Three-body force and the tetraquark interpretation of light scalar mesons *

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Abstract We study the possible tetraquark interpretation of light scalar meson states $a_0(980)$, $f_0(980)$, κ, σ within the framework of the non-relativistic potential model. The wave functions of tetraquark states are obtained in a space spanned by multiple Gaussian functions. We find that the mass spectra of the light scalar mesons can be well accommodated in the tetraquark picture if we introduce a three-body quark interaction in the quark model. Using the obtained multiple Gaussian wave functions, the decay constants of tetraquarks are also calculated within the "fall apart" mechanism.

Key words scalar meson, tetraquark, three-body force

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1 Introduction

Tetraquarks were proposed decades ago. Early in 1977, Jaffe made a calculation using the color-magnetic interaction in the bag model[1, 2]. He suggested that the light scalar mesons below 1 GeV, $a_0(980)$, $f_0(980)$, κ and σ, be interpreted as a nonet of light tetraquarks.

In recent years, the light scalar mesons were observed in decays of charmed mesons. The σ is observed as a peak in the decay $D^+ \to \pi^+\pi^-\pi^+$[3, 4] and $f_0(980)$ in $D_s^+ \to \pi^+\pi^-\pi^+$[5]. From the process $J/\psi \to \omega\pi^+\pi^-$, the BES collaboration determined the pole position of σ to be $M - i\Gamma/2 = (541\pm39) - i(252\pm42)$ MeV[6]. The BES collaboration found also a κ like structure in the decay $J/\psi \to \bar{K}^* K^+ \pi^-$[7]. The accumulation of experimental data allows us to study the structure of the light scalar nonet based on the decay properties of its members[8—10].

As a many-body system, a tetraquark state is quite different from a baryon or a conventional $q\bar{q}$ meson. The color structure is no longer trivial. It is quite sensitive to the hidden color structure of the QCD interaction. A tetraquark state, if its existence is confirmed, may provide us important information about the QCD interaction that is absent from the ordinary baryons or the $q\bar{q}$ mesons. For instance, some authors had investigated the tetraquark system with the three-body $qq\bar{q}$ and $q\bar{q}\bar{q}$ interaction, whose existence has no direct effect on the ordinary hadronic states[11—13]. The newly updated experimental data can shed more light on the relation between the possible tetraquark states and the QCD interaction.

In this article, we will study the possible tetraquark state within the framework of the non-relativistic potential model. We will calculate mass spectra and wave functions of the light tetraquark using the Bhaduri potential[14]. To fit the experimental masses, we will extend the model with the three-body $qq\bar{q}$ and $q\bar{q}\bar{q}$ interaction. Using the wave functions of tetraquarks, we will determine the coupling constants of tetraquarks to mesons under the "fall apart" mechanism.

The article is organized as follows: In Sec. 2, we introduce the model Hamiltonian and the multiple Gaussian function method which is used to obtain the tetraquark wave functions. In Sec. 3, we present the "fall apart" decay calculation with tetraquark wave function. In Sec. 4, we present the numerical results. Finally we will give a brief summary.

2 Hamiltonian and wave functions

In a non-relativistic quark model, usually the potentials are limited to the two-body interaction, which mainly consists of two parts: the $\boldsymbol{\lambda}^c\cdot\boldsymbol{\lambda}^c$ color inter-

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action including the confinement and the Coulomb interaction of one-gluon exchange, and the $\boldsymbol{\lambda}^{\rm c}\cdot\boldsymbol{\lambda}^{\rm c}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}$ color-magnetic interaction. The Hamiltonian reads as

$$H = \sum_i \left(m_i + \frac{\boldsymbol{P}_i^2}{2m_i}\right) - \frac{3}{4}\sum_{i<j}\left[\boldsymbol{F}_i\cdot\boldsymbol{F}_j V^{\rm C}(r_{ij}) + \boldsymbol{F}_i\cdot\boldsymbol{F}_j \boldsymbol{S}_i\cdot\boldsymbol{S}_j V^{\rm SS}(r_{ij})\right] \tag{1}$$

where the m_i's are the quark masses, $\boldsymbol{F}_i^{\rm c} = \frac{\boldsymbol{\lambda}_i^{\rm c}}{2}$, and r_{ij} is the distance between quark i and quark j.

