11.66	1.68,11.14	1.20,10.60
9	4.90,14.93	4.39,14.43	3.90,13.93	3.39,13.43	2.89,12.94	2.40,12.43	1.90,11.92
10	5.63,16.19	5.13,15.69	4.63,15.20	4.13,14.70	3.63,14.20	3.13,13.70	2.63,13.19
11	6.38,17.44	5.88,16.95	5.38,16.44	4.88,15.95	4.38,15.45	3.88,14.95	3.38,14.44
12	7.14,18.68	6.64,18.18	6.14,17.68	5.64,17.18	5.14,16.69	4.64,16.18	4.14,15.69
13	7.91,19.91	7.41,19.41	6.91,18.91	6.41,18.41	5.91,17.92	5.41,17.41	4.91,16.91
14	8.69,21.13	8.19,20.63	7.69,20.13	7.19,19.63	6.69,19.14	6.19,18.63	5.68,18.13
15	9.47,22.35	8.97,21.85	8.47,21.35	7.97,20.85	7.47,20.35	6.97,19.85	6.47,19.35
16	10.26,23.55	9.77,23.05	9.26,22.55	8.77,22.05	8.26,21.55	7.76,21.05	7.27,20.56
17	11.06,24.75	10.57,24.25	10.06,23.75	9.57,23.25	9.06,22.75	8.56,22.25	8.06,21.75
18	11.87,25.95	11.37,25.45	10.87,24.95	10.37,24.45	9.87,23.95	9.37,23.45	8.87,22.95
19	12.68,27.14	12.18,26.64	11.68,26.14	11.18,25.64	10.68,25.14	10.18,24.64	9.68,24.14
20	13.50,28.32	13.00,27.82	12.49,27.32	11.99,26.82	11.49,26.32	10.99,25.82	10.49,25.33

$n\backslash b$	3.5	4.0	5.0	6.0	7.0	8.0	9.0
0	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30
1	0.00,2.78	0.00,2.74	0.00,2.67	0.00,2.62	0.00,2.58	0.00,2.55	0.00,2.53
2	0.00,3.39	0.00,3.29	0.00,3.12	0.00,3.00	0.00,2.92	0.00,2.85	0.00,2.79
3	0.00,4.15	0.00,3.97	0.00,3.68	0.00,3.48	0.00,3.32	0.00,3.20	0.00,3.10
4	0.00,5.04	0.00,4.78	0.00,4.36	0.00,4.04	0.00,3.80	0.00,3.61	0.00,3.46
5	0.00,6.06	0.00,5.72	0.00,5.15	0.00,4.71	0.00,4.37	0.00,4.10	0.00,3.89
6	0.00,7.16	0.00,6.76	0.00,6.06	0.00,5.49	0.00,5.04	0.00,4.67	0.00,4.38
7	0.15,8.46	0.00,7.88	0.00,7.07	0.00,6.37	0.00,5.80	0.00,5.34	0.00,4.96
8	0.75,9.99	0.33,9.31	0.00,8.15	0.00,7.35	0.00,6.67	0.00,6.09	0.00,5.62
9	1.42,11.38	0.96,10.79	0.15,9.41	0.00,8.40	0.00,7.62	0.00,6.95	0.00,6.37
10	2.14,12.68	1.66,12.14	0.75,10.95	0.00,9.51	0.00,8.65	0.00,7.88	0.00,7.21
11	2.88,13.94	2.39,13.43	1.43,12.34	0.57,11.08	0.00,9.73	0.00,8.89	0.00,8.13
12	3.64,15.18	3.14,14.68	2.16,13.64	1.23,12.51	0.40,11.19	0.00,9.95	0.00,9.11
13	4.41,16.42	3.91,15.91	2.92,14.90	1.95,13.84	1.05,12.66	0.26,11.28	0.00,10.16
14	5.19,17.64	4.69,17.13	3.69,16.13	2.70,15.10	1.75,14.02	0.88,12.79	0.12,11.36
15	5.97,18.85	5.47,18.35	4.47,17.35	3.48,16.34	2.50,15.30	1.57,14.17	0.73,12.90
16	6.76,20.05	6.26,19.55	5.26,18.55	4.27,17.55	3.27,16.53	2.31,15.47	1.40,14.31
17	7.56,21.26	7.06,20.75	6.06,19.75	5.06,18.76	4.07,17.75	3.08,16.72	2.13,15.64
18	8.37,22.45	7.87,21.95	6.87,20.95	5.87,19.95	4.87,18.95	3.88,17.93	2.90,16.89
19	9.18,23.64	8.68,23.14	7.68,22.14	6.68,21.14	5.68,20.14	4.68,19.13	3.69,18.11
20	9.99,24.82	9.49,24.32	8.49,23.32	7.49,22.32	6.49,21.32	5.50,20.32	4.50,19.31

