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Quantal symmetries in the non-linear σ model with Maxwell and non-Abelian Chern-Simons terms

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Abstract The quantal symmetry property of the CP^1 nonlinear σ model with Maxwell non-Abelian Chern-Simons terms in (2+1) dimension is studied. In the Coulomb gauge, the system is quantized by using the Faddeev-Senjanovic (FS) path-integral formalism. Based on the quantaum Noether theorem, the quantal conserved angular momentum is derived and the fractional spin at the quantum level in this system is presented.

Key words non-linear sigma model, non-Abelian Chern-Simons, fractional spin

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1 Introduction

The non-linear sigma model, which was introduced by Schwinger [1] is widely used in statistical mechanics and quantum field theory. It effectively describes the asymptoticly free theroy, and has nontrivial relations with Yang-Mills gauge theory [2–4]. The theoretically possibile particles, namely anyons, exhibit the properties of fractional spin and statistics in 2+1 dimensions. Anyons model was studied intensively in expectation to explain the fractional quantum Hall effect and high-T_c superconductivity [5, 6]. The $O(3)$ nonlinear sigma model with Hopf and Chern-Simons (CS) terms was studied, because fractional spin and statistics also occur in such a kind of model [7, 8]. The CP^1 nonlinear sigma model, as a low energy effective model for vortices, has applications in ferromagnets physics. It is also a toy-model displaying many important features of gauge field theories, which is intimately related to the $O(3)$ nonlinear sigma model in the long-range limit. A lot of recent works on (2+1)-dimensional Abelian Chern-Simons gauge theories revealed the existence of fractional spin property [9, 10]. The classical angular momentum for non-Abelian Chern-Simons was discussed, and it was pointed out that the non-Abelian Chern-Simons term in certain models could change the property of fractional statistics. The spin property of the CP^1 non-linear sigma model at the quantum level has also been studied [11–13]. In our paper, the quantal symmetry properties of the CP^1 non-linear sigma models in a non-Abelian case are studied. According to the path integral quantization rule for the constrained Hamilton system，we first quantize this system in the Faddeev-Senjanovic scheme. Based on the quantal Noether theorem, the conserved angular momentum has been calculated at the quantum level, and the fractional spin property of this system is presented.

2 Faddeev-Senjanovic (FS) path-integral quantization of $O(3)$ σ model

In (2+1)-dimensions, the lagrangian density of the CP^1 nonlinear σ model with the Maxwell term and non-Abelian Chern-Simons term is given by [7, 8]

$$\mathcal{L}=\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a+(D_\mu Z^a_k)^*(D^\mu Z^a_k)+\frac{\kappa}{4}\varepsilon^{\mu\nu\rho}(\partial_\mu A^a_\nu A^a_\rho+\frac{1}{3}\varepsilon^{abc}A^a_\mu A^b_\nu A^c_\rho)\ , \tag{1}$$

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where $D_\mu = \partial_\mu - \mathrm{i}T^a A^a_\mu, T^a$ are generators of non-abelian gauge groups $O(3)$, satisfying $[T^a, T^b] = \mathrm{i}\varepsilon^{abc} T^c$, $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. The gauge field strength is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$. ε^{abc} and $\varepsilon^{\mu\nu\rho}$ are the totally anti-symmetric Levi-Civita tensor. In conventional Latin symbols $a, b, c = 1, 2, 3$. Greek symbol $\mu, \nu, \rho = 0, 1, 2$. κ is a Chern-Simons coefficient. Coupling constant is assumed to be unity, and Z^a_k are complex fields which satisfy:

$$Z^a_k Z^{a*}_k = |Z^a_1|^2 + |Z^a_2|^2 = 1 . \tag{2}$$

The canonical momentums conjugate to the fields A^a_μ, Z^a_k, $Z^{a*}_k$are defined as

$$\pi^a_0 = 0, \tag{3a}$$

$$\pi^a_i = F^a_{0i} + \frac{\kappa}{4}\varepsilon_{ij} A^a_j , \tag{3b}$$

$$\pi^a_k = (D_0 Z^a_k)^* , \tag{3c}$$

$$\pi^{a*}_k = D_0 Z^a_k , \tag{3d}$$

respectively. where $\varepsilon^{ij} = \varepsilon^{0ij}$ in shorthand. The primary constraints of the system are

$$\Lambda_1 = \pi^a_0 \approx 0,\ \theta_1 = Z^a_k Z^{a*}_k - 1 \approx 0 , \tag{4}$$

