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Courant-Snyder invariant density screening method for emittance analysis*

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Abstract: Emittance is an important characteristic of describing charged particle beams. In hadron accelerators, we often meet irregular beam distributions that are not appropriately described by a single rms emittance or 95% emittance or total emittance. In this paper, it is pointed out that in many cases a beam halo should be described with very different Courant-Snyder parameters from the ones used for the beam core. A new method – the Courant-Snyder invariant density screening method – is introduced for analyzing emittance data clearly and accurately. The method treats the emittance data from both measurements and numerical simulations. The method uses the statistical distribution of the beam around each particle in phase space to mark its local density parameter, and then uses the density distribution to calculate the beam parameters such as the Courant-Snyder parameters and emittance for different beam boundary definitions. The method has been used in the calculations for beams from different sources, and shows its advantages over other methods. An application code based on the method including the graphic interface has also been designed.

Key words: emittance, Courant-Snyder invariant density, Courant-Snyder parameters, beam halo

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1 Introduction

The emittance and its shape of a beam are important physical parameters that express the beam quality and matching to the beam line it will pass through. They are important in designing and commissioning accelerators and beam transport lines. Generally the emittance of a particle beam is defined as the occupied volume in six-dimensional phase space, but is usually expressed in the occupied areas of three two-dimensional projection phase planes, especially when the beam is decoupled in the phase planes. In electron accelerators where the beam distribution is almost Gaussian, one usually uses rms emittance that represents the occupied area of the rms particle. However, in hadron accelerators where beam distributions are often irregular, one uses different definitions to better represent the beam distribution in different applications, for example, rms emittance, 95% emittance and total emittance. In this paper, we introduce a new method to evaluate all kinds of beam distributions for both simulated and measured distribution data.

2 Emittance expression of a beam distribution

In decoupled linear beam transport systems, each particle follows tracks along ellipses in the phase planes, which can be expressed by the following formula (taking the horizontal phase plane as an example):

$$\gamma x^2+2\alpha xx'+\beta x'^2=I, \tag{1}$$

where α, β, γ are the Courant-Snyder parameters (hereafter referred to as the C-S parameters), and there is a relationship between them: $\beta\gamma-\alpha^2=1$; I is the Courant-Snyder invariant (hereafter referred to as the C-S invariant) which is constant during beam transporting without acceleration. The emittance ε is defined as the area of the boundary ellipse, which is usually defined as the area divided by π and with

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a unit of πmm·mrad. The emittance can be the rms emittance that represents the second-moment of all particles, 95% emittance that contains 95% particles or 100% emittance that contains all particles. A simulated beam by most simulation codes is a distribution of macro-particles in phase planes or a six-dimensional phase space. One can calculate the rms emittance directly from the phase points by using a statistical method. For example, in the horizontal phase plane, the rms emittance can be calculated by the following formulae [1]:

$$\varepsilon=\sqrt{\mu_{20}\mu_{02}-\mu_{11}^2}, \tag{2}$$

where

$$\begin{aligned}\mu_{20}&=\overline{(x-\overline{x})^2}=\overline{x^2}-\overline{x}^2,\\ \mu_{02}&=\overline{(x'-\overline{x}')^2}=\overline{x'^2}-\overline{x'}^2,\\ \mu_{11}&=\overline{(x-\overline{x})(x'-\overline{x'})}=\overline{xx'}-\overline{x}\overline{x'}.\end{aligned}$$