Among the various potential forms used in different quark models, the Bhaduri potential[14] is rather simple and gives a unified description of conventional hadron spectroscopy. Also it is often used to discuss the tetraquark system[13, 15—18]. The potentials read as

$$V_{ij}^{\rm C} = -\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a_0^2} - D,\quad V_{ij}^{\rm SS} = \frac{4\kappa}{m_i m_j}\frac{1}{r_0^2 r_{ij}}{\rm e}^{-r_{ij}/r_0}\ .$$

The parameters have the following values:

$$\begin{aligned}&\kappa = 102.67\ \text{MeV}\cdot\text{fm},\quad a_0 = 0.0326\ (\text{MeV}^{-1}\cdot\text{fm})^{\frac{1}{2}},\\ &D = 913.5\ \text{MeV},\quad r_0 = 0.4545\ \text{fm},\\ &m_{\rm u} = m_{\rm d} = 337\ \text{MeV},\quad m_{\rm s} = 600\ \text{MeV},\\ &m_{\rm c} = 1870\ \text{MeV},\quad m_{\rm b} = 5259\ \text{MeV}.\end{aligned} \tag{2}$$

In a tetraquark, some new interactions which have no direct effect on the ordinary hadrons may have significant contribution. For instance, one can introduce the following three-body qq$\bar{\rm q}$ and q$\bar{\rm q}\bar{\rm q}$ interactions[11—13]

$$\begin{aligned}V_{{\rm qq}\bar{\rm q}}(\boldsymbol{r}_i,\boldsymbol{r}_j,\boldsymbol{r}_k) =& d^{\rm abc}F_i^{\rm a}F_j^{\rm b}F_k^{\rm c*}U_0\times\\ &\exp[-(r_{ij}^2+r_{jk}^2+r_{ki}^2)/r_0^2],\\ V_{{\rm q}\bar{\rm q}\bar{\rm q}}(\boldsymbol{r}_i,\boldsymbol{r}_j,\boldsymbol{r}_k) =& d^{\rm abc}F_i^{\rm a}F_j^{\rm b*}F_k^{\rm c*}U_0\times\\ &\exp[-(r_{ij}^2+r_{jk}^2+r_{ki}^2)/r_0^2].\end{aligned}$$

In this article, since we will only treat the ground states of tetraquarks, the spatial dependence of the three-body interaction is less important. So we will only add the following simplified interaction into the model Hamiltonian (1)

$$V_{\rm 3b} = U_0(d^{\rm abc}F_i^{\rm a}F_j^{\rm b}F_k^{\rm c*} + d^{\rm abc}F_i^{\rm a}F_j^{\rm b*}F_k^{\rm c*}). \tag{3}$$

This interaction is diagonal in the diquark–anti-diquark color base of the tetraquark

$$\langle[{\rm qq}]_{\bar{3}}[\bar{\rm q}\bar{\rm q}]_3 \mid V_{\rm 3b} \mid [{\rm qq}]_{\bar{3}}[\bar{\rm q}\bar{\rm q}]_3\rangle = -\frac{20}{9}U_0, \tag{4a}$$

$$\langle[{\rm qq}]_6[\bar{\rm q}\bar{\rm q}]_{\bar{6}} \mid V_{\rm 3b} \mid [{\rm qq}]_6[\bar{\rm q}\bar{\rm q}]_{\bar{6}}\rangle = +\frac{10}{9}U_0. \tag{4b}$$

An immediate consequence is that this three-body interaction has no direct contribution to any meson-meson coupling channel.

To explain our calculation method, we first define some convenient coordinates for the tetraquark system as illustrated in Fig. 1[17],

$$\boldsymbol{x}_1 = \boldsymbol{r}_1 - \boldsymbol{r}_2, \tag{5a}$$
$$\boldsymbol{x}_2 = \boldsymbol{r}_3 - \boldsymbol{r}_4, \tag{5b}$$
$$\boldsymbol{x}_3 = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} - \frac{m_3\boldsymbol{r}_3 + m_4\boldsymbol{r}_4}{m_3+m_4}\ , \tag{5c}$$

$$\boldsymbol{y}_1 = \boldsymbol{r}_1 - \boldsymbol{r}_3, \tag{6a}$$
$$\boldsymbol{y}_2 = \boldsymbol{r}_2 - \boldsymbol{r}_4, \tag{6b}$$
$$\boldsymbol{y}_3 = \frac{m_1\boldsymbol{r}_1 + m_3\boldsymbol{r}_3}{m_1+m_3} - \frac{m_2\boldsymbol{r}_2 + m_4\boldsymbol{r}_4}{m_2+m_4}\ , \tag{6c}$$

$$\boldsymbol{z}_1 = \boldsymbol{r}_1 - \boldsymbol{r}_4, \tag{7a}$$
$$\boldsymbol{z}_2 = \boldsymbol{r}_2 - \boldsymbol{r}_3, \tag{7b}$$
$$\boldsymbol{z}_3 = \frac{m_1\boldsymbol{r}_1 + m_4\boldsymbol{r}_4}{m_1+m_4} - \frac{m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3}{m_2+m_3}\ . \tag{7c}$$

