Matrix method for the solution of RF field perturbations due to local frequency shifts

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Abstract To tune the accelerating field to the design value in a periodical radio frequency accelerating structure, Slater's perturbation theorem is commonly used. This theorem solves a second-order differential equation to obtain the electrical field variation due to a local frequency shift. The solution becomes very difficult for a complex distribution of the local frequency shifts. Noticing the similarity between the field perturbation equation and the equation describing the transverse motion of a particle in a quadrupole channel, we propose in this paper a new method in which the transfer matrix method is applied to the field calculation instead of directly solving the differential equation. The advantage of the matrix method is illustrated in examples.

Key words electrical field perturbation, frequency shift, transfer matrix

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1 Introduction

In a radio-frequency accelerating structure, a specified electrical field distribution along the longitudinal axis is designed in longitudinal beam dynamics. Usually the distribution is simply flat, linear tilted, or a combination of these two. On the one hand, the RF structure design must follow the specified field distribution. On the other hand, due to mechanical errors in machining and assembly of the structure cells the field distribution usually does not coincide with the design curve. So, in the design and in the cavity tuning, Slater's theorem is commonly applied to calculate the field distribution according to the local frequency shift. The theorem is described by a resonant differential equation. This equation becomes very difficult to solve if the local frequency shift represents a complex function. However, the similarity of the equation to that of the particle motion in a quadrupole channel gives us a hint that the beam matrix transfer method can be applied for the RF cavity field calculation. In this paper we will propose this new approach.

2 Traditional algorithm of the perturbation theorem

In a RF cavity the local frequency perturbation along the longitudinal direction can be described as a function of z

$$\omega(z)=\omega_0+\delta\omega(z)\,, \tag{1}$$

where ω_0 is the resonant angular frequency of the cavity , while $\delta\omega(z)$ is the local frequency shift. The local frequency shift induces a longitudinal electrical field distribution, as given by the equation:

$$\frac{\mathrm{d}^2E_z}{\mathrm{d}z^2}+\mu(z)\varepsilon(z)[\omega_0^2-\omega^2(z)]E_z=0\,. \tag{2}$$

In a quasi-static approximation the variation of μ and ε is negligible:

$$\mu(z)\approx\mu_0,\quad \varepsilon(z)\approx\varepsilon_0\,. \tag{3}$$

For small perturbations we assume $E_z=E_0+\delta E_z$, where E_0 stands for the average electrical field of the whole cavity, and δE_z is the field perturbation along the cavity, with $\delta E_z\ll E_0$. For a first order small frequency shift, we have $\omega^2-\omega_0^2=2\omega_0\delta\omega+(\delta\omega)^2\approx2\omega_0\delta\omega$, and therefore Eq. (2) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[\frac{\delta E_z}{E_0}\right]=\frac{8\pi^2}{\lambda_0^2}\frac{\delta f}{f_0}\,. \tag{4}$$

Here $f_0=\omega_0/2\pi=c/\lambda_0$. Eq. (4) is called Slater's theorem.

Applying the following boundary condition for a resonant cavity:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\delta E_z}{E_0}\right]=0\quad(z=0\text{ and }z=L), \tag{5}$$

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we get

$$\frac{1}{L_T}\int_0^{L_T}\delta f \mathrm{d}z=0\,. \tag{6}$$

Eq. (6) means that the average frequency shift over the whole cavity is zero. From this relation one can derive the resonant frequency of the cavity:

$$f_0=\frac{1}{L_T}\int_0^{L_T} f(z)\mathrm{d}z\,. \tag{7}$$

The solution of Eq. (4) with the boundary condition leads to a relation between the local frequency shifts and the deviation of the electrical field distribution along the cavity. Some solution examples for the step function of the local frequency shift can be found in Refs. [1] and [4]. Inversely, it can also be used to adjust the electrical field by changing the local frequency while keeping the total frequency unchanged.