For a measured beam, the distribution is usually discrete. One obtains a density distribution $c(x_i,x'_j)$ for a discrete phase point (x_i,x'_j). Thus, the rms emittance is defined as [2]:

$$\varepsilon_{\rm rms}=\sqrt{\langle x_i^2\rangle\langle x_j'^2\rangle-\langle x_i\cdot x'_j\rangle^2}, \tag{3}$$

where

$$\langle x_i^2\rangle=\frac{\sum\limits_{i,j}(x_i-\overline{x})^2\cdot c(x_i,x'_j)}{\sum\limits_{i,j}c(x_i,x'_j)},\quad \overline{x}=\frac{\sum\limits_{i,j}x_i\cdot c(x_i,x'_j)}{\sum\limits_{i,j}c(x_i,x'_j)},$$

$$\langle x_j'^2\rangle=\frac{\sum\limits_{i,j}(x'_j-\overline{x}')^2\cdot c(x_i,x'_j)}{\sum\limits_{i,j}c(x_i,x'_j)},\quad \overline{x'}=\frac{\sum\limits_{i,j}x'_j\cdot c(x_i,x'_j)}{\sum\limits_{i,j}c(x_i,x'_j)},$$

$$\langle x_i\cdot x'_j\rangle=\frac{\sum\limits_{i,j}(x_i-\overline{x})(x'_j-\overline{x}')\cdot c(x_i,x'_j)}{\sum\limits_{i,j}c(x_i,x'_j)}.$$

The C-S parameters for the rms ellipse can be calculated as:

$$\alpha=-\frac{\langle x_i\cdot x'_j\rangle}{\varepsilon_{\rm rms}},\quad \beta=\frac{\langle x_i^2\rangle}{\varepsilon_{\rm rms}},\quad \gamma=\frac{\langle x_j'^2\rangle}{\varepsilon_{\rm rms}}.$$

As mentioned in Section 1, the rms emittance is good to describe beam distributions of a Gaussian type which one often sees in electron accelerators. However, the rms emittance does not give sufficient information about the beam distribution that is irregular as one often sees in hadron accelerators. For high-intensity hadron accelerators, one has to deal with the sparse part of a beam (or beam halo) very carefully to avoid significant beam losses, as beam losses will have a very important impact on equipment safety and machine maintenance.

Halo particles with large C-S invariants have a great influence on calculating the rms emittance using the statistical method. If the halo particles are irregularly populated, the calculated rms emittance and the C-S parameters do not reflect the distribution of the beam core. In addition, sometimes we are more concerned with the emittance of different beam fractions. In this case, it is not appropriate to simply scale the rms emittance. Therefore, it is better to use a more general expression of emittance that can give more detailed information on a beam distribution. In the next section, a new method is introduced to give different C-S parameters and emittance for different beam fractions or boundaries.

3 Methods of evaluating beam emittance

There are usually different methods to evaluate the emittance for either a simulated beam or a measured beam. A simulated beam means a beam distribution generated by applying a multi-particle simulation code in a section or sometimes end-to-end of an accelerator. It usually has a continuous distribution in spite of limited particles. The emittance of such a beam can be coupled or decoupled in the projection phase planes. Although a usual emittance analysis in the phase planes shows only the geometrical emittance, it is possible to derive the decoupled emittance. For a measured beam, either from a double-slit system or a pepper-pot system or other methods, the measured data of beam distribution is discrete or represented by signals in meshes. It usually contains only the geometrical emittance information. In addition, the measurement noise and a typically small bias are contained in the measured signals. Depending on the measurement device and the beam quality, a measured beam distribution is usually less reproducible and rough due to the mesh size.

3.1 Simulated beams

For a simulated beam, the particle distribution in the phase space is given from simulation codes and the rms emittance can be calculated by Eq. (2) within or outside the codes. This statistical method of calculating the rms emittance is widely used in accelerator simulation software, for instance PARMILA [3], TRACE 3-D [4] and ORBIT [5]. For different particle distributions, emittance with different beam fractions can be calculated through the rms emittance by zooming in/out and keeping the same C-S parameters that are calculated together with the rms emittance. In TRACE 3-D, the total emittance in each phase plane is five times the rms emittance in that plane. In most hadron accelerators, 95% beam emittance that contains 95% of the total particles is used. However, for high beam power hadron accelerators it is better to use total beam emittance as halo particles are also important in the operation [6].