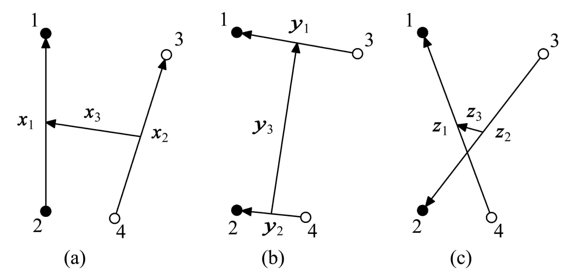

Fig. 1. Three ways to define the relative coordinates for a tetraquark system. Filled and open circles represent quarks and anti-quarks respectively.

The basis wave functions for the tetraquark will be the product of color, spin, flavor and spatial wave functions. The color and spin $SU_{\rm c}(3)\otimes SU_{\rm s}(2)$ basis functions we use, is of the following diquark anti-diquark coupling form:

1) $S=0$

$$\begin{aligned}&\alpha_1 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|0_{12}0_{34}\rangle_{\rm s},\ \alpha_2 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s},\\ &\alpha_3 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|0_{12}0_{34}\rangle_{\rm s},\ \alpha_4 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s}.\end{aligned} \tag{8}$$

2) $S=1$

$$\begin{aligned}&\alpha_1 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|0_{12}1_{34}\rangle_{\rm s},\ \alpha_2 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|1_{12}0_{34}\rangle_{\rm s},\\ &\alpha_3 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s},\ \alpha_4 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|0_{12}1_{34}\rangle_{\rm s},\\ &\alpha_5 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|1_{12}0_{34}\rangle_{\rm s},\ \alpha_6 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s}.\end{aligned} \tag{9}$$

3) $S=2$

$$\alpha_1 = |\bar{3}_{12}3_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s},\ \alpha_2 = |6_{12}\bar{6}_{34}\rangle_{\rm c}\otimes|1_{12}1_{34}\rangle_{\rm s}. \tag{10}$$

Here the color wave function of the two (anti-)quarks is labeled by the $SU_{\rm c}(3)$ dimension and the spin wave function by the total spin.

The anti-symmetric diquarks [ud], [us], [ds] form the $\bar{3}$ representation of flavor $SU_{\rm f}(3)$. The $\bar{3}$ diquarks and 3 anti-diquarks further form a tetraquark nonet. They are assumed to be the light scalar mesons[1, 8, 19]. So the flavor wave functions are:

$$a_0(I=1,I_3=0)=\frac{1}{\sqrt{2}}([{\rm us}][\bar{\rm u}\bar{\rm s}]-[{\rm ds}][\bar{\rm d}\bar{\rm s}]), \tag{11a}$$

$$f_0(I=0)=\frac{1}{\sqrt{2}}([{\rm us}][\bar{\rm u}\bar{\rm s}]+[{\rm ds}][\bar{\rm d}\bar{\rm s}]), \tag{11b}$$

$$\sigma_0(I=0)=[{\rm ud}][\bar{\rm u}\bar{\rm d}], \tag{11c}$$

$$\kappa^+=[{\rm ud}][\bar{\rm s}\bar{\rm d}]. \tag{11d}$$

As for the spatial wave functions, we will start from the multi-dimensional Gaussian function

$$g^{\rm s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)=\exp\left(-\sum_{i,j}^{3}A^{\rm s}_{ij}\boldsymbol{x}_i\cdot\boldsymbol{x}_j\right), \tag{12}$$

where $A^{\rm s}_{ij}$ are the function parameters. The wave function of this form is well convergent and there exists many analytical expressions for different matrix elements. We will use it to construct the spatial basis wave functions[20, 21].