3 Particle transport equation

It is well know that the particle transport in a quadrupole channel is described by the following equation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2}+\Omega_U(z)u=0\,. \tag{8}$$

Here we define $u=(\beta_s\gamma_s)^{1/2}U$, where U means the transverse direction, either x or y and

$$\Omega_U(z)=\begin{cases} k_q^2=\dfrac{eG_m}{m_0c\beta_s\gamma_s}, & (x:\text{focus};y:\text{defocus})\\[2mm] -k_q^2=-\dfrac{eG_m}{m_0c\beta_s\gamma_s}, & (x:\text{defocus};y:\text{focus})\\[2mm] 0, & (x,y:\text{drifting})\end{cases}\,. \tag{9}$$

The coefficient $\Omega_U(z)$ is a piecewise constant in the transport channel. For this kind of equation, the transfer matrix is chosen to express the solution. For example, the transfer matrix in a focusing magnet is:

$$\begin{pmatrix} x\\ x'\end{pmatrix}=M_f\begin{pmatrix} x_0\\ x_0'\end{pmatrix}, \tag{10}$$

with

$$M=\begin{bmatrix}\cos(k_qz) & \dfrac{1}{k_q}\sin(k_qz)\\ -k_q\sin(k_qz) & \cos(k_qz)\end{bmatrix}. \tag{11}$$

Here k_q is the focusing coefficient.

For a transport channel with N elements, the particle transverse movement can be easily calculated by the following equation:

$$\begin{pmatrix} x\\ x'\end{pmatrix}=M\begin{pmatrix} x_0\\ x_0'\end{pmatrix}=\prod_{j=N}^{j=1}M_j\cdot\begin{pmatrix} x_0\\ x_0'\end{pmatrix}. \tag{12}$$

4 Transfer matrix for the electrical field solution

It is obvious that the transfer matrix method is much easier than solving the differential equation. So the electrical field problem should also have the same kind of transfer matrix solution as particle transport in a quadrupole channel.

Without the assumption of a small perturbation in both the field and the local frequency, Eq. (2) can be written as:

$$\frac{\mathrm{d}^2E_z}{\mathrm{d}z^2}-\frac{4\pi^2\delta f}{c^2}[2f_0-\delta f]E_z=0\,. \tag{13}$$

Here δf is a function of z. We define a coefficient $R_E(z)$:

$$R_E(z)=\begin{cases} -k^2=-\dfrac{4\pi^2\delta f}{c^2}[2f_0-\delta f]= \\ \qquad -\left|\dfrac{4\pi^2\delta f}{c^2}[2f_0-\delta f]\right|, \quad (\delta f>0)\\ k^2= \quad -\dfrac{4\pi^2\delta f_0}{c^2}[2f_0-\delta f]= \\ \qquad \left|\dfrac{4\pi^2\delta f}{c^2}[2f_0-\delta f]\right|, \quad (\delta f<0)\\ 0, \qquad (\delta f=0)\end{cases}, \tag{14}$$

with the parameter $k=\left|\dfrac{4\pi^2\delta f}{c^2}[2f_0-\delta f]\right|^{1/2}$.

It represents a RF cavity composed of multi-cells which have different frequency shifts due to various errors. Comparing with the particle transverse movement equation, the coefficient here is also piecewise constant. Consequently, the equation of the electrical field has a transfer matrix solution.

Corresponding to the positive, negative and zero coefficient, the particle transverse movement has a focusing matrix, a defocusing matrix and a drift matrix, respectively, while in the perturbation of the electrical field these correspond to a frequency down shift, a frequency up shift and to no change.

1) Transfer matrix of the "frequency down shift" section:

The electric field perturbation equation can be written as follows:

$$\begin{pmatrix} E\\ E'\end{pmatrix}=M(z)\begin{pmatrix} E_0\\ E_0'\end{pmatrix}. \tag{15}$$

Here the suffix 0 stands for the initial electrical field and

$$M(z)=\begin{bmatrix}\cos(kz) & \dfrac{1}{k}\sin(kz)\\ -k\sin(kz) & \cos(kz)\end{bmatrix}. \tag{16}$$