3.2 Measured beams

As mentioned above, the measurement noise and a typically small bias are contained in the measured signals. Thus, the contribution from the background must be minimized before calculating the rms emittance by Eq. (3). Here two influential methods to calculate the rms emittance of a measured beam distribution are described and commented on.

1) Threshold analysis method

Applying a threshold commonly means that all values of a distribution above the threshold remain unchanged while all values below the threshold are set to zero [7]. Various methods have been developed to select the most appropriate threshold. For example, a given percentage of the maximum peak signal, a given percentage of the summed beam current and so on can be used for the threshold definition.

In addition, the small biases contained in the measurement data need to be subtracted to obtain a better calculation accuracy. Usually, bias estimates can be obtained using a frequency-weighted average of the small measured signals that are far from the beam core.

2) SCUBEEx analysis method

For the self-consistent unbiased elliptical exclusion (SCUBEEx) [8, 9] method, the C-S parameters are calculated first from the data after thresholding them at a high percentage of the peak signal (20% for instance) to exclude all background signals. These C-S parameters define the shape of the exclusion ellipse, and then zoom into the exclusion ellipse starting at zero. The average of the signals measured outside the exclusion ellipse is taken as the bias and needs to be subtracted from the raw data. The rms emittance can be evaluated from the data within the ellipse after subtracting the average current signal found outside the ellipse. As the exclusion ellipse's area is increased, both the average outside current and the inside rms emittance form plateaus when all data containing part of the particle beam are inside the boundary. These plateau edges mark the smallest acceptable exclusion ellipse and provide unbiased estimates for the average background and rms emittance (as Fig. 7 in Ref. [9] shows).

3.3 C-S invariant density screening method

The methods discussed above are good to evaluate the rms emittance of regular beam distributions that are symmetric and have the same contour shape for the inner and outer parts. In these cases, when calculating the emittance of different beam fractions, one can zoom in/out of the rms ellipse. However, for some beams with irregular distributions simple rms emittance calculation methods are not appropriate. For example, for a beam with a large halo that may be non-symmetric or incomplete due to the acceptance limit, errors can arise in calculating the rms emittance, because the halo particles with large C-S invariants have very important weights to the calculations. Another example is for a beam that has different orientations for the beam core and the beam halo in the phase plane. In this case, it is better to use different C-S parameters to express the emittance shapes of the beam core and the beam halo. In addition, to better evaluate the importance of a macro-particle or a signal by minimizing the statistical errors or noise, one can use its neighboring particles or signals. This can be done by counting the particles or signals in a small phase ellipse defined by a given C-S invariant, namely, the C-S invariant density is used to mark the particle or signal. The emittance of different beam fractions can be calculated by including particles or signals with different density thresholds. This is called the C-S invariant density screening method. More details about the method are described below.

Usually a beam distribution has a denser core and sparser halo, and this is the case to be dealt with by the new method. Certainly it is possible to see beams with irregular distributions of poor symmetry, e.g. those having several islands, and they cannot be dealt with by the method very effectively.

Before the C-S parameters have been calculated, an initial set of C-S parameters ($\beta=1\ m$, α=0) that are quite arbitrary are given. In order to determine the C-S invariant density for each particle, one can count the number of macro-particles within a small ellipse that is centered on the particle with a pre-set ellipse size and uses the initial C-S parameters. Then after, all the particles are labeled by their C-S invariant densities. There are two ways to use the density distribution: 1) Calculate the rms emittance by ruling out the outermost halo particles. One just needs to set a threshold on density to rule out sparse the irregular halo particles, and takes the usual statistical method to obtain the rms emittance and the C-S parameters. This part is similar to the two methods discussed above. 2) Calculate the C-S invariant boundary and the C-S parameters for any given beam fraction. Or one can calculate the beam fraction for a given C-S invariant boundary. For the latter, by setting a density and a narrow window, one can calculate the fit ellipse for the selected particles and the beam fraction with all the particles within the ellipse. In this way, one can obtain the beam fraction and the C-S parameters for the beam core, the beam halo or any given C-S invariant boundary. One needs to repeat the above procedures again by replacing the initially-set C-S parameters with the calculated ones so that the orientation of the small ellipse for calculating the C-S invariant density can be better presented. The test experience shows that the calculation results are not susceptible to the definition of the small ellipse; therefore, repeating once is usually good enough to obtain a good definition of the ellipse. In addition, the size of the small ellipse is chosen by input tests according to the total emittance size to balance between the statistical effect and the fine contour definition.