Under the hypothesis of Jaffe, the color-spin wave function of a "good" diquark is the symmetric one, $|\bar{3}_{12}\rangle_{\rm c}\otimes|0_{12}\rangle_{\rm s}$. As the flavor wave function of the scalar nonet state is anti-symmetric, so the spatial wave function should be symmetric. That is, the spatial wave function of the tetraquark state should be invariant under $\boldsymbol{x}_1\to-\boldsymbol{x}_1$ and/or $\boldsymbol{x}_2\to-\boldsymbol{x}_2$. If we use the Gaussian function (12) as the basis wave function, it is easy to see that[17]

$$A_{12}=A_{23}=A_{31}=0\ .$$

We will use the following symmetric combination as a basis function

$$\psi^{\rm s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)=\frac{1}{4}[g^{\rm s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)+g^{\rm s}(-\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)+ g^{\rm s}(\boldsymbol{x}_1,-\boldsymbol{x}_2,\boldsymbol{x}_3)+g^{\rm s}(\boldsymbol{x}_1,\boldsymbol{x}_2,-\boldsymbol{x}_3)]. \tag{13}$$

If the non-diagonal parameters $A_{ij}(i\neq j)$ are small, we have

$$\psi^{\rm s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\approx\exp[-(A^{\rm s}_{11}\boldsymbol{x}_1^2+A^{\rm s}_{22}\boldsymbol{x}_2^2+A^{\rm s}_{33}\boldsymbol{x}_3^2)]\times \left[1+2A^{\rm s2}_{12}(\boldsymbol{x}_1\cdot\boldsymbol{x}_2)^2+2A^{\rm s2}_{13}(\boldsymbol{x}_1\cdot\boldsymbol{x}_3)^2+ 2A^{\rm s2}_{23}(\boldsymbol{x}_2\cdot\boldsymbol{x}_3)^2\right]. \tag{14}$$

This allow us to study the correlations in the quark alignment.

We will choose n independent symmetric Gaussian functions (13), $s=1,2,...,n$, to span an n-dimensional nonorthogonal basis. The n independent Gaussian functions are obtained by the following process. First, we use one such symmetric Gaussian function as a test wave function in the variation to determine a base parameter set A_{ij}. The matrix (A_{ij}) will be specified by its three principal values denoted by $A^{(0)}_{11}$, $A^{(0)}_{22}$, $A^{(0)}_{33}$ and three Euler angels (α,β,γ) which specify the orientation. Then a complete parameter set $A^{\rm s}_{ij}(s=1,2,...,n)$ is generated by first scaling to the principal values[17]

$$A^{{\rm s}(0)}_{ii}=A^{(0)}_{ii}d^{s_i} \tag{15}$$

where $s_i=-k,-k+1,...,k-1,k$, $(2k+1)^3=n$, and d is a scaling factor. Then we make an Euler rotation (α,β,γ).

By diagonalizing the Hamiltonian in the above nonorthogonal basis, we will obtain the masses and wave functions of tetraquark states. The wave function can be expressed in the above basis functions as

$$|T\rangle=\phi_{\rm f}\sum_{is}C_{is}\alpha_i\psi^{\rm s}, \tag{16}$$

where $\phi_{\rm f}$ is the flavor wave function and C_{is} are the superposition coefficients.

Similar to the case in pseudo-scalar mesons, the $I=0$ members f_0, σ_0 in the scalar nonet will mix with each other. To consider the mixing, we further introduce a mixing angle ϕ[9]

$$f=f_0\cos\phi+\sigma_0\sin\phi,\quad \sigma=-\sin\phi f_0+\cos\phi\sigma_0\ . \tag{17}$$

Then f and σ are the physically observable states. In this article, we do not discuss the underlying mechanism of this mixing. So we will merely treat the mixing angle ϕ as one additional parameter.

3 Decay properties of tetraquark states

Several authors have used the effective Lagrangian with $SU_{\rm f}(3)$ symmetry to discuss the decay of the light scalar nonet[8, 9]. Here we can calculate the coupling constants using the tetraquark wave functions. The general coupling Lagrangian reads as

$$\begin{aligned}\mathcal{L}=&f_0\left[g_{{\rm f}_0\pi\pi}\frac{\pi\cdot\pi}{2}+g_{{\rm f}_0\bar{\rm K}{\rm K}}\bar{\rm K}{\rm K}+\cdots\right]+\\ &\sigma_0\left[+g_{\sigma_0\pi\pi}\frac{\pi\cdot\pi}{2}+g_{\sigma_0\bar{\rm K}{\rm K}}\bar{\rm K}{\rm K}+\cdots\right]+\\ &a\cdot\left[g_{{\rm a}_0\bar{\rm K}{\rm K}}\bar{\rm K}\tau{\rm K}+g_{{\rm a}\eta_{\rm s}\pi}\eta_{\rm s}\pi+c_{{\rm a}\eta_{\rm q}\pi}\eta_{\rm q}\pi+\cdots\right]+\\ &g_{\kappa\bar{\rm K}\pi}\left(\bar{\rm K}\tau\kappa\cdot\pi+{\rm h.c.}\right)+\cdots\end{aligned} \tag{18}$$