2) Transfer matrix of the "frequency up shift" section:

$$M(z)=\begin{bmatrix}\mathrm{ch}(kz) & \dfrac{1}{k}\mathrm{sh}(kz)\\ k\mathrm{sh}(kz) & \mathrm{ch}(kz)\end{bmatrix}. \tag{17}$$

3) Transfer matrix of the "frequency unchanged" section:

$$M_{\mathrm{O}}(z)=\begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}. \tag{18}$$

For a cavity with multi-cells, the electrical field distribution in the N-th cell can be calculated by multiplying the transfer matrices of the N sections as follows:

$$\begin{pmatrix}E\\ E'\end{pmatrix}=M(z)\begin{pmatrix}E_0\\ E_0'\end{pmatrix}=M_N(z)\prod_{j=N-1}^{j=1}M_j(L_j)\cdot\begin{pmatrix}E_0\\ E_0'\end{pmatrix}. \tag{19}$$

Successive calculation for each cell leads to the whole distribution of the field in the cavity.

Apart from the formal similarity between the equations describing the movement of a particle in a quadrupole channel and the field perturbation in a RF cavity, we should notice that there is an obvious difference between them. For the particle transfer the initial phase space coordinates x_0, x_0' and the focusing coefficient k_q are given parameters. But the three constants E_0, E_0' and k are to be determined in the case of a field perturbation in a cavity. From the resonant boundary condition we have

$$E_0'=0\,,\quad E'(L_T)=0\,. \tag{20}$$

This means that the transfer element of the total matrix for the cavity is $M_{21}(L_T)=0$. This relation can be solved for k and the resonant frequency f_0 be determined.

To determine the initial field E_0 we assume the condition that the stored energy in a cavity is a constant, independent of the perturbation. So the integration of E^2 along the cavity should be $L_TE_0^2$:

$$\int_0^{L_T}E^2\mathrm{d}z=L_TE_0^2\,. \tag{21}$$

Here E_0 stands for the initial field amplitude without perturbation, which is a given parameter.

5 Two examples

In this section two simple examples are given to verify the transfer matrix method. The first example shows the calculation of the electric field for a defined frequency perturbation. The second example shows how to apply the method for tuning a required electric field using a frequency perturbation.

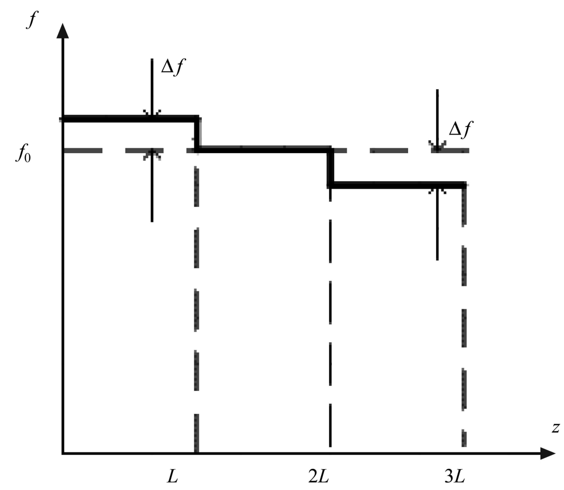

Fig. 1. The simplest frequency shift approximation: 3 sections with the same length.

Example 1 A given frequency shift distribution as shown in Fig. 1 can be expressed as:

$$f=\begin{cases}f_0+\Delta f & 0\leqslant z\leqslant L\\ f_0 & L<z\leqslant 2L\\ f_0-\Delta f & 2L<z\leqslant 3L\end{cases}. \tag{22}$$

The transfer matrix for the whole cavity can be obtained by:

$$M\ =\ M_3\cdot M_2\cdot M_1=\begin{bmatrix}\cos(kL) & \dfrac{1}{k}\sin(kL)\\ -k\sin(kL) & \cos(kL)\end{bmatrix}\cdot$$

$$\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\mathrm{ch}(kL) & \dfrac{1}{k}\mathrm{sh}(kL)\\ k\mathrm{sh}(kL) & \mathrm{ch}(kL)\end{bmatrix}. \tag{23}$$