With the C-S invariant density screening method one can describe the emittance and the C-S parameters for different beam fractions more accurately, and it can be applied to both the simulated and measured data. To some extent, the C-S invariant density method is similar to the threshold method and the SCUBEEx method, but it is applicable to both the simulated beam distributions and the measured beam distributions whereas the other two are applicable only to the measured beam distributions. Fig. 1 shows the procedures in applying the C-S invariant density screening method.

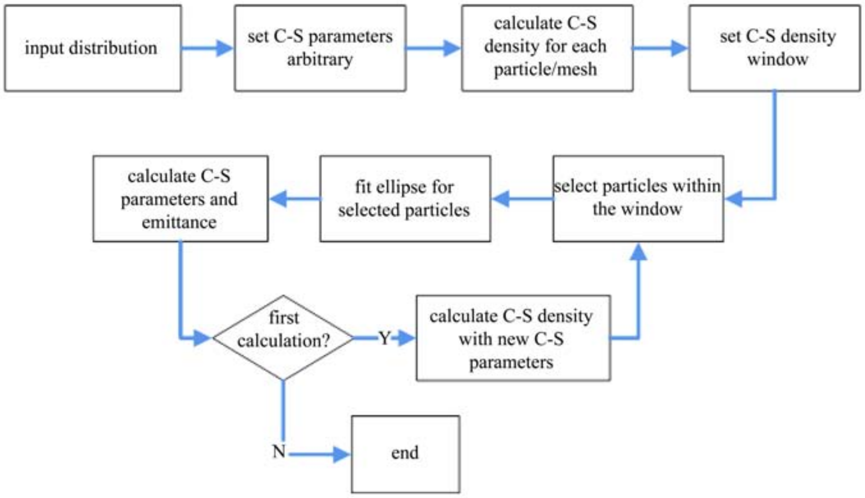

Fig. 1. Procedures of the C-S invariant density screening method for emittance analysis.

4 Practical applications of the C-S invariant density screening method

4.1 For a simulated beam distribution

A simulated beam with a normal beam core and a halo part is used for the study of the effectiveness of different emittance evaluation methods. It comes from the study of the scattering effect of the proton beam crossing through a PBW (Proton Beam Window) in the RTBT beam transport line at CSNS [10, 11]. Fig. 2 shows the calculated 95% emittance ellipse of the simulated beam at the CSNS target by using the rms ellipse zooming method and the C-S invariant density screening method. The parameters of the ellipses from the two methods are compared in Table 1. One can find that there is a large difference

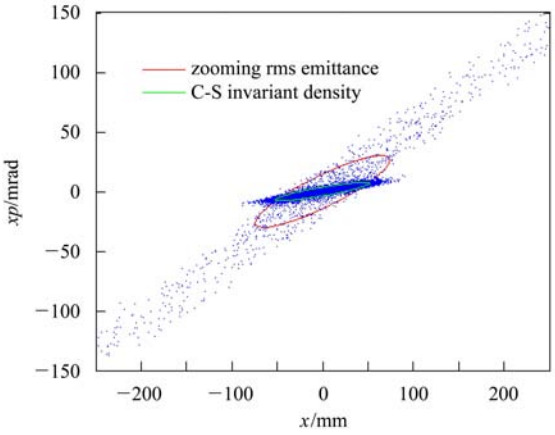

Fig. 2. Comparison of the 95% emittance ellipses between the rms ellipse zooming method and the C-S invariant density screening method for a scattered beam through a PBW.