where symbol "≈" means weakly equality in Dirac sense. The canonical Hamiltonian density is given by

$$\mathcal{H}_c = \pi^a_i \dot{A}^a_i + \pi^a_k \dot{Z}^a_k + \pi^{a*}_k \dot{Z}^{a*}_k - \mathcal{L} = \mathcal{H}_0 - A^a_0[(\partial_i \pi^i + \frac{\kappa}{4}\varepsilon^{ij}\partial_i A_j) - J^a_0] , \tag{5}$$

with

$$\mathcal{H}_0 = \frac{1}{2}\pi^a_i \pi^a_i + \frac{1}{4}F^a_{ij}F^{ij}_a + \frac{\kappa^2}{16}A^a_j A^a_j + \pi^{a*}_k \pi^a_k + (D_i Z^a_k)^*(D^i Z^a_k) - \frac{\kappa}{4}\varepsilon^{ij}\pi^a_i A^a_j , \tag{6a}$$

$$J^a_0 = -\mathrm{i}\pi^c_k T^c_{ab} Z^b_k + \mathrm{i}\pi^{c*}_k T^c_{ab} Z^{b*}_k . \tag{6b}$$

The total Hamiltonian is written as

$$H_{\mathrm{T}} = \int \mathrm{d}^2x(\mathcal{H}_c + \lambda^a_1 \Lambda^a_1 + \mu^a_1 \theta^a_1) . \tag{7}$$

The consistency conditions $\dot{\Lambda}_1 = \{\Lambda_1, H_{\mathrm{T}}\} \approx 0$ and $\dot{\theta}_1 = \{\theta_1, H_{\mathrm{T}}\} \approx 0$ lead to secondary constraints

$$\Lambda_2 = J^0 - \partial_i \pi^i - \frac{\kappa}{4}\varepsilon^{ij}\partial_i A_j \approx 0 , \tag{8a}$$

$$\theta_2 = \pi^a_k Z^a_k + \pi^{a*}_k Z^{a*}_k \approx 0 , \tag{8b}$$

respectively, and no further constraints are generated by this iterative procedure. It is easy to check that the constraints Λ_1 and Λ_2 are first-class constraints, and θ_1 and θ_2 are second-class constraints.

According to the Faddeev-Senjanovic (FS) path-integral quantization scheme, for each first-class constraint, one must choose a gauge condition. We choose to work in Coulomb gauge $\Omega^a_1 = \partial_i A^a_i \approx 0$. The consistentence of Coulomb gauge requires, $\partial_i \dot{A}_i \approx \{\Omega_1, H_{\mathrm{T}}\} \approx 0$, this leads to another gauge constraint

$$\Omega^a_1 = \nabla^2 A^a_0 + \partial^i \pi^a_i - \varepsilon^{abc} A^b_i \partial^i A^c_0 \approx 0 . \tag{9}$$

The phase-space generating functional of Green function for the model (1) reads

$$Z[J,K] = \int \mathcal{D}\varphi^\alpha_a D\pi^a_\alpha \prod_{\mathrm{i}} \delta(\Lambda^a)\delta(\Omega^a)\delta(\theta^a_i) \times \det|\{\Lambda^a, \Omega^a\}| \cdot (\det|\{\theta^a_i, \theta^a_j\}|)^{1/2} \times \exp\{\mathrm{i}\int \mathrm{d}^3x(\pi^a_\alpha \dot{\varphi}^\alpha_a - \mathcal{H}_c + J^a_\alpha \varphi^\alpha_a + K^\alpha_a \pi^a_\alpha)\} , \tag{10}$$

where φ^α represents all fields, π_α are the canonical momenta conjugate to φ^α, and $J^\alpha, K^\alpha$are exterior sources with respect to φ^α, π_α respectively. Calculating the factors in (10), we get

$$\det|\{\theta^a_i, \theta^a_j\}| = 4(Z^a_k \cdot Z^{a*}_k)^2 ,$$
$$\det|\{\Lambda^a, \Omega^b| = \det M^{ab}\delta^{(2)}(x-y) ,$$
$$M^{ac} = (\delta^{ac}\nabla^2 - \varepsilon^{abc} A^b_i \partial^i)\delta(x-y) . \tag{11}$$

The factor $\det|\{\Lambda^a, \Omega^b\}|\delta(\partial^i A^a_i)$can be replaced by $\det M_{\mathrm{L}}\delta(\partial^\mu A^a_\mu)$.