At present, the quark interaction underlying the meson decay couplings is still unclear to us. Here we will assume that the mechanism underlying the decay and the fusion process is the same and can be depicted by the "fall apart" mechanism in Fig. 2. More specific, we assume that the coupling constant of a tetraquark T to two mesons $\rm M_1$ and $\rm M_2$ is proportional to the wave function overlap

$$g_{\rm TMM}\propto\langle{\rm M_1M_2}\,|\,{\rm T}\rangle\ . \tag{19}$$

The meson wave functions will also be approximated by multiple Gaussian wave functions determined by a similar variation process

$$|M\rangle_{\boldsymbol{r}}=\phi_{\rm f}\sum_s C_s\psi^{\rm s}(\boldsymbol{r}) \ , \tag{20}$$

where $\phi_{\rm f}$ is the meson flavor wave function, and the spatial basis function is

$$\psi_{\rm s}(\boldsymbol{r})={\rm e}^{-A^{\rm s}\boldsymbol{r}^2} \ . \tag{21}$$

A tetraquark system $q_1q_2\bar{q}_3\bar{q}_4$ can fall apart into two different flavor combinations $q_1\bar{q}_3+q_2\bar{q}_4$ and $q_1\bar{q}_4+q_2\bar{q}_3$, and the corresponding final meson-meson states are different

$$|M_1M_2\rangle_1=|M_1\rangle_{\boldsymbol{y}_1}|M_2\rangle_{\boldsymbol{y}_2}, \tag{22}$$
$$|M_1M_2\rangle_2=|M_1\rangle_{\boldsymbol{z}_1}|M_2\rangle_{\boldsymbol{z}_2}. \tag{23}$$

The spatial wave functions are defined in the coordinates $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$ of Eqs. (6) and (7) respectively.

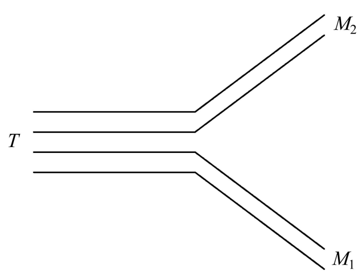

Fig. 2. "fall apart" mechanism for decays of $q^2\bar{q}^2$ tetraquark states.

In the decay of the light scalars to pseudo-scalar mesons, we need to consider the η-η' mixing

$$\eta=\cos\theta\eta_{\rm q}+\sin\theta\eta_{\rm s}, \tag{24a}$$
$$\eta'=-\sin\theta\eta_{\rm q}+\cos\theta\eta_{\rm s}, \tag{24b}$$

where $\eta_{\rm q}=\frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})$, $\eta_{\rm s}=s\bar{s}$ and $\sin\theta=-0.608$[22]. We obtain the following expressions for the coupling constants (a proportionality constant is dropped)

$$g_{{\rm f}_0\to\eta\eta}=\sin\theta\cos\theta A_{{\rm f}_0\to\eta_{\rm q}+\eta_{\rm s}}, \tag{25a}$$
$$g_{{\rm f}_0\to\eta\eta'}=\frac{1}{\sqrt{2}}(\cos\theta^2-\sin\theta^2)A_{{\rm f}_0\to\eta_{\rm q}+\eta_{\rm s}}, \tag{25b}$$
$$g_{{\rm f}_0\to\eta'\eta'}=-\sin\theta\cos\theta A_{{\rm f}_0\to\eta_{\rm q}+\eta_{\rm s}}, \tag{25c}$$
$$g_{{\rm f}_0\to{\rm KK}}=\frac{1}{\sqrt{2}}A_{{\rm f}_0\to{\rm K}^++{\rm K}^-}. \tag{25d}$$

$$g_{\sigma_0\to\pi\pi}=\frac{\sqrt{3}}{2}A_{\sigma_0\to\pi^++\pi^-}, \tag{26a}$$
$$g_{\sigma_0\to\eta\eta}=\frac{1}{2}\cos\theta^2A_{\sigma_0\to\eta_{\rm q}+\eta_{\rm q}}, \tag{26b}$$
$$g_{\sigma_0\to\eta\eta'}=-\frac{1}{\sqrt{2}}\sin\theta\cos\theta A_{\sigma_0\to\eta_{\rm q}+\eta_{\rm q}}, \tag{26c}$$
$$g_{\sigma_0\to\eta'\eta'}=\frac{1}{2}\sin\theta^2A_{\sigma_0\to\eta_{\rm q}+\eta_{\rm q}}. \tag{26d}$$