Table 1. Parameters of the 95% emittance ellipse by the rms ellipse zooming method and the C-S invariant density screening method.

method	emittance/(π·mm·mrad)	alpha	beta/m
RMS ellipse zooming	1075.4	−1.82	5.10
C-S invariant density screening	191.5	−1.86	14.15

between the two methods in evaluating the C-S parameters when the distribution is irregular. With the C-S invariant density method, one can define the phase ellipse shape much more accurately than the rms zooming method. This is important when one designs a transverse matching system in a beam line to a beam with an irregular distribution; or one can estimate the maximum accepted portion for a given acceptance in the same situation.

4.2 For a measured beam distribution

A set of measured emittance data from ISIS, Rutherford Appleton Lab (RAL) is also used to evaluate the C-S invariant density method. These data were obtained by applying a double-slit emittance measurement system to a 65 keV H^- beam extracted from a PIG H^- ion source. Fig. 3 shows the contours of the raw data in the vertical phase plane. Fig. 4 shows the C-S invariant density contours for the same data. One can find that the two contours are different, as the C-S invariant density contour shows more hiberarchy. This is because the density at each mesh point is calculated with the intensities in the neighboring meshes defined by a given C-S invariant ellipse having the C-S parameters fit to the distribution.

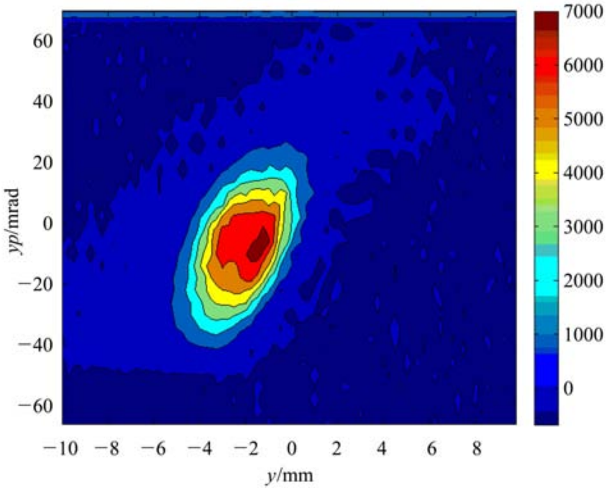

Fig. 3. Contour plot in the vertical phase plane of the measured emittance data for a beam extracted from a PIG H^- ion source in ISIS, RAL.

By applying the threshold method, the SCUBEEx method and the C-S invariant density screening method to the same data, one can make a comparison. Fig. 5 shows the emittance as a function of beam fraction by applying the three methods. The dashed and dotted lines are obtained by zooming the rms emittance ellipse, and the solid line is obtained by the data fitting with different C-S invariant densities. One can find that the three methods give almost the same result for the beam core and quite different ones for the halo part that has a beam fraction larger than 0.85. This can be explained by the fact that they are equally effective in dealing with a regular distribution such as the beam core in the data, but the hiberarchy with the C-S invariant density method has the advantage in clarifying the beam halo.

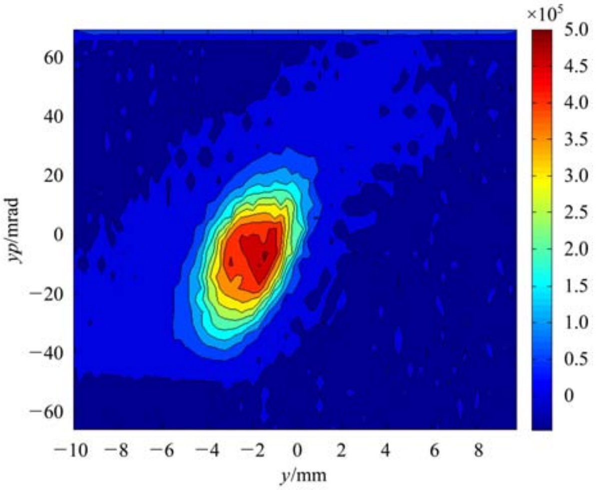

Fig. 4. C-S invariant density contour plot for the same data as shown in Fig. 3.

