On the Next-leading order Heavy-Quark Potential from AdS/CFT*

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Abstract Applying the AdS/CFT correspondence, the expansion of the heavy-quark potential of $\mathcal{N}=4$ supersymmetric Yang-Mills theory at large N_c is carried out to the next-leading term in the large 't Hooft coupling at zero temperature. The strong coupling corresponds to the semi-classical expansion of the string-sigma model, the gravity dual of the Wilson loop operator, with the next-leading term expressed in terms of functional determinants of fluctuations. The singularities of these determinants are examined and their contributions are evaluated numerically. We find the next-leading order correction is negative and suppressed by minus square root of the 't Hooft coupling relative to the leading order.

Key words holographic QCD, heavy quarkonium

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1 Introduction

AdS/CFT duality [1–4] remains an active field of research. Motivated by the isomorphism between the isometry group of AdS_5 and the conformal group in four dimensions, it was conjectured by Maldacena that a string theory in $\text{AdS}_5\times S^5$ corresponds to a four dimensional conformal field theory on the boundary. A prominent implication of the conjecture is the correspondence between the type IIB superstring theory formulated on $\text{AdS}_5\times S^5$ and $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) with the isometry group $O(6)$ of S^5 dual to the R-symmetriy group $SU(4)$ of SYM. In particular, the supergravity limit of the string theory corresponds to the leading behavior of SYM at large N_c and large 't Hooft coupling

$$\lambda\equiv g^2_{\text{YM}}N_c=\frac{L^4}{\alpha'^2}. \tag{1}$$

with L the AdS radius and α' the reciprocal of the string tension. This relation thereby opens a new avenue to explore the strong coupling properties of SYM and sheds new lights on strongly coupled QGP created in RHIC in spite of the difference between SYM and QCD. Among notable successes on the RHIC phenomenology are the equation of state, the viscosity ratio [5] and jet quenching parameters and the energy loss [6].

The heavy quark potential (the potential energy between a heavy quark and its anti-particle) of QCD is an important quantity that probes the confinement mechanism in the hadronic phase and the meson melting in the plasma phase. It is extracted from the expectation of a Wilson loop operator, which can be measured on a lattice. In the case of $\mathcal{N}=4$ SYM, the AdS/CFT duality relates the Wilson loop expectation value to the path integral of the string-sigma action developed in Ref. [7] for the worldsheet in the $\text{AdS}_5\times S^5$ bulk spanned by the loop on the boundary. To the leading order of strong coupling, the path integral is given by its classical limit, which is the minimum area of the world sheet. From the Wilson loop of a pair of parallel lines, Maldacena extracted the potential function in $\mathcal{N}=4$ SYM at zero temperature [8],

$$V(r)=-\frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)}\frac{\sqrt{\lambda}}{r}\simeq-0.2285\frac{\sqrt{\lambda}}{r}, \tag{2}$$

with r the distance between the quark and the an-

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tiquark. Introducing a black hole in AdS bulk, the potential at nonzero temperature as well as that for moving quarks have been obtained by a number of authors [9, 10]. The field theoretic aspects of the potential (2) and its finite temperature counterpart as well as their implications on RHIC physics were discussed in Ref. [10–12]. As was pointed out in Ref. [8], the "heavy quarks" underlying the Wilson loop (2) in $\mathcal{N}=4$ SYM are actually heavy W bosons resulted in a Higgs mechanism, which implement the fundamental representation of $SU(N_c)$. Since the function (2) measures the force between two static fundamental color objects, we shall borrow the terminology of QCD by naming it the heavy quark potential throughout this paper.