$$g_{{\rm a}\to\pi\eta}=\frac{1}{\sqrt{2}}\sin\theta A_{{\rm a}_0^+\to\pi^++\eta_{\rm s}}, \tag{27a}$$
$$g_{{\rm a}\to\pi\eta'}=\frac{1}{\sqrt{2}}\cos\theta A_{{\rm a}_0^+\to\pi^++\eta_{\rm s}}, \tag{27b}$$
$$g_{{\rm a}\to{\rm KK}}=\frac{1}{\sqrt{2}}A_{{\rm a}^+\to{\rm K}^++\bar{\rm K}^0}. \tag{27c}$$

$$g_{\kappa\to\pi{\rm K}}=\frac{\sqrt{3}}{2}A_{\kappa^+\to\pi^++{\rm K}^0}, \tag{28a}$$
$$g_{\kappa\to\eta{\rm K}}=\frac{1}{2}\cos\theta A_{\kappa^+\to\eta_{\rm q}}, \tag{28b}$$
$$g_{\kappa\to\eta'{\rm K}}=-\frac{1}{2}\sin\theta A_{\kappa^+\to{\rm K}^++\eta_{\rm q}} \tag{28c}$$

Besides the explicit flavor overlap factors, $A_{\rm T\to MM}$ is the overlap of the color, spin and spatial wave function.

After considering the σ-f_0 mixing effect, The coupling constants $g_{\rm T\to MM}$ for the decays of σ and f_0 are further modified to

$$g_{\rm f\to MM}=\cos\phi g_{{\rm f}_0\to\rm MM}+\sin\phi g_{\sigma_0\to\rm MM}, \tag{29a}$$
$$g_{\sigma\to\rm MM}=-\sin\phi g_{{\rm f}_0\to\rm MM}+\cos\phi g_{\sigma_0\to\rm MM}. \tag{29b}$$

4 Numerical results

The Bhaduri potential gives a unified description of the spectroscopy of ordinary mesons and baryons. The Hamiltonian (1) itself is an eigenvalue problem of a differential equation which can be solved numerically. However, the multiple Gaussian function method can still give an impressively good approximation of the mesonic ground state and, in addition, the Gaussian wave function is rather simple to use. In Table 1, we show some results of the pseudo-scalar meson calculation. We see that the multiple Gaussian function method greatly improves the single Gaussian function approximation.

Table 1. Pseudo-scalar meson calculations. Col. 1: direct solution of the Schrödinger equation, col. 2: use of the variational method with a single Gaussian function, col. 3: use of the multiple Gaussian function method with 7 Gaussian functions.

$m_\pi=m_{\eta_{\rm q}}$/MeV	$m_{\rm K}$/MeV	$m_{\eta_{\rm s}}$/MeV
136	520	758
250	582	800
137	521	758

Now we turn to the tetraquark calculation. In our calculation, the scaling factor is fixed to be $d=2$. We will take $k=1$, i.e., the wave function space is spanned by $3^3=27$ Gaussian functions. In the light scalar tetraquark, as we assume that the flavor parts of the diquark and anti-diquark wave functions are antisymmetric and the spatial wave function is symmetric, so the color and spin wave function must be the symmetric ones, α_1 and α_4 in Eq. (8).

First, we will consider the original Bhaduri potential without the three-body quark interaction (3). We obtain the following mass values

$$M_\sigma = 687\ \text{MeV},\quad M_\kappa = 1067\ \text{MeV},$$
$$M_{a_0} = M_{f_0} = 1371\ \text{MeV}, \tag{30}$$

which are about 300 MeV higher than the experimental values. One can calculate the probability to find a tetraquark state $|\Psi\rangle$ in different color-spin structures α_k

$$P_{\alpha_k} = \int \prod_{i=1}^{3} \mathrm{d}\boldsymbol{x}_i \, |\langle \alpha_k | \Psi \rangle|^2 . \tag{31}$$

The color-spin contents of the tetraquark nonet without three quark interaction are presented in Table 2. In this case the color content is mainly the $6\times\bar{6}$ component which disagrees with Jaffe's "good" diquark hypothesis.

Table 2. Contents of tetraquarks without three-body interaction.

	σ	κ	a_0, f_0
P_{α_1}	0.30	0.30	0.29
P_{α_4}	0.70	0.70	0.71

Next, if the three body interaction with $U_0 = 0.333$ GeV is turned on, we find that light tetraquark masses are

$$M_\sigma = 443\ \text{MeV},\quad M_\kappa = 744\ \text{MeV},$$
$$M_{a_0} = M_{f_0} = 985\ \text{MeV}, \tag{32}$$

which are in agreement with the experimental values[23]:

$$M_\sigma = 800 \pm 400\ \text{MeV},\quad M_\kappa = 840 \pm 80\ \text{MeV},$$
$$M_{a_0} = 984.7 \pm 1.2\ \text{MeV},\quad M_{f_0} = 980 \pm 10\ \text{MeV}. \tag{33}$$

The color-spin contents of the nonet are shown in Table 3 and also agree with the "good" diquark picture[8, 24].