Then the electrical field can be written as:

$$\begin{pmatrix}E\\ E'\end{pmatrix}_f=M\begin{pmatrix}E\\ E'\end{pmatrix}_i=M_3M_2M_1\begin{pmatrix}E\\ E'\end{pmatrix}_i. \tag{24}$$

Now we introduce two restrictions:

a) The electric field is fixed at the discrete points as calculated with the transfer matrix. We define a linear change of the field between each 2 adjacent points, which is similar to the case of transverse particle motion.

b) The stored energy in the cavity is unchanged. This means $\int_0^{L_T}E^2\mathrm{d}l=$constant, which is used to calculate E_0. The other initial parameter E_0' is defined as 0 according Eq. (20).

For the given parameters $f_0 = 324$ MHz, $\Delta f = 1$ MHz, $L = 1$ m, we get the following transfer matrix:

$$M_3 = \begin{bmatrix} \cos(kL) & \frac{1}{k}\sin(kL) \\ -k\sin(kL) & \cos(kL) \end{bmatrix} = \begin{bmatrix} 0.8616 & 0.9534 \\ -0.2702 & 0.8616 \end{bmatrix}, \tag{25}$$

$$M_2 = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{26}$$

$$M_1 = \begin{bmatrix} \mathrm{ch}(kL) & \frac{1}{k}\mathrm{sh}(kL) \\ k\mathrm{sh}(kL) & \mathrm{ch}(kL) \end{bmatrix} = \begin{bmatrix} 1.145 & 1.0479 \\ 0.3081 & 1.145 \end{bmatrix}. \tag{27}$$

Then the electric field can be calculated as follows:

$$\begin{aligned} &E_T = M_3 \cdot M_2 \cdot M_1 \cdot (E_0, E_0')^{\mathrm{T}} \\ &(E_1\ E_2\ E_T) = (1.145\ 1.4531\ 1.5457)\,E_0\,, \end{aligned} \tag{28}$$

which finally leads to

$$E_T = \begin{cases} (1+0.145x)E_0 & 0 \leqslant x < 1 \\ (0.8369+0.3081x)E_0 & 1 \leqslant x < 2 \\ (1.2679+0.0926x)E_0 & 2 \leqslant x \leqslant 3 \end{cases}. \tag{29}$$

Since the integration of E^2 along the cavity is a constant, we get for the electric field with perturbation:

$$\int_0^{3L} E_T^2 \mathrm{d}x = 1^2 \cdot (3L). \tag{30}$$

From Eqs. (29) and (30) we then obtain $E_0 = 0.77$. This means that the electric field with perturbation is reduced to 77% of the field without perturbation.

The electrical field along the cavity is shown in Fig. 2.

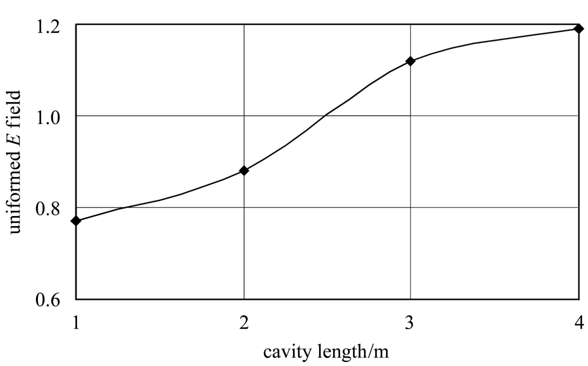

Fig. 2. The electric field with perturbation obtained by the transfer matrix.

A higher number of sections will increase the precision of the calculated field distribution.

Example 2 In Fig. 3 the electric field distribution designed for the DTL tank-1 in CSNS is shown. But the mechanical structure generated by PARMILA can only provide a linear electric field distribution, which is shown in Fig. 4. Here we will use the transfer matrix method to calculate the required frequency perturbation which can change the electric field such that it coincides with the designed one.