The strong coupling expansion of the SYM Wilson loop corresponds to the semi-classical expansion of the string-sigma action and reads

$$V(r)=-\frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)}\frac{\sqrt{\lambda}}{r}\left[1+\frac{\kappa}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)\right] \quad (3)$$

for the heavy quark potential. Computing the coefficient κ is the main subject of the present report [13]. κ comes from the one loop effective action of the world sheet fluctuations around its minimum area. This effective action has been obtained explicitly for some simple Wilson loops including parallel lines [14, 15] and is expressed in terms of functional determinants. Evaluating these determinants, we end up with the numerical value of κ,

$$\kappa\simeq-1.33460. \quad (4)$$

The classical solution of the string-sigma model and the one loop effective action underlying κ is briefly reviewed in the next section. There we also outline our strategy of computation, which is along the line suggested in [15]. We parametrize the string world sheet of the single Wilson line or parallel lines by conformal coordinates. Then a scaling transformation is made that leaves the measure of the spectral problem of the functional determinants trivial. Instead of solving the eigenvalue problem of the operators underlying the determinants, we use the method employed in [16], which amounts to solve a set of ordinary differential equations. Unlike the straight Wilson line and the circular Wilson loop dealt with in [16], some of differential equations for the parallel lines are not analytically tractable. The presence of various singularities makes numerical works highly nontrivial. It is critical to isolate the singularities analytically in order to obtain a robust numerical result.

2 The one-loop effective action

Let us begin with a brief review of the classical limit that leads to the leading order potential (2). The string-sigma action in this limit reduces to the Nambu-Goto action

$$S_{\rm NG}=\frac{1}{2\pi\alpha'}\int{\rm d}^2\sigma\sqrt{g}, \quad (5)$$

with g the determinant of the induced metric on the string world sheet embedded in the target space, i.e.

$$g_{\alpha\beta}=G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}, \quad (6)$$

where X^μ and $G_{\mu\nu}$ are the target space coordinates and the metric, and σ^α with $(\alpha=0,1)$ parametrize the world sheet. The target space here is $\mathrm{AdS}_5\times S^5$, whose metric may be written as

$${\rm d}s^2=\frac{1}{z^2}({\rm d}t^2+{\rm d}\vec{x}^2+{\rm d}z^2)+{\rm d}\Omega_5^2, \quad (7)$$

with ${\rm d}\Omega_5$ the element of the solid angle of S^5. The physical 3-brane resides on the AdS boundary $z=0$. The string world sheets considered in this paper are all projected onto a point of S^5 in the classical limit. The Wilson loop of a static heavy quark, denoted by $\mathcal{C}_1$, is a straight line winding up the Euclidean time t periodically at the AdS boundary. The corresponding world sheet in the AdS bulk can be parametrized by t and z with $\vec{x}$ constant and extends all the way to AdS horizon, $z\to\infty$. The induced metric is that of AdS_2, given by

$${\rm d}s^2[\mathcal{C}_1]=\frac{1}{z^2}({\rm d}t^2+{\rm d}z^2), \quad (8)$$

with the scalar curvature

$$R=-2. \quad (9)$$

Substituting the metric (8) into (5), we find the self-energy of the heavy quark

$$E[\mathcal{C}_1]=\frac{1}{T}S_{\rm NG}[\mathcal{C}_1]=\frac{1}{2\pi\alpha'}\int_\delta^\infty\frac{{\rm d}z}{z^2}. \quad (10)$$

with $T\to\infty$ the time period. Notice that we have pulled the physical brane slightly off the boundary to the radial coordinate $z=\delta$, as a regularization of the divergence pertaining the lower limit of the integral (10).

The total energy of a pair of a heavy quark and a heavy antiquark separated by a distance r, can be extracted from the Wilson loop consisting of two parallel lines each winding up the Euclidean time at the boundary. This Wilson loop will be denoted by $\mathcal{C}_2$ and the world sheet in the bulk can be parametrized by t and z with $x^1=\xi(z)$ and $x^2,x^3=\text{const.}$. The function $\xi(z)$ is determined by substituting the induced metric

$$\mathrm{d}s^2[\mathcal{C}_2]=\frac{1}{z^2}\left\{\mathrm{d}t^2+\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^2+1\right]\mathrm{d}z^2\right\}, \tag{11}$$

into the action (5) and minimizing it. We have

$$\xi=\pm\int_z^{z_0}\mathrm{d}z'\frac{z'^2}{\sqrt{z_0^4-z'^4}}. \tag{12}$$

The maximum bulk extension of the world sheet, z_0, is determined by the distance r between the two lines at the boundary and we find that