$n\backslash b$	10.0	11.0	12.0	13.0	14.0	15.0
0	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30	0.00,2.30
1	0.00,2.51	0.00,2.49	0.00,2.48	0.00,2.46	0.00,2.45	0.00,2.44
2	0.00,2.74	0.00,2.71	0.00,2.67	0.00,2.65	0.00,2.62	0.00,2.60
3	0.00,3.02	0.00,2.96	0.00,2.90	0.00,2.86	0.00,2.82	0.00,2.78
4	0.00,3.34	0.00,3.24	0.00,3.16	0.00,3.09	0.00,3.03	0.00,2.98
5	0.00,3.71	0.00,3.57	0.00,3.46	0.00,3.36	0.00,3.27	0.00,3.20
6	0.00,4.15	0.00,3.95	0.00,3.80	0.00,3.66	0.00,3.55	0.00,3.45
7	0.00,4.65	0.00,4.40	0.00,4.18	0.00,4.01	0.00,3.86	0.00,3.74
8	0.00,5.23	0.00,4.90	0.00,4.63	0.00,4.41	0.00,4.22	0.00,4.06
9	0.00,5.89	0.00,5.48	0.00,5.15	0.00,4.86	0.00,4.62	0.00,4.42
10	0.00,6.63	0.00,6.14	0.00,5.73	0.00,5.38	0.00,5.08	0.00,4.83
11	0.00,7.46	0.00,6.88	0.00,6.39	0.00,5.96	0.00,5.60	0.00,5.30
12	0.00,8.36	0.00,7.70	0.00,7.12	0.00,6.62	0.00,6.19	0.00,5.82
13	0.00,9.33	0.00,8.59	0.00,7.93	0.00,7.35	0.00,6.84	0.00,6.41
14	0.00,10.36	0.00,9.55	0.00,8.81	0.00,8.15	0.00,7.57	0.00,7.06
15	0.00,11.43	0.00,10.56	0.00,9.75	0.00,9.02	0.00,8.36	0.00,7.78
16	0.59,13.00	0.00,11.61	0.00,10.75	0.00,9.95	0.00,9.22	0.00,8.57
17	1.25,14.44	0.46,13.09	0.00,11.79	0.00,10.94	0.00,10.15	0.00,9.42
18	1.97,15.78	1.10,14.56	0.34,13.17	0.00,11.97	0.00,11.12	0.00,10.34
19	2.73,17.06	1.81,15.92	0.97,14.66	0.23,13.25	0.00,12.14	0.00,11.30
20	3.52,18.29	2.56,17.21	1.66,16.05	0.84,14.76	0.12,13.32	0.00,12.31

Table 4. 95% CL.

$n\backslash b$	0.0	0.5	1.0	1.5	2.0	2.5	3.0
0	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00
1	0.04,4.75	0.00,4.35	0.00,4.11	0.00,3.94	0.00,3.81	0.00,3.72	0.00,3.64
2	0.31,6.33	0.00,5.77	0.00,5.38	0.00,5.06	0.00,4.80	0.00,4.60	0.00,4.43
3	0.73,7.87	0.23,7.36	0.00,6.75	0.00,6.34	0.00,5.97	0.00,5.66	0.00,5.39
4	1.22,9.38	0.72,8.89	0.24,8.31	0.00,7.67	0.00,7.23	0.00,6.83	0.00,6.48
5	1.77,10.83	1.27,10.34	0.77,9.83	0.30,9.22	0.00,8.53	0.00,8.08	0.00,7.66
6	2.36,12.23	1.86,11.74	1.36,11.25	0.86,10.72	0.40,10.12	0.00,9.36	0.00,8.90
7	2.98,13.61	2.48,13.12	1.98,12.62	1.48,12.12	0.99,11.59	0.53,10.99	0.11,10.27
8	3.63,14.96	3.13,14.47	2.63,13.97	2.13,13.47	1.63,12.96	1.14,12.44	0.67,11.85
9	4.30,16.29	3.80,15.79	3.30,15.30	2.79,14.80	2.29,14.30	1.80,13.79	1.31,13.26
10	4.98,17.60	4.48,17.10	3.98,16.61	3.48,16.11	2.98,15.61	2.48,15.11	1.98,14.60
11	5.68,18.90	5.18,18.40	4.68,17.90	4.18,17.40	3.68,16.91	3.18,16.40	2.68,15.90
12	6.40,20.18	5.90,19.69	5.40,19.19	4.90,18.69	4.40,18.19	3.90,17.69	3.40,17.19
13	7.12,21.46	6.62,20.96	6.12,20.46	5.62,19.96	5.12,19.46	4.62,18.96	4.12,18.46
14	7.86,22.72	7.36,22.22	6.86,21.72	6.36,21.22	5.86,20.72	5.36,20.22	4.86,19.72
15	8.60,23.97	8.11,23.47	7.61,22.98	7.10,22.47	6.61,21.98	6.11,21.48	5.60,20.98
16	9.36,25.22	8.86,24.72	8.36,24.22	7.86,23.72	7.36,23.22	6.86,22.72	6.36,22.22
17	10.12,26.46	9.62,25.96	9.12,25.46	8.62,24.96	8.12,24.46	7.62,23.96	7.12,23.46
18	10.89,27.69	10.39,27.19	9.89,26.69	9.39,26.19	8.89,25.69	8.39,25.19	7.89,24.69
19	11.66,28.92	11.16,28.41	10.66,27.91	10.16,27.41	9.66,26.92	9.16,26.42	8.66,25.92
20	12.44,30.14	11.94,29.64	11.44,29.14	10.94,28.64	10.44,28.14	9.94,27.64	9.44,27.14