First we will introduce a frequency shift as a perturbation. Anticipating it is better to change the frequency smoothly, we select the first 3 cells and cell no.22, and 23&24 as the 2 perturbation sections[5].The 2 sections have the length 0.228 m and 0.336 m. The stored energy is obtained from the electric field shown in Fig. 3.

$$\int_0^8 E_T^2 \mathrm{d}x = \int_0^{2.11} (2.2+0.4265x)^2 \mathrm{d}x + 3.1^2 \times 5.86 = 71.274\,. \tag{31}$$

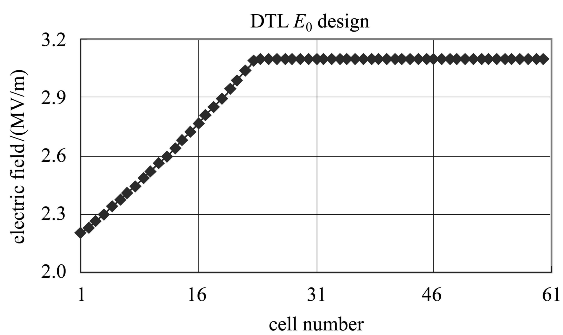

Fig. 3. The designed DTL electric field distribution in CSNS.

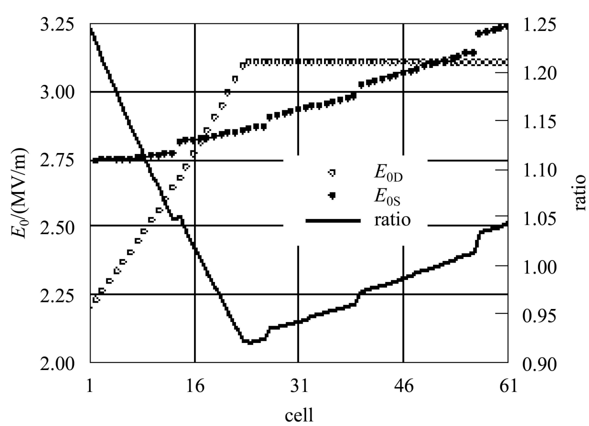

Fig. 4. The electric field obtained from structure generated by PARMILA.

Now the cavity is separated into 4 sections: from 0 to 0.228 m (frequency changing part), from 0.228 to 1.888 m (drift part); from 1.888 to 2.224 m (another frequency changing part); and from 2.224 to 7.988 m (drift part). If we get the transfer matrix of each part, the electric field along the cavity will be fixed from them. (The geometry data used here are taken from the DTL tank-1 design in CSNS.)

Comparing with the linear distributed electric field, we see that one needs to "defocus" the initial electric field to get a tilt and "focus" the field on the changing point. So one introduces the frequency shift f_1 and $-f_2$ (in MHz) separately in the 2 sections.

Now the transfer matrix along the cavity can be written as follows:

$$M_1=\begin{bmatrix}\mathrm{ch}(0.5331\sqrt{f_1}\times0.228) & \mathrm{sh}(0.12155\sqrt{f_1})/0.5331\sqrt{f_1}\\ \mathrm{sh}(0.12155\sqrt{f_1})\times0.5331\sqrt{f_1} & \mathrm{ch}(0.5331\sqrt{f_1}\times0.228)\end{bmatrix},\quad M_2=\begin{bmatrix}1 & 1.66\\ 0 & 1\end{bmatrix},$$

$$M_3=\begin{bmatrix}\cos(0.5331\sqrt{f_2}\times0.336) & \sin(0.1791\sqrt{f_2})/0.5331\sqrt{f_2}\\ -\sin(0.1791\sqrt{f_2})\times0.5331\sqrt{f_2} & \cos(0.5331\sqrt{f_2}\times0.336)\end{bmatrix},\quad M_4=\begin{bmatrix}1 & 5.764\\ 0 & 1\end{bmatrix}.$$