Using the properties of δ-function and integral properties of Grassman variables, the phase-space generating functional of Green function for the model (1) can be written as [14]

$$Z[J^\alpha_a, K^a_\alpha, \bar{\xi}_a, \xi_a, U^l_a, V^n_a, W^i_a] = \int D\phi^a_\alpha D\pi^a_\alpha D\bar{C}^a \times DC^a D\lambda^a_l D\mu^a_n D\omega^a_i \exp\{\mathrm{i}\int \mathrm{d}^3x(\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}} + J^\alpha_a \phi^a_\alpha + K^a_\alpha \pi^a_\alpha + \bar{\xi}_a C^a + \bar{C}^a \xi_a + U^l_a \lambda^a_l + V^n_a \mu^a_n + W^i_a \omega^a_i)\}, \tag{12}$$

where

$$\mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} = \mathcal{L}^{\mathrm{p}} + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_{\mathrm{gh}} , \tag{13}$$

$$\mathcal{L}^{\mathrm{p}} = \pi^\mu \dot{A}_\mu + \pi^a_k \dot{Z}^a_k + \pi^{a*}_k \dot{Z}^{a*}_k + \bar{P}_a \dot{C}^a + \dot{\bar{C}}^a P_a - \mathcal{H}_c \tag{14}$$

$$\mathcal{L}_{\mathrm{m}} = \lambda^a_l \Lambda^a_l + \mu^a_n \Omega^a_n + \omega^a_i \theta^a_i , \tag{15}$$

$$\mathcal{L}_{\mathrm{gh}} = -\partial^\mu \bar{C}^a D^a_{b\mu} C^b , \tag{16}$$

λ_l, μ_n and ω_i are the multiplier fields, $\bar{C}^a$ and C^a are the ghost fields (Grassmann variables), and $\bar{P}_a$, P_a are the canonical momentums conjugate to $\bar{C}^a$ and C^a respectively.

3 Angular momentum and fractional spin term

We first formulate the results of the quantal canonical Noether theorem [15, 16]: If the effective canonical action $I^{\mathrm{P}}_{\mathrm{eff}}=\int \mathrm{d}^2x\,\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}}$ is invariant under the following global transformation in extended phase space

$$\begin{cases} x^{\mu'}=x^{\mu}+\Delta x^{\mu}=x^{\mu}+\varepsilon_{\sigma}\tau^{\mu\sigma}(x,\varphi,\pi) \\ \varphi'(x')=\varphi(x)+\Delta\varphi(x)=\varphi(x)+\varepsilon_{\sigma}\xi^{\sigma}(x,\varphi,\pi) \\ \pi'(x')=\pi(x)+\Delta\pi(x)=\pi(x)+\varepsilon_{\sigma}\eta^{\sigma}(x,\varphi,\pi) \end{cases}. \quad (17)$$

In the model to be discussed, φ and π denote: $\varphi=(Z^a_k,Z^{a*}_k,A_{\mu},\lambda_l,\mu_n,\omega_i)$, $\pi=(\pi^a,\pi^{\mu},\overline{P}^aP^a)$, and $\varepsilon_{\sigma}(\sigma=1,2,\cdots,r)$ are infinitesimal arbitrary parameters, $\tau^{\mu\sigma},\xi^{\sigma},\eta^{\sigma}$ are some smoothed functions of canonical variables and time. If the Jacobian of the transformation (17) is equal to unity, according to the canonical Noether theorem in quantum formalism, there are conserved quantities at the quantum level [15, 16]

$$Q^{\sigma}=\int_V \mathrm{d}^2x[\pi(\xi^{\sigma}-\varphi_{,k}\tau^{k\sigma})-\mathcal{H}_{\mathrm{eff}}\tau^{0\sigma}]=\text{const},\ \sigma=(1,2,\cdots,r) \quad (18)$$

where $\mathcal{H}_{\mathrm{eff}}$ is an effective Hamiltonian density corresponding to $\mathcal{L}^{\mathrm{P}}_{\mathrm{eff}}$. Now, we consider the spatial rotation, $\tau^{0\sigma}=0$, and the Jacobian is equal to unity. The $\mathcal{L}_{\mathrm{gh}}$ term does not involve the time derivative of field variables. Thus, using (18) the quantal conserved angular momentum for this system is given by