Table 3. Contents of tetraquarks with three-body interaction.

	σ	κ	a_0, f_0
P_{α_1}	0.80	0.88	0.92
P_{α_4}	0.20	0.12	0.08

In our calculation, the tetraquark wave function is symmetric under the coordinate reflections $\boldsymbol{x}_1 \to -\boldsymbol{x}_1$ and/or $\boldsymbol{x}_2 \to -\boldsymbol{x}_2$. It is easy to see that the expectation values

$$\langle \boldsymbol{x}_i \cdot \boldsymbol{x}_j \rangle = \langle \boldsymbol{x}_i^2 \rangle \delta_{ij} \ . \tag{34}$$

$\sqrt{\langle \boldsymbol{x}_1^2 \rangle}$ and $\sqrt{\langle \boldsymbol{x}_2^2 \rangle}$ are the mean square radii(RMS) of the diquark and anti-diquark respectively. The quark and anti-quark RMS of the tetraquark is given by

$$R^2 \equiv \frac{\left\langle \sum_{i=0}^{4} m_i (\boldsymbol{r}_i - \boldsymbol{R}_{\mathrm{CM}})^2 \right\rangle}{\sum_{i=0}^{4} m_i} = \frac{\mu_{12}\langle \boldsymbol{x}_1^2\rangle + \mu_{34}\langle \boldsymbol{x}_2^2\rangle + \mu_{12,34}\langle \boldsymbol{x}_3^2\rangle}{m_1+m_2+m_3+m_4}, \tag{35}$$

where

$$\boldsymbol{R}_{\mathrm{CM}} = \sum_{i=0}^{4} m_i \boldsymbol{r}_i \Big/ \sum_{i=0}^{4} m_i, \tag{36}$$

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j}, \tag{37}$$

$$\mu_{ij,kl} = \frac{(m_i+m_j)(m_k+m_l)}{m_i+m_j+m_k+m_l}. \tag{38}$$

The RMS values are tabulated in Table 4.

Table 4. The RMS values in fm.

	σ	κ	a_0, f_0
$\sqrt{\langle \boldsymbol{x}_1^2 \rangle}$	0.70	0.72	0.70
$\sqrt{\langle \boldsymbol{x}_2^2 \rangle}$	0.70	0.69	0.70
$\sqrt{\langle \boldsymbol{x}_3^2 \rangle}$	0.54	0.58	0.56

However, the spatial wave function (13) is beyond the usual tetraquark assumption. Usually a tetraquark is assumed to be constructed from "good diquarks". The inner orbital angular momentum of the (anti-)diquark in a tetraquark is zero. So the relative angular momentum between the diquark and anti-diquark in the scalar tetraquark is also zero. The spatial wave function will have the form

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \psi(\boldsymbol{x}_1^2, \boldsymbol{x}_2^2, \boldsymbol{x}_3^2), \tag{39}$$

i.e., all the $\boldsymbol{x}_i$ are in S-waves. Our choice (13) is beyond the above assumption (this can be easily seen from Eq. (14)). If Eq. (39) holds, then the following identity is valid:

$$\langle (\boldsymbol{x}_i \cdot \boldsymbol{x}_j)^2 \rangle = \frac{1}{3} \langle \boldsymbol{x}_i^2 \rangle \langle \boldsymbol{x}_j^2 \rangle, \qquad (i \neq j). \tag{40}$$

The deviation of a tetraquark state from (39) can be measured by

$$\epsilon_{ij} = \frac{3\langle (\boldsymbol{x}_i \cdot \boldsymbol{x}_j)^2 \rangle}{\langle \boldsymbol{x}_i^2 \rangle \langle \boldsymbol{x}_j^2 \rangle} - 1. \tag{41}$$

The numerical ϵ_{ij} values are listed in Table 5. The small nonzero ϵ values means that the tetraquark states are indeed not pure S-waves. There is always some D-wave mixing.

Table 5. The ϵ_{ij} values of tetraquark wave function.