The required tilt is 0.4265 MV/m (see Fig. 3), while the designed initial electric field is E_0 = 2.2 MV/m. So the frequency shift f_1 is obtained as 2.985 MHz and the matrix M_1 is then given by:

$$M_1=\begin{bmatrix}1.022 & 0.2297\\ 0.1940 & 1.022\end{bmatrix}.$$

The frequency shift $-f_2$ should "focus" the electric field, i.e. it must change the tilt from 0.4265 to 0. The electric field E_2 after the first drift section is calculated from the matrices M_1 and M_2. From this we get the frequency shift $f_2=1.488$ MHz. The matrix M_3 is then given by:

$$M_3=\begin{bmatrix}0.9762 & 0.3338\\ -0.1409 & 0.9762\end{bmatrix},$$

By now we have obtained all transfer matrices and the electric field can be calculated as follows:

$$\begin{pmatrix}E_T\\ E'_T\end{pmatrix}=\prod_4^1 M_i\cdot\begin{pmatrix}1\\ 0\end{pmatrix}E_0\,.$$

$$(E_1\ E_2\ E_3\ E_4)=(1.022\ 1.344\ 1.377\ 1.377)\,E_0\,.$$

Using the condition of Eq. (31), we obtain the initial electric field $E_0=2.2485$ MHz. The electric field distribution is depicted in Fig. 5.

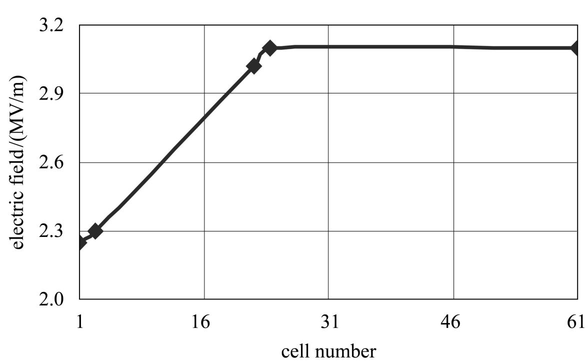

Fig. 5. The electric field obtained by the transfer matrix method.

The result is also simulated with the MDTFISH code to check our answer and is shown in Fig. 6. The result shows that the electric field in the first two sections is exactly equal to the designed one. But in the 4th drift section it is affected by a phase change. From Fig. 6 we see that the perturbation from the phase change can also be compensated for by the transfer matrix method if necessary.

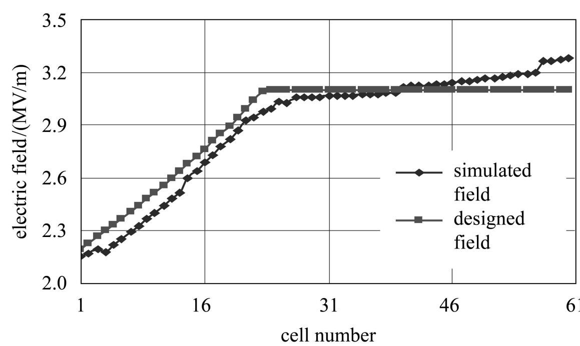

Fig. 6. The electric field simulated with the MDTFISH code compared with the designed field.

Letting the first perturbation f_1 be unchanged and moving the second perturbation $-f_2$ 4 m far away to the cells 39, 40&41, keeps the tilt unchanged. The initial value E_0 of 1.58 MV/m is obtained from the stored energy of 71.27. Then the peak electric field is 3.38 MV/m with a tilt of 0.45. In order to compensate for the tilt to zero, a second perturbation of −1.1 MHz is needed. Using these parameters as input for the MDTFISH code, we get the electric field distribution shown in Fig. 7. The simulation greatly confirms the results calculated with the transfer matrix method.

The two examples show that the transfer matrix method is consistent with the simulation results. On the other hand, if we measure and obtain a deformed RF electric field, the transfer matrix method can also be used to calculate the frequency shift required to modify the field such that it coincides with the designed one.