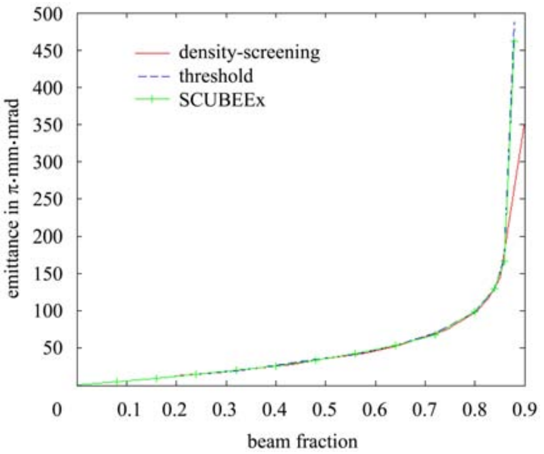

Fig. 5. Emittance as a function of beam fraction with the three methods using the data shown in Fig. 3.

Figure 6 shows the C-S parameters calculated by the C-S invariant density screening method. It shows clearly that the C-S parameters change from the beam core to the beam halo, while they are supposed to be constant in the other two methods. This shows the important advantage of the C-S invariant density screening method in calculating the shapes of the emittance ellipses for different beam fractions.

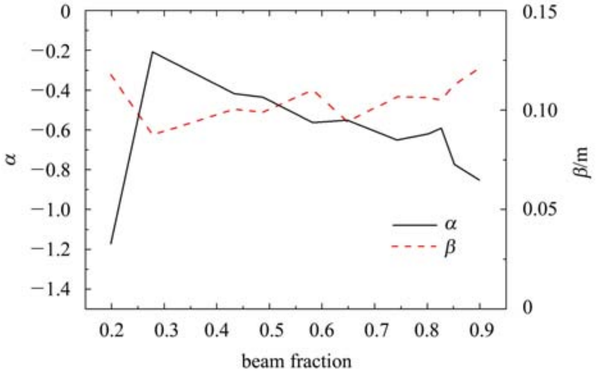

Fig. 6. C-S parameters evaluated by applying the C-S invariant density screening method to the data shown in Fig. 3 as a function of beam fraction.

The imitated data are used for further testing to evaluate the C-S invariant density method. The imitated beam is obtained by applying a double-slit measurement system to a simulated distribution that is from a simulation study on the beam halo development after RFQ acceleration using the PARMILA code [12], and small random data are also added as noise background as shown in Fig. 7. The top left and right plots are the contours of the imitated beam and the C-S invariant density. In the top right plot one can see more details of the beam distribution such as the beam halo as well as the noise, as the contours can be made in smaller steps.

The bottom left plot in Fig. 7 shows emittance as a function of beam fraction by applying the threshold method, the SCUBEEx method and the C-S invariant density screening method to the imitated data. Smaller emittance for a given beam faction means that the method is more effective, and this means the C-S invariant density screening method is the most effective one among the three methods. It should be pointed out that for a selected density window situated at a peak, the selected particles populate a filled ellipse instead of a ring-type ellipse, thus it is better