$$L=\int \mathrm{d}^2x\varepsilon^{ij}[x_i\pi^a_k\partial_j Z^a_k+x_i\pi^{a*}_k\partial_j Z^{a*}_k+(\pi_{a\mu}S^{\mu\nu}_{ij}A^a_{\nu}+x_i\pi^{\mu}_a\partial_j A^a_{\mu})]+x_i\bar{P}_a\partial_j C^a+x_i\bar{C}^a\partial_j P_a], \quad (19)$$

where $S^{kl}_{ij}=\delta^k_i\delta^l_j-\delta^k_j\delta^l_i$. Substituting (3b) into (19), using the relations $\varepsilon^{jk}\varepsilon_{il}=\delta^j_i\delta^k_l-\delta^j_l\delta^k_i$, this Eq. (19) can be simplified to

$$L=\int \mathrm{d}^2x\varepsilon^{ij}(x_i\pi^a_k\partial_j Z^a_k+x_i\pi^{a*}_k\partial_j Z^{a*}_k)+\varepsilon^{ij}[x_i\bar{P}_a\partial_j C^a+x_i\bar{C}^a\partial_j P_a]+\int \mathrm{d}^2x\varepsilon^{ij}[F_{k0}S^{kl}_{ij}A_l+x_iF^{k0}\partial_j A_k]+\kappa\int \mathrm{d}^2x[\varepsilon^{ij}x_iA^a_j(\varepsilon^{lk}\partial_l A^a_k)]. \quad (20)$$

The Lagrange equation of motion for A^a_{μ} is given by

$$-\partial_{\mu}F^{a\mu\nu}+\frac{1}{2}\varepsilon^{abc}F^{\mu\nu}_bA^{\mu}_c+\frac{\kappa}{4}\varepsilon^{\nu\mu\lambda}\times(2\partial_{\mu}A^a_{\lambda}+\varepsilon^{abc}A^b_{\nu}A^c_{\lambda})=J'^{a\nu}. \quad (21)$$

In Coulomb gauge $\partial_iA^a_i=0$, and seting $\nu=0$, (21) reads

$$-\partial_iF^{ai0}+\frac{1}{2}\varepsilon^{abc}F^{i0}_bA^i_c+\frac{\kappa}{4}\varepsilon^{ij}(2\partial_iA^a_j+\varepsilon^{abc}A^b_iA^c_j)=J'^{a0}. \quad (22)$$

The asymptotic form of the non-abelian vortex configuration is structurally identical to [17]

$$A^a_i(x)=-\frac{2Q'^a}{\pi\kappa}\varepsilon_{ij}\frac{x^j}{x^2}, \quad (23)$$

where $Q'^a=\int \mathrm{d}^2xj'^a_0(x)$ is the non-Abelian charge. The Eq. (20) is reduced to

$$L=\int \mathrm{d}^2x\varepsilon^{ij}(x_i\pi_a\partial_j Z^a_k+x_i\pi_a\partial_j Z^{a*}_k)+\int \mathrm{d}^2x\varepsilon^{ij}[F_{k0}S^{kl}_{ij}A_l+x_iF^{k0}\partial_j A_k]+\int \mathrm{d}^2x\varepsilon^{ij}[x_i\bar{P}_a\partial_j C^a+x_i\bar{C}^a\partial_j P_a]+\frac{2(Q'^a)^2}{\pi\kappa}. \quad (24)$$

Then we can find that the angular momentum has a anomalous term. The first two terms appearing on the right hand side of (24) contains both the orbital angular momentum which generates rotation in the ordinary space and spin term which generates rotations in the internal space. A notibale term is the Ghost term due to the quantization. The last term is independent of the origin of coordinates, and interpreted as a spin operator [18]. It is different with the abelian case, because it has non-abelian group index. So anyons still survive in our model.

4 Conclusion and discussion

The property of fractional spin is presented in the CP^1 nonolinear σ model with non-abelian Chern-Simons term at the quantum level. The angular momentum can take any arbitrary value which is determined by the CS parameter κ. Here we first quantize this system in the Faddeev-Senjanovic path-integral formalism. Then we rigously calculated the angular momentum based on the quantal Noether theorem. Fractional spin term is contained in the total angular momentum. The additional Maxwell kinetic term in the model does not change the property of fractional spin. But the non-abelian Chern-Simons term dominates in 2+1 dimension. Without this term the anomaly spin will not occur [18].

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