$$z_0=\frac{\Gamma^2\left(\frac{1}{4}\right)}{(2\pi)^{\frac{3}{2}}}r. \tag{13}$$

Substituting (12) into (11), we end up with the induced metric

$$\mathrm{d}s^2[\mathcal{C}_2]=\frac{1}{z^2}\left(\mathrm{d}t^2+\frac{z_0^4}{z_0^4-z^4}\mathrm{d}z^2\right), \tag{14}$$

and the scalar curvature

$$R=-2\left(1+\frac{z^4}{z_0^4}\right) \tag{15}$$

The energy of the heavy quark pair is therefore given by,

$$E[\mathcal{C}_2]=\frac{1}{T}S_{\mathrm{NG}}[\mathcal{C}_2]=\frac{1}{\pi\alpha'}z_0^2\int_\delta^{z_0}\frac{\mathrm{d}z}{z^2\sqrt{z_0^4-z^4}}, \tag{16}$$

where the same regularization is applied to the lower limit of the integral.

The heavy quark potential is obtained by subtracting from (16) the self energy of each quark(antiquark), i.e.

$$V=\lim_{\delta\to0^+}(E[\mathcal{C}_2]-2E[\mathcal{C}_1]) \tag{17}$$

and is divergence free. Carrying out the integral and substituting in the relations (13), we derive (2).

The one loop effective action, W is obtained by expanding the string-sigma action of Ref. [7] to the quadratic order of the fluctuating coordinates around the minimum area and carrying out the path integral [14, 15]. We have

$$W[\mathcal{C}_1]=-\ln\left[\frac{\det^4(-\mathrm{i}\gamma^\alpha\nabla_\alpha+\tau_3)}{\det^{\frac{3}{2}}(-\nabla^2+2)\det^{\frac{5}{2}}(-\nabla^2)}\right], \tag{18}$$

for the static quark or antiquark and

$$W[\mathcal{C}_2]=-\ln\left[\frac{\det^4(-\mathrm{i}\gamma^\alpha\nabla_\alpha+\tau_3)\det^{-\frac{5}{2}}(-\nabla^2)}{\det^{\frac{1}{2}}(-\nabla^2+4+R)\det(-\nabla^2+2)}\right], \tag{19}$$

for the quark pair.

The one loop correction to the heavy quark potential is then

$$\Delta V=\lim_{T\to\infty}\frac{1}{T}\lim_{\delta\to0^+}(W[\mathcal{C}_2]-2W[\mathcal{C}_1]). \tag{20}$$

The effective action $W[\mathcal{C}_1]$ or $W[\mathcal{C}_2]$ suffers from the usual logarithmic UV divergence, which is proportional to the volume part of the Euler character

$$\int_{z>\delta}\mathrm{d}t\mathrm{d}z\sqrt{g}R \tag{21}$$

of each world sheet with the same coefficient of proportionality [15]. Therefore the UV divergence as well as the conformal anomaly cancel in the combination of (20) in the limit $\delta\to0$. As a contrast, the volume integral $\int\mathrm{d}^2\sigma\sqrt{g}$ of the parallel lines differs from twice of that of a straight line by a finite quantity in the same limit. The UV divergence associated to the volume integral cancels within each effective action of (18) and (19).

3 Results and discussion

We start with the determinant ratio for a single Wilson line and that for parallel lines in the static gauge, in which the fluctuations come from eight transverse bosonic coordinates and eight 2d Majorana fermions. Then we scaled the operators underlying the determinants, leaving a trivial measure for the associated spectral problem. The subleading term of the heavy quark potential is extracted from the combination (20), which consists of the spectral and the measure parts. A robust numerical result of the former is obtained and the contributions from measure change of each determinant cancel [13]. We have,

$$V(r)=-\frac{a(\lambda)}{r}, \tag{22}$$

with

$$a(\lambda)=\begin{cases}\dfrac{4\pi^2\sqrt{\lambda}}{\Gamma^4\left(\frac{1}{4}\right)}\left[1-\dfrac{1.33460}{\sqrt{\lambda}}+O\left(\dfrac{1}{\lambda}\right)\right], & \text{for }\lambda\gg1\\[2ex] \dfrac{\lambda}{4\pi}\left[1-\dfrac{\lambda}{2\pi^2}\left(\ln\dfrac{2\pi}{\lambda}-\gamma_{\mathrm{E}}+1\right)+O(\lambda^2)\right], & \text{for }\lambda\ll1\end{cases}, \tag{23}$$

where the weak coupling expansion obtained in [11, 17] from field theory is also included for completeness. The authors of [11] also worked out the strong coupling expansion under the ladder approximation in field theory,

$$V_{\rm ladder}(r)=-\frac{\sqrt{\lambda}}{\pi r}\left(1-\frac{\pi}{\sqrt{\lambda}}\right). \tag{24}$$