	σ	κ	a_0, f_0
ϵ_{12}	0.14	0.23	0.08
ϵ_{13}	0.21	0.22	0.13
ϵ_{23}	0.21	0.24	0.13

With the obtained wave functions we can calculate the wave function overlap in Eqs. (25) — (28) to get the coupling constants. The results are collected in Table 6. According to Ref. [9], the scalar isoscalar mixing angle ϕ in Eq. (17) will be fixed by the ratio $g^2_{f\to\bar{K}K}/g^2_{f\to\pi\pi} = 4.21$ with Eq. (29). This gives $\phi = 16.8°$. The ratios of coupling constants for scalar meson decays are listed in Table 7. Similar to Bugg's calculation[9], although most of the experimental ratios can be fitted within a factor 2, $g^2_{f\to\eta\eta}/g^2_{f\to\pi\pi}$ is far above the experimental value.

Table 6. Tetraquark-meson-meson wave function overlap in color, spin, spatial space $A_{\mathrm{T}\to\mathrm{M_1M_2}}$ (unit $\mathrm{GeV}^{-3/2}$).

$\sigma_0\to\pi^+ + \pi^-$	$\kappa^+\to\pi^+ + \mathrm{K}^0$	$\kappa^+\to\mathrm{K}^+ + \mathrm{d}\bar{\mathrm{d}}$	$\mathrm{a}^+\to\mathrm{K}^+ + \bar{\mathrm{K}}^0$	$\mathrm{a}^+\to\pi^+ + \eta_s$
10.75	9.37	9.37	8.16	8.38

Table 7. Ratios of coupling constants for light scalar meson decays, with $\phi=16.8^\circ$.

	analysis of Ref. [25]	analysis of Ref. [9]	our results	Expt.[6, 26—28]
$g^2_{\mathrm{a_0}\to\pi\eta}/g^2_{\mathrm{a_0}\to\bar{\mathrm{K}}\mathrm{K}}$	0.60	0.40 ± 0.03	0.39	0.75 ± 0.11
$g^2_{\mathrm{f}\to\bar{\mathrm{K}}\mathrm{K}}/g^2_{\mathrm{f}\to\pi\pi}$	4.21	4.21 ± 0.46	4.21	4.21 ± 0.46
$g^2_{\mathrm{f}\to\bar{\mathrm{K}}\mathrm{K}}/g^2_{\mathrm{a_0}\to\bar{\mathrm{K}}\mathrm{K}}$	2.28	0.93 ± 0.01	0.92	2.15 ± 0.4
$g^2_{\mathrm{a_0}\to\pi\eta'}/g^2_{\mathrm{a_0}\to\pi\eta}$	0.16	–	1.71	–
$g^2_{\mathrm{f}\to\eta\eta}/g^2_{\mathrm{f}\to\pi\pi}$	1.35	1.07 ± 0.18	1.15	<0.33
$g^2_{\sigma\to\bar{\mathrm{K}}\mathrm{K}}/g^2_{\sigma\to\pi\pi}$	4.8×10^{-7}	0.03 ± 0.01	0.04	0.6 ± 0.1
$g^2_{\sigma\to\eta\eta}/g^2_{\sigma\to\pi\pi}$	0.05	0.23 ± 0.02	0.25	0.20 ± 0.04
$g^2_{\kappa\to\pi\mathrm{K}}/g^2_{\sigma\to\pi\pi}$	0.78	0.58	0.83	(2.14 ± 0.28) to (1.35 ± 0.10)
$g^2_{\kappa\to\eta\mathrm{K}}/g^2_{\kappa\to\pi\mathrm{K}}$	0.12	0.20 ± 0.01	0.21	0.06 ± 0.02
$g^2_{\kappa\to\eta'\mathrm{K}}/g^2_{\kappa\to\pi\mathrm{K}}$	0.006	0.13 ± 0.01	0.12	0.29 ± 0.29

5 Summary

In summary, we have performed a tetraquark calculation of light scalar mesons using the quark potential model. If we only consider the two-body quark interaction as in the conventional hadron calculation, the masses of the tetraquark states will be several hundred MeV higher than the experimental data. In addition the major component of the light tetraquark wave functions consists of the color sextet diquark and anti-diquark. After including a three-body interaction in the Hamiltonian, the masses of the light tetraquark nonet agree with the experimental data and their wave functions are composed of mainly the "good" diquarks and anti-diquarks. We have used multiple Gaussian functions to approximate the tetraquark wave functions and noticed that there is a small mixing of D-waves in the wave functions. With our wave functions, we also calculate the coupling constants for scalar meson decays according to the "fall apart" mechanism. By introducing the isoscalar mixing angle ϕ, we obtain a fit of the ratios of coupling constants for scalar meson decays similar to other analysis based on the tetraquark picture.

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