Analytical transfer matrix of a quadrupole fringe*

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Abstract: The analytical linear transfer matrices for different quadrupole fringes including quadratic, high order power and exponential models are deduced in this paper. As an example, the transfer matrices of the quadrupole BEPCⅡ 105Q are computed for the above three models and compared with hard edge and slice-by-slice models in cases of near 60° and 90° FODO cells. These models' results are much better than the hard edge model's, and can meet the requirement of accurate calculation.

Key words: quadrupole, transfer matrix, particles

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1 Introduction

There are many quadrupole magnets in modern accelerators. They focus the charged particle beam, so we regard them as lenses. The linear transfer matrices of a thick lens quadrupole and a thin lens quadrupole are given in [1, 2]. However, these matrices are only used for delta fringe or so-called hard-edge magnets. There is an approximation linear map for a linear fringe in SAD [3]. Recently, there have been many studies about quadrupole fringes in high order maps [4, 5]. In this paper, we will deduce the analytical linear transfer matrix for quadratic, shifted higher order and exponential fringes. For the solenoid fringe [6], these solutions are all a hyper-geometric function. Finally, taking BEPCⅡ 105Q as an example, the transfer matrices are computed according to the 3 models and compared with hard edge and slice-by-slice models. The differences for all these models are shown also by the tune and twiss parameters in near 60° and 90° FODO cells.

2 Equation of motion

We used a Cartesian coordinates system. s is the longtitudinal coordinate, x and y are the transverse coordinates. For the charged particle, the Lorenz force is

$$\vec{F} = e\vec{v}\times\vec{B} = e\begin{vmatrix} \vec{x} & \vec{y} & \vec{s} \\ \dot{x} & \dot{y} & \dot{s} \\ B_x & B_y & B_s \end{vmatrix} = e[(\dot{y}B_s-\dot{s}B_y)\vec{x} +(\dot{s}B_x-\dot{x}B_s)\vec{y}+(\dot{x}B_y-\dot{y}B_x)\vec{s}]. \tag{1}$$

Here "$\cdot$" is to show the derivation of time "t".

If we neglect the radiation when a particle is accelerated, γ is a constant in a static magnetic field. Following $\vec{F}=\gamma m_0\vec{a}$, the equation of particle motion is

$$\begin{cases} \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = \dfrac{e(\dot{y}B_s-\dot{s}B_y)}{\gamma m_0} \\ \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = \dfrac{e(\dot{s}B_x-\dot{x}B_s)}{\gamma m_0} \end{cases}. \tag{2}$$

Here γ is the relativistic factor, and m_0 is the rest mass of the particle. In the accelerator, $\dot{s}\gg\dot{x}, \dot{s}\gg\dot{y}$, so the total momentum of the particle $P=\gamma m_0\dfrac{\mathrm{d}s}{\mathrm{d}t}$, and

$$\begin{cases} \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = \dfrac{e(\dot{y}\dot{s}B_s-(\dot{s})^2B_y)}{p} \\ \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} = \dfrac{e((\dot{s})^2B_x-\dot{x}\dot{s}B_s)}{p} \end{cases}. \tag{3}$$

Transforming the variable time t to position s, we can

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obtain

$$x'' = \frac{e(y'B_s - B_y)}{p}; \tag{4a}$$

$$y'' = \frac{e(B_x - y'B_s)}{p}. \tag{4b}$$

3 The expression of magnetic field in a quadrupole

We assume the expression of a quadrupole field is

$$B_x(x,y,s) = y\sum_{i=0}^{n} y^i \frac{i}{i+1} fy(i,s); \tag{5a}$$

$$B_y(x,y,s) = x\sum_{i=0}^{n} y^i fy(i,s); \tag{5b}$$

$$B_s(x,y,s) = xy\sum_{i=0}^{n} y^i \frac{1}{1+i}\frac{\partial fy(i,s)}{\partial s}. \tag{5c}$$

The field in (5) should satisfy the Maxwell Equations

$$\nabla\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_s}{\partial s} + \frac{\partial B_y}{\partial y} = 0; \tag{6a}$$

$$\nabla\times\vec{B} = \left(\frac{\partial B_y}{\partial s} - \frac{\partial B_s}{\partial y}\right)\hat{x} + \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right)\hat{s} + \left(\frac{\partial B_s}{\partial x} - \frac{\partial B_x}{\partial s}\right)\hat{y} = 0, \tag{6b}$$

and we have

$$fy(i,z) = \frac{-1}{i(i-1)}\frac{\partial^2(fy(i-2,z))}{i(i-1)}. \tag{7}$$

Let $fy(0,s) = f_{y0}(s)$, the field in a quadrupole can be expressed as

$$B_x = y\sum_{i=0}^{n}\frac{(-1)^i}{(2i+1)!} y^{2i} f_{y0}^{(2i)}(s); \tag{8a}$$

$$B_y = x\sum_{i=0}^{n}\frac{(-1)^i}{(2i)!} y^{2i} f_{y0}^{(2i)}(s); \tag{8b}$$

$$B_s = xy\sum_{i=0}^{n}\frac{(-1)^i}{(2i+1)!} y^{2i} f_{y0}^{(2i+1)}(s). \tag{8c}$$