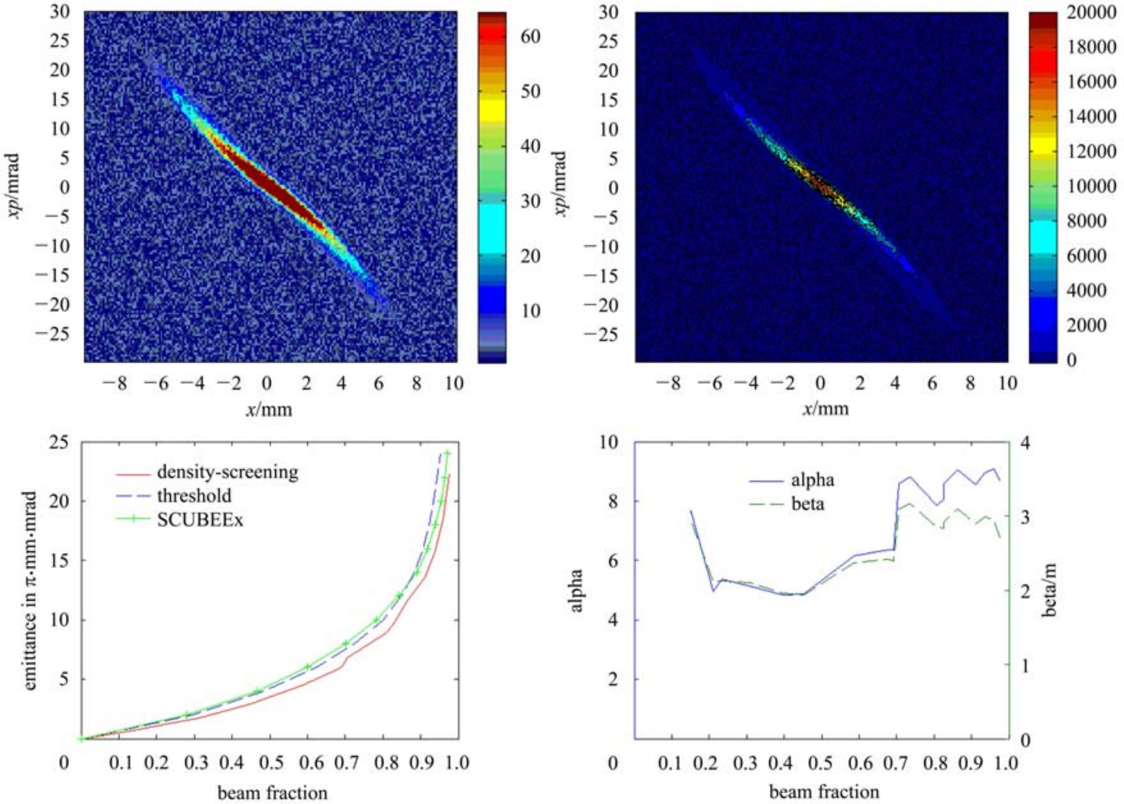

Fig. 7. The contours of the imitated data and some analysis results.

to use a selected density slightly lower than the peak density and zoom out from the ellipse when a smaller beam fraction is needed.

The bottom right plot in Fig. 7 shows the C-S parameters calculated by the C-S invariant density screening method. The beam fraction of about 0.7 separating the beam core and the beam halo can be seen in the curves related to the C-S invariant density screening method in the bottom left plot. Here, one can see a large difference between the C-S parameters for the beam core and the beam halo. However, the halo definition using the abrupt change in the C-S parameters is perhaps useful in many applications, although it is quite different from the usual one using a beam fraction greater than 0.95.

These results evaluate the C-S invariant density screening method further and indicate that this method is effective.

4.3 IDMEA code

An emittance analysis code - IDMEA (Invariant Density Method for Emittance Analysis) based on the C-S invariant density screening method including the graphic interface has been developed by using Matlab [13]. With this code, one can perform the emittance analysis intuitively.

The three examples mentioned above show that the C-S invariant density screening method is effective in analyzing the emittance for most beam distributions coming from either a simulation study or an emittance measurement or discretized data as the imitation of emittance measurements.

5 Conclusions

A new emittance analysis method “C-S invariant density screening” has been developed. The method can be used to evaluate the emittance and the C-S parameters for different beam fractions for both simulated and measured data, either in simulation codes or in a control room. The comparisons show that it has significant advantages over the usually used emittance analysis methods based on the rms emittance calculation, especially when irregular distributions are concerned. A practical code based on the method – IDMEA – confirms its effectiveness. It also stresses that different C-S parameters are needed to express the core and the halo of a given beam distribution in many applications.

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