$n\backslash b$	3.5	4.0	5.0	6.0	7.0	8.0	9.0
0	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00
1	0.00,3.58	0.00,3.53	0.00,3.45	0.00,3.39	0.00,3.34	0.00,3.31	0.00,3.28
2	0.00,4.29	0.00,4.17	0.00,3.99	0.00,3.86	0.00,3.75	0.00,3.67	0.00,3.60
3	0.00,5.16	0.00,4.97	0.00,4.66	0.00,4.42	0.00,4.24	0.00,4.10	0.00,3.99
4	0.00,6.16	0.00,5.89	0.00,5.43	0.00,5.09	0.00,4.82	0.00,4.60	0.00,4.43
5	0.00,7.27	0.00,6.92	0.00,6.33	0.00,5.85	0.00,5.48	0.00,5.18	0.00,4.94
6	0.00,8.46	0.00,8.05	0.00,7.33	0.00,6.73	0.00,6.24	0.00,5.84	0.00,5.52
7	0.00,9.70	0.00,9.25	0.00,8.42	0.00,7.70	0.00,7.10	0.00,6.60	0.00,6.18
8	0.25,11.16	0.00,10.48	0.00，9.57	0.00,8.76	0.00,8.05	0.00,7.45	0.00,6.94
9	0.84,12.69	0.41,12.03	0.00,10.77	0.00,9.89	0.00,9.09	0.00,8.38	0.00,7.77
10	1.50,14.07	1.02,13.51	0.18,12.15	0.00,11.05	0.00,10.18	0.00,9.39	0.00,8.69
11	2.19,15.40	1.70,14.87	0.77,13.71	0.00,12.25	0.00,11.33	0.00,10.47	0.00,9.69
12	2.90,16.69	2.40,16.18	1.43,15.11	0.55,13.87	0.00,12.50	0.00,11.59	0.00,10.74
13	3.62,17.96	3.12,17.46	2.14,16.44	1.19,15.32	0.35,13.99	0.00,12.75	0.00,11.85
14	4.36,19.22	3.86,18.72	2.86,17.72	1.89,16.67	0.97,15.50	0.18,14.09	0.00,12.99
15	5.10,20.48	4.60,19.98	3.60,18.98	2.61,17.96	1.65,16.88	0.77,15.65	0.02,14.17
16	5.86,21.72	5.36,21.22	4.36,20.22	3.36,19.22	2.38,18.18	1.44,17.07	0.59,15.78
17	6.62,22.96	6.12,22.46	5.12,21.46	4.12,20.46	3.13,19.44	2.16,18.39	1.24,17.23
18	7.39,24.19	6.88,23.69	5.89,22.69	4.89,21.69	3.89,20.69	2.90,19.66	1.94,18.59
19	8.16,25.42	7.66,24.92	6.66,23.92	5.66,22.92	4.66,21.92	3.66,20.91	2.69,19.87
20	8.94,26.63	8.44,26.14	7.44,25.14	6.44,24.14	5.44,23.14	4.44,22.13	3.45,21.12