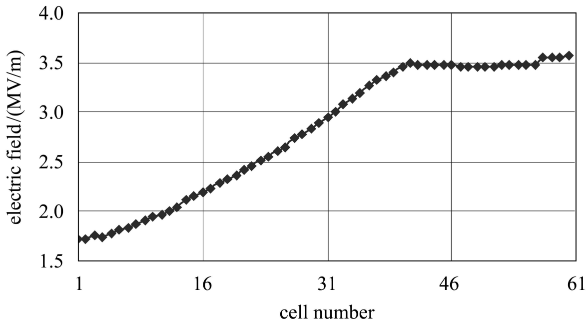

Fig. 7. Moving the perturbation$-f_2$ backwards. The electric field is generated by MDT-FISH.

6 Deviation analysis

From Fig. 6 we see that the simulated field is close to the designed one, except for the cells located between those having different phase angles. Applying the matrix method with a more detailed modification, one can obtain the improved result shown in Fig. 8. The ratio of the simulated field to the designed one is shown in Fig. 9, having a deviation of no more than 3%.

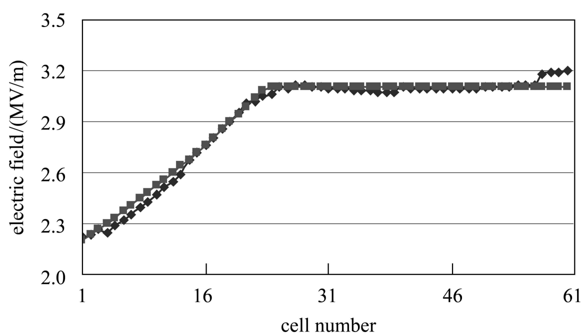

Fig. 8. Simulation result after applying the matrix method twice.

From the deviation analysis we see that the simulation result is consistent with the design. This will greatly decrease the difficulty of the field tuning after mechanical fabrication. From the technique report of SNS and JPARC we see that the traditional radio frequency design has not the step of the simulation-based modification. The measured field has a deviation of 50% in the entrance of DTL, which is shown in Fig. 10. In the case of a more complicated axial radio frequency design such as in CSNS, it will be easier to operate the radio frequency electric field tuning by applying the modification by the matrix method.

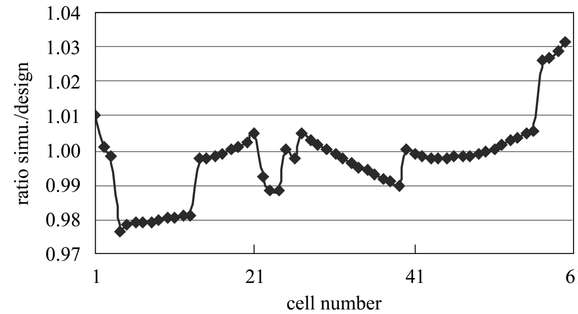

Fig. 9. Ratio of the simulated field to the designed one.

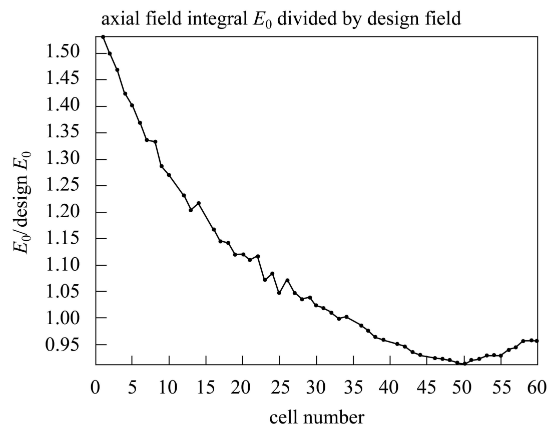

Fig. 10. Measured field (normalized) in SNS[6]. (With slug tuners penetrating to identical depths, no post couplers).

7 Conclusions

The equations for the electric field distribution in a radio frequency cavity have the same form as the equations of motion of a particle in a quadupole. So the transfer matrix can be used in both cases. With the matrix solutions, the small Δf approximation is not necessarily supposed, as is done in the commonly used Slater's theorem. So comparing with the solutions of the differential equations, the transfer matrix method is more precise and can handle more complex problems, especially when modifying the cavity to fit a special electric field distribution.

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