4 Deducing the transfer matrix

In this part we will deduce the transfer matrix of the x direction; the same process will be followed for the y direction. We deal with the three models: the quadratic, the high order power and the exponential model. Let $s=0$ at the entrance of the quadrupole. $x[0]=x_0$, $x'[0]=x'_0$ are the position and the angle deviation in the x direction respectively. Solving the linear differential equations with initial conditions, we get x function with s, x_0 and x'_0. The coefficient of x_0 is m_{11}, the coefficient of x'_0 is m_{12}, the derivative of m_{11}, m_{12} to s is the m_{21}, m_{22}. These elements make up a transfer matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

4.1 The quadratic model

Assuming $f_{y0}(s) = G(1+as+bs^2)$ from Eq. (8), we can get the magnetic field for the quadrupole

$$B_x = Gy(1+as+bs^2) - \frac{bGy^3}{3}, \tag{9a}$$

$$B_y = Gx(1+as+bs^2) - bGxy^2, \tag{9b}$$

$$B_s = Gxy(a+2bs). \tag{9c}$$

We only consider the linear terms. The nonlinear terms are neglected, and then we get the equation of particle motion

$$x'' = -\frac{e}{p}Gx(1+as+bs^2) = -G1x(1+as+bs^2). \tag{10}$$

Let $x[s]=u[t]$, where $t = \frac{G_1^{\frac{1}{4}}(a+2bs)}{\sqrt{2}b^{\frac{3}{4}}}$. Then Eq. (10) becomes

$$\left[\left(\frac{a^2}{8b^{\frac{3}{2}}} - \frac{1}{2\sqrt{b}}\right)\sqrt{G_1} - \frac{t^2}{4}\right]u[t] + u''[t] = 0. \tag{11}$$

Eq. (11) is a Weber Eqs. (7, 8), the general solution of (11) is

$$u[t] = C_1 D_{\frac{-4b^{\frac{3}{2}}+a^2\sqrt{G_1}-4b\sqrt{G_1}}{8b^{\frac{3}{2}}}}(t) + C_2 D_{\frac{-4b^{\frac{3}{2}}-a^2\sqrt{G_1}+4b\sqrt{G_1}}{8b^{\frac{3}{2}}}}(t). \tag{12}$$

The elements of the transfer matrix are

$$m_{11} = -\frac{1}{2}(-1)^{\frac{1}{4}}\mathrm{e}^{\frac{ig\pi}{4}}\left[\left(AD_{-\frac{1+g}{2}}(iA) + 2iD_{\frac{1-g}{2}}(iA)\right)D_{\frac{-1+g}{2}}(A+Bs) + \left(AD_{\frac{-1+g}{2}}(A) - 2D_{\frac{1+g}{2}}(A)\right)D_{-\frac{1+g}{2}}(i(A+Bs))\right]; \tag{13}$$

$$m_{12} = \frac{1}{B}(-1)^{\frac{1}{4}}\mathrm{e}^{\frac{ig\pi}{4}}\left[D_{-\frac{1+g}{2}}(i(A+Bs))D_{\frac{-1+g}{2}}(A) - D_{-\frac{1+g}{2}}(iA)D_{\frac{-1+g}{2}}(A+Bs)\right]; \tag{14}$$