$n\backslash b$	10.0	11.0	12.0	13.0	14.0	15.0
0	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00	0.00,3.00
1	0.00,3.25	0.00,3.23	0.00,3.22	0.00,3.20	0.00,3.19	0.00,3.18
2	0.00,3.55	0.00,3.50	0.00,3.46	0.00,3.43	0.00,3.40	0.00,3.38
3	0.00,3.89	0.00,3.81	0.00,3.75	0.00,3.69	0.00,3.64	0.00,3.60
4	0.00,4.29	0.00,4.17	0.00,4.07	0.00,3.98	0.00,3.91	0.00,3.85
5	0.00,4.73	0.00,4.57	0.00,4.43	0.00,4.31	0.00,4.22	0.00,4.13
6	0.00,5.25	0.00,5.03	0.00,4.85	0.00,4.69	0.00,4.55	0.00,4.44
7	0.00,5.84	0.00,5.55	0.00,5.31	0.00,5.11	0.00,4.94	0.00,4.78
8	0.00,6.51	0.00,6.15	0.00,5.84	0.00,5.58	0.00,5.36	0.00,5.17
9	0.00,7.25	0.00,6.81	0.00,6.44	0.00,6.12	0.00,5.84	0.00,5.61
10	0.00,8.08	0.00,7.56	0.00,7.11	0.00,6.72	0.00,6.38	0.00,6.10
11	0.00,8.99	0.00,8.38	0.00,7.85	0.00,7.39	0.00,6.99	0.00,6.64
12	0.00,9.97	0.00,9.27	0.00,8.66	0.00,8.12	0.00,7.65	0.00,7.25
13	0.00,11.01	0.00,10.24	0.00,9.55	0.00,8.93	0.00,8.39	0.00,7.91
14	0.00,12.10	0.00,11.26	0.00,10.50	0.00,9.81	0.00,9.20	0.00,8.65
15	0.00,13.22	0.00,12.34	0.00,11.51	0.00,10.75	0.00,10.07	0.00,9.45
16	0.00,14.37	0.00,13.45	0.00,12.57	0.00,11.75	0.00,11.00	0.00,10.31
17	0.43,15.89	0.00,14.59	0.00,13.67	0.00,12.80	0.00,11.99	0.00,11.24
18	1.05,17.38	0.27,15.98	0.00,14.79	0.00,13.88	0.00,13.02	0.00,12.21
19	1.75,18.76	0.88,17.51	0.14,16.07	0.00,15.00	0.00,14.10	0.00,13.24
20	2.48,20.06	1.56,18.92	0.72,17.63	0.01,16.14	0.00,15.20	0.00,14.30

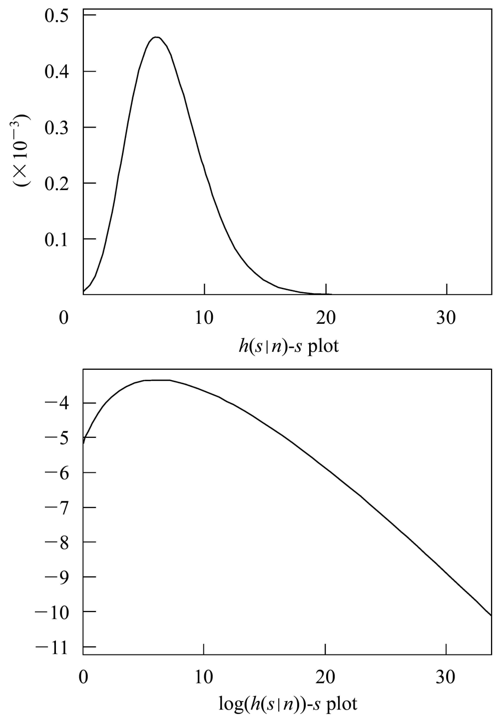

Fig. 1. The posterior density $h(s|n)$ as a function of s for n=8, b=2 with Gaussian systematic uncertainty σ_b/b=0.3 using uniform prior.

4 Summary

The Bayesian HPD credible interval construction for a Poisson variable with or without inclusion of the systematic uncertainties for background expectation and signal efficiency has been illustrated. Such intervals have several desirable properties. Like the unified interval, Bayesian HPD intervals lie in the physical region and automatically change from credible bounds to two-sided intervals. They avoid the drawback of background dependence of unified intervals when zero events are observed. In addition, the Bayesian HPD intervals are optimal on their own terms, and minimize the length among all Bayesian intervals. Introducing the conditioning produces shorter intervals in the case of incorporating the systematic uncertainty of background expectation (and also the systematic uncertainty of signal efficiency simultaneously), while keeps unchanged otherwise. Although the Bayesian HPD intervals are conceptually different from the Frequentist intervals, they are close numerically, that is, the frequentist coverage of the Bayesian intervals is quite close to the Bayesian posterior credible level[9]. The code BPOCI provides a convenient tool to calculate the Bayesian HPD intervals with or without conditioning and systematic uncertainties.

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18 Consider that the $q(n|s)_b$ represented by Eqs. (10), (11) is $p(n|s)_b$ smeared by pdf of $f_{b'}(b,\sigma_b)$ and/or $f_\varepsilon(1,\sigma_\varepsilon)$, in the case of f being Gaussian, Log-Gaussian or uniform (in limited region) pdf, the $h(s|n)$ with or without inclusion of the systematic errors has similar shape as the function of s