It is interesting to notice that our subleading term is of the same sign as theirs but the magnitude relative to the leading order is smaller in our result. In view of the range of the 't Hooft coupling which was used for the RHIC phenomenology,

$$5.5<\lambda<6\pi, \tag{25}$$

the correction to the leading order of the strong coupling may be significant in magnitude. One may define an effective coupling

$$\sqrt{\lambda'}=\sqrt{\lambda}-1.33460. \tag{26}$$

If λ of (25) is replaced by λ', the range of the 't Hooft coupling is shifted to

$$13.54<\lambda<32.22. \tag{27}$$

At a nonzero temperature T, however, the order $O(\lambda^{-1/2})$ is not merely a redefinition of the coupling and the strong coupling expansion of the heavy quark potential becomes

$$V(r)\simeq-\frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)}\frac{\sqrt{\lambda}}{r}\left[g_0(rT)-\frac{1.33460g_1(rT)}{\sqrt{\lambda}}+O\left(\frac{1}{\lambda}\right)\right], \tag{28}$$

with $g_0(x)$ and $g_1(x)$ two functions satisfying the conditions $g_0(0)=g_1(0)=1$. The function $g_0(x)$ have been determined by the minimum area of the world sheet in the Schwarzschild-AdS$_5\times S^5$ target space [9]

$$ds^2=\frac{1}{z^2}\left(f(z)dt^2+d\vec{x}^2+\frac{1}{f(z)}dz^2\right)+d\Omega_5^2 \tag{29}$$

with $f(z)=1-\pi^4T^4z^4$ The one loop effective action underlying the function $g_1(x)$ has been developed in [18] and the methodology employed in this work can be readily generalized there.

As AdS/CFT has become an important reference to understand the observation of the strongly interacting quark-gluon plasma created by heavy ion collisions, it is critical to asses the robustness of the leading order prediction by exploring the next order correction in the expansion according to the inverse powers of the large 't Hooft coupling $\lambda=N_{\rm c}g_{\rm YM}^2$. The subleading terms of the expansion have been addressed in the literature in the context of the equation of state [19] and the shear viscosity [20]. This type of corrections comes from the α'^3 correction of the target space metric [21]. Its contribution is of the order $O(\lambda^{-3/2})$ relative to the leading order in the $\mathcal{N}=4$ SYM and is present only at nonzero temperature. In case of the expectation value of a Wilson loop operator, however, the dominant correction stems from the fluctuation of the world sheet around its minimum area and is suppressed only by $O(\lambda^{-1/2})$ relative to the leading order. It shows up at all temperatures and is more difficult to compute. The only attempts made in the literature in this regard include the strong coupling expansion of a single line, a circular loop and a spinning line at zero temperature [14–16, 22]. These Wilson loops, though theoretically important, do not carry direct phenomenological implications.

In this work, we have extended the method in [16] to the fluctuations of the world sheet dual to a pair of parallel Wilson lines and have derived the next term of the strong coupling expansion of the heavy quark-antiquark potential in $\mathcal{N}=4$ SYM at zero temperature.

While simple in practice, the static gauge we worked with suffers a problem. Though the combination (20) gives rise to a finite result, neither the UV divergence nor the conformal anomaly of each term on RHS of (20) vanishes. A less problematic gauge is the conformal gauge, in which the world sheet metric is not set to the induced metric at the beginning. One has to include the determinant of the longitudinal fluctuations and that of the ghost and an appropriate measure of the path integral. The contributions from the transverse bosons and fermions obtained in this paper will remain there, but other contributions including the measure change may be subtle to collect. It is important to carry out the parallel analysis in the conformal gauge to ascertain that our result in this paper is complete. Another alternative is the canonical quantization method employed in [22]. We hope to report our progress in this direction in near future.

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