$$
\begin{aligned}
m_{21} = & -\frac{1}{4}(-1)^{\frac{1}{4}} B \mathrm{e}^{\frac{ig\pi}{4}} \Big\{ -(A+Bs) D_{-\frac{1+g}{2}}(i(A+Bs)) \Big[A D_{\frac{-1+g}{2}}(A) - 2 D_{\frac{1+g}{2}}(A) \Big] \\
& -2iD_{\frac{1-g}{2}}(i(A+Bs)) \Big[A D_{\frac{-1+g}{2}}(A) - 2 D_{\frac{1+g}{2}}(A) \Big] + \Big[A D_{-\frac{1+g}{2}}(iA) + 2iD_{\frac{1-g}{2}}(iA) \Big] \\
& \times \Big[(A+Bs) D_{\frac{-1+g}{2}}(A+Bs) - 2 D_{\frac{1+g}{2}}(A+Bs) \Big] \Big\}; \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
m_{22} = & -\frac{1}{2}(-1)^{\frac{1}{4}} \mathrm{e}^{\frac{ig\pi}{4}} \Big\{ (A+Bs) D_{-\frac{1+g}{2}}(i(A+Bs)) D_{\frac{-1+g}{2}}(A) + 2iD_{\frac{1-g}{2}}(i(A+Bs)) D_{\frac{-1+g}{2}}(A) \\
& + D_{-\frac{1+g}{2}}(iA) \Big[(A+Bs) D_{\frac{-1+g}{2}}(A+Bs) - 2 D_{\frac{1+g}{2}}(A+Bs) \Big] \Big\}, \qquad (16)
\end{aligned}
$$

where $g=\dfrac{a^2\sqrt{G_1}-4b\sqrt{G_1}}{4b^{3/2}}$, $A=\dfrac{aG_1^{\frac{1}{4}}}{\sqrt{2}b^{\frac{3}{4}}}$, $B=\sqrt{2}b^{\frac{1}{4}}G_1^{\frac{1}{4}}$, $D_v(x)$ is the parabolic cylinder function. The transfer matrix of the quadratic fringe $G(1+as+bs^2)$ from 0 to s for Eq. (10) can be expressed as

$$
\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} (13) & (14) \\ (15) & (16) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}. \qquad (17)
$$

4.2 The high order power of the $(b+as)$ model

Assuming $f_{y0}(s)=(b+as)^n$, neglecting the high-order terms, we obtain the equation of motion

$$
x'' = -\frac{e}{p}x(b+as)^n = G_1 x(b+as)^n. \qquad (18)
$$

Let $x[s]=u[t]\sqrt{b+as}$ [7, 9, 10], where $t=(b+as)^{\frac{2+n}{2}}$, Eq. (18) becomes

$$
\left(-\frac{1}{(2+n)^2}-\frac{4G_1t^2}{a^2(2+n)^2}\right)u[t]+t(u'[t]+tu''[t])=0, \qquad (19)
$$

Eq. (19) is a Bessel equation, the general solution is

$$
u[t]=C_1J_{\frac{1}{2+n}}\left(\frac{-2i\sqrt{G_1}t}{a(2+n)}\right)+C_2Y_{\frac{1}{2+n}}\left(\frac{-2i\sqrt{G_1}t}{a(2+n)}\right). \qquad (20)
$$

The elements of the transfer matrix are

$$
m_{11} = \frac{g\pi\sqrt{b+as}[J_A(gB)J_{(1+n)A}(g)+J_{-A}(gB)J_{-1+A}(g)]\csc(\pi A)}{2\sqrt{b}}; \qquad (21)
$$

$$
m_{12} = -\frac{\sqrt{b}\pi\sqrt{b+as}[J_{-A}(gB)J_A(g)-J_{-A}(g)J_A(gB)]\csc(\pi A)}{a(2+n)}; \qquad (22)
$$

$$
m_{21} = -\frac{ag^2(2+n)\pi\sqrt{(b+as)^{1+n}}[J_{(1+n)A}(gB)J_{-1+A}(g)-J_{(1+n)A}(g)J_{-1+A}(gB)]\csc(\pi A)}{4b^{\frac{(3+n)}{2}}}; \qquad (23)
$$

$$
m_{22} = \frac{g\pi\sqrt{(b+as)^{1+n}}[J_A(g)J_{(1+n)A}(gB)+J_{-A}(g)J_{-1+A}(gB)]\csc(\pi A)}{2b^{\frac{(1+n)}{2}}}, \qquad (24)
$$

where $\dfrac{g^2(2+n)^2a^2}{4b^{2+n}}=G_1=-\dfrac{e}{p}$, $A=\dfrac{1}{2+n}$, $B=b^{-1-\frac{n}{2}}(b+as)^{1+\frac{n}{2}}$.

The transfer matrix of the fringe $(b+as)^n$ from 0 to s can be expressed as

$$
\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} (21) & (22) \\ (23) & (24) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}. \qquad (25)
$$

4.3 The exponential model

Assuming $f_{y0}(s)=G(a+e^{bs})$, neglecting the high-order terms, we obtain the equation of motion

$$x''=-\frac{e}{p}Gx(a+e^{bs})=G_1x(a+e^{bs}). \quad (26)$$

Let $x[s]=u[t]$, where

$$t=\frac{2\sqrt{e^{bs}G_1}}{b},$$

and Eq. (26) becomes

$$\left(\frac{4aG_1}{b^2}+t^2\right)u[t]+t(u'[t]+tu''[t])=0. \quad (27)$$

Eq. (27) is a Bessel equation, the general solution is

$$u[t]=C_1J_{\frac{2\sqrt{aG_1}}{b}}(t)+C_2Y_{\frac{2\sqrt{aG_1}}{b}}(t), \quad (28)$$

The elements of the transfer matrix are

$$m_{11}=\frac{\pi\sqrt{G_1}\left[(I_{-1+A}(B)+I_{1+A}(B))I_{-A}(\sqrt{e^{bs}}B)-(I_{-1-A}(B)+I_{1-A}(B))I_A(\sqrt{e^{bs}}B)\right]\csc[A\pi]}{2b}; \quad (29)$$

$$m_{12}=-\frac{\pi\left[I_{-A}(\sqrt{e^{bs}}B)I_A(B)-I_{-A}(B)I_A(\sqrt{e^{bs}}B)\right]\csc[A\pi]}{b}; \quad (30)$$

$$m_{21}=\frac{\sqrt{e^{bs}}G_1\pi}{4b}\left\{\left[I_{-1-A}\left(\sqrt{e^{bs}}B\right)+I_{1-A}(\sqrt{e^{bs}}B)\right][I_{-1+A}(B)+I_{1+A}(B)]\right.$$
$$\left.-[I_{-1-A}(B)+I_{1-A}(B)]\left[I_{-1+A}(\sqrt{e^{bs}}B)+I_{1+A}(\sqrt{e^{bs}}B)\right]\right\}\csc[\pi A], \quad (31)$$

$$m_{22}=\frac{\sqrt{e^{bs}G_1}\pi}{2b}\left\{\left[I_{-1+A}\left(\sqrt{e^{bs}}B\right)+I_{1+A}\left(\sqrt{e^{bs}}B\right)\right]I_{-A}(B)\right.$$
$$\left.-\left[I_{-1-A}\left(\sqrt{e^{bs}}B\right)+I_{1-A}\left(\sqrt{e^{bs}}B\right)\right]I_A(B)\right\}\csc[\pi A], \quad (32)$$

where

$$A=\frac{2\sqrt{a}\sqrt{G_1}}{b},\ B=\frac{2\sqrt{G_1}}{b}.$$

$I_n(\nu)$ is the modified Bessel function of the first kind.

The transfer matrix of the exponential fringe $G(a+e^{bs})$ from 0 to s can be expressed as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s=\begin{pmatrix} (29) & (30) \\ (31) & (32) \end{pmatrix}\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}. \quad (33)$$

5 Fitting fringe field and calculating the transfer matrix

We take the quadrupole magnet Q105 in BEPCⅡ for example. By fitting the left fringe, the matrix of right fringe can be obtained from the symmetry. If the transfer matrix of the left fringe is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the transfer matrix of right fringe is

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

[11]. The optimized energy of BEPCⅡ is 1.89 GeV, so

$$\frac{e}{p}=0.1586.$$

The effective length of Q105 is 0.3114 m , and the maximum field gradient is 13.3256T/m.

The fringe models and the transfer matrix of Q105 with slice-by-slice, hard-edge, linear, quadratic and exponential fringe models are given in Table 1 and 2 respectively.

Table 1 gives the fitting effect, the abscissas of pictures are the longitudinal positions of the quadrupole, in m, and the perpendicular coordinates are the quadrupole's grads (T/m^2). The dished lines are measured values, and the real lines are the fitting curves.

Table 1. Fringe fitting five models.

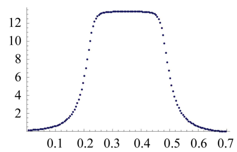
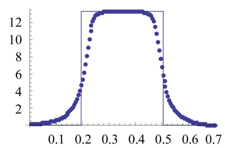
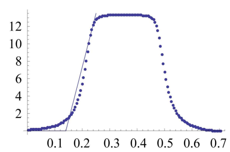
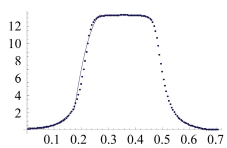
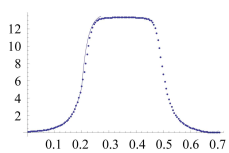
model	fitting effect	fitting function
slice-by-slice		Take each point as a slice, the gradient value is measured value. The first and the last slice lengths are 0.0025 m, the other's length is 0.005 m
hard-edge		$\begin{cases} 0 & s \leqslant 0.1943 \\ 13.3269 & 0.1943 \leqslant s \leqslant 0.5057 \\ 0 & 0.5057 \leqslant s \leqslant 0.7 \end{cases}$
linear		$\begin{cases} 0 & s \leqslant 0.14 \\ 122.702(s-0.14) & 0.14 \leqslant s \leqslant 0.25 \\ 13.3265 & 0.25 \leqslant s \leqslant 0.45 \end{cases}$
quadratic		$\begin{cases} 0.3707-10.7693s+127.479s^2 & s \leqslant 0.175 \\ 2.3902+203.915(s-0.175)-930.989(s-0.175)^2 & 0.175 \leqslant s \leqslant 0.27 \\ 13.3266 & 0.27 \leqslant s \leqslant 0.43 \end{cases}$
exponential		$\begin{cases} 0.2863+0.0231e^{26.43s} & s \leqslant 0.2 \\ 13.8162-8.9789e^{-46.02(s-0.2)} & 0.2 \leqslant s \leqslant 0.27 \\ 13.3266 & 0.27 \leqslant \mathrm{s} \leqslant 0.43 \end{cases}$

Table 2. The transfer matrices of five models.

model	transfer matrix for focusing	transfer matrix for defocusing
slice-by-slice	$\begin{pmatrix} 0.7750 & 0.6277 \\ -0.6336 & 0.7772 \end{pmatrix}$	$\begin{pmatrix} 1.2381 & 0.7756 \\ 0.6832 & 1.2357 \end{pmatrix}$
hard-edge	$\begin{pmatrix} 0.7757 & 0.6263 \\ -0.6359 & 0.7757 \end{pmatrix}$	$\begin{pmatrix} 1.2365 & 0.7770 \\ 0.6809 & 1.2365 \end{pmatrix}$
linear	$\begin{pmatrix} 0.7759 & 0.6270 \\ -0.6347 & 0.7759 \end{pmatrix}$	$\begin{pmatrix} 1.2368 & 0.7763 \\ 0.6822 & 1.2368 \end{pmatrix}$
quadratic	$\begin{pmatrix} 0.7761 & 0.6279 \\ -0.6334 & 0.7761 \end{pmatrix}$	$\begin{pmatrix} 1.2370 & 0.7754 \\ 0.6837 & 1.2370 \end{pmatrix}$
exponential	$\begin{pmatrix} 0.7761 & 0.6280 \\ -0.6332 & 0.7761 \end{pmatrix}$	$\begin{pmatrix} 1.2370 & 0.7752 \\ 0.6838 & 1.2370 \end{pmatrix}$

Assuming a FODO cell and its elements are Q105 (for focusing)–drift–Q105 (for defocusing)–drift. The transfer matrix of a cell is

$$\begin{pmatrix} \cos(\Delta\varphi)+\alpha\sin(\Delta\varphi) & \beta\sin(\Delta\varphi) \\ -\gamma\sin(\Delta\varphi) & \cos(\Delta\varphi)-\alpha\sin(\Delta\varphi) \end{pmatrix}. \tag{34}$$

Assuming the phase advance $\Delta\varphi$ in a slice-by-slice model is

$$\frac{\pi}{2}\ \left(\frac{\pi}{3}\right),$$

and the length of drift is L, we obtain the other model's phase advance in the same cell. The results are given in Table 3. Table 3 also gives the minimum beta and maximum beta of every model in the

$$\frac{\pi}{2}\ \left(\frac{\pi}{3}\right)$$

cell.

Table 3. Difference of the results of four models compared with the slice-by-slice result in a FODO cell.

model	phase advance(°)	maximum beta/m	minimum beta/m	phase advance(°)	maximum beta/m	minimum beta/m
slice-by-Slice	90	7.6113	1.3494	60	5.5846	1.9219
hard-edge	90.6226	7.5733	1.3423	60.5137	5.5449	1.9081
linear	90.2918	7.5936	1.3460	60.2388	5.5663	1.9155
quadratic	89.9303	7.6162	1.3501	59.9375	5.5900	1.9235
exponential	89.8886	7.6194	1.3505	59.9028	5.5931	1.9243

6 Conclusions

The transfer matrices of three quadrupole fringe models are presented. The expressions look messy, but they can be used very easily. If the field is given we can calculate the parameters in transfer matrices, and then by substituting the real parameters into the elements of the transfer matrix, we can get the precise value. It is very useful for some accurate calculations. We compare the three new transfer matrices with the hard-edge and slice-by-slice matrices. The results have very small differences that can verify the correctness of our transfer matrices calculation and they are more accurate.

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