Symmetries in a very special relativity and isometric group of Finsler space*

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Abstract: We present an explicit connection between the symmetries in a Very Special Relativity (VSR) and isometric group of a specific Finsler space. It is shown that the line element that is invariant under the VSR symmetric group is a Finslerian one. The Killing vectors in Finsler space are constructed in a systematic way. The Lie algebras corresponding to the symmetries of VSR are obtained from a geometric famework. The dispersion relation and the Lorentz invariance violation effect in the VSR are discussed.

Key words: isometric group, Killing vector, Lorentz violation

PACS: 02.40.-k, 11.30.-j **DOI:** 10.1088/1674-1137/35/6/004

1 Introduction

In the past few years, two interesting theories have been proposed for investigating the violation of Lorentz Invariance (LI). One is the so called Doubly Special Relativity (DSR) [1–5]. This theory takes Planck-scale effects into account by introducing an invariant Planckian parameter into the theory of special relativity. Another is the Very Special Relativity (VSR) developed by Cohen and Glashow [6]. This theory suggested that the exact symmetry group of nature may be isomorphic to a subgroup $SIM(2)$ of the Poincare group. And the $SIM(2)$ group semi-direct product with the spacetime translation group gives an 8-dimensional subgroup of the Poincare group called $ISIM(2)$ [7]. Under the symmetry of $ISIM(2)$, the CPT symmetry is preserved and many empirical successes of special relativity are still functioned.

Recently, physicists found that both of the theories mentioned above are related to Finsler geometry. Girelli, Liberati and Sindoni [8, 9] showed that the modified dispersion relation (MDR) in DSR can be incorporated into the framework of Finsler geometry. The symmetry of the MDR was described in the Hamiltonian formalism. Also, Gibbons, Gomis and Pope [10] showed that the Finslerian line element $\mathrm{d}s=(\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu)^{(1-b)/2}(n_\rho\mathrm{d}x^\rho)^b$ is invariant under the transformations of the group $DISIM_b(2)$ (1-parameter family of deformations of $ISIM(2)$).

Finsler geometry as a natural generalization of Riemann geometry could provide new insight into modern physics. The model of gravity and cosmology based on Finsler geometry is in good agreement with recent astronomical observations. An incomplete list includes the following: the flat rotation curves of spiral galaxies can be deduced naturally without invoking dark matter [11]; the anomalous acceleration [12] in the solar system observed by Pioneer 10 and 11 spacecrafts could account for a special Finsler space-Randers space [13]; the secular trend in the astronomical unit [14, 15] and the anomalous secular eccentricity variation of the Moon's orbit [16] could account for the effect of the length change of unit circle in Finsler geometry [17].

Thus, the symmetry of Finslerian spacetime is

Received 26 September 2010, Revised 12 October 2010
* Supported by National Natural Science Foundation of China (10525522, 10875129, 10771004)

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worth investigating. The way of describing spacetime symmetry in a covariant language (the symmetry should not depend on any particular choice of coordinate system) involves the concept of isometric transformation. In fact, the symmetry of spacetime is described by the so-called isometric group. The generators of the isometric group are directly connected with the Killing vectors [18]. Actually, the symmetry of deformed relativity has been studied by investigating the Killing vectors [19, 20]. In this paper, we present an explicit connection between the symmetries in the VSR and isometric group of Finsler space. It is shown that the line element that is invariant under the VSR symmetric group is a Finslerian one. The Killing vectors in Finsler space are constructed in a systematic way. The Lie algebras corresponding to the symmetries of VSR are obtained from a geometric framework. The dispersion relation and the Lorentz invariance violation effect in the VSR are discussed.

Throughout this paper, the index is lowered and raised by the Minkowskian metric $\eta_{\mu\nu}=\mathrm{diag}\,(1,-1,-1,-1)$ and its matrix reverse, respectively.

2 Killing vectors in Finsler space

Instead of defining an inner product structure over the tangent bundle in Riemann geometry, Finsler geometry is based on the so-called Finsler structure F with the property $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$, where x represents the position and $y\equiv\dfrac{\mathrm{d}x}{\mathrm{d}\tau}$ represents the velocity. The Finsler metric is given as [21]

$$g_{\mu\nu}\equiv\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\left(\frac{1}{2}F^2\right). \tag{1}$$

Finsler geometry has its genesis in integrals of the form

$$\int_s^r F\left(x^1,\cdots,x^n;\frac{\mathrm{d}x^1}{\mathrm{d}\tau},\cdots,\frac{\mathrm{d}x^n}{\mathrm{d}\tau}\right)\mathrm{d}\tau. \tag{2}$$

So the Finsler structure represents the length element of Finsler space.

To investigate the Killing vector, we should construct the isometric transformations of the Finsler structure. It is convenient to investigate the isometric transformations under an infinitesimal coordinate transformation for x,

$$\bar{x}^\mu=x^\mu+\epsilon V^\mu, \tag{3}$$

together with a corresponding transformation for y,

$$\bar{y}^\mu=y^\mu+\epsilon\frac{\partial V^\mu}{\partial x^\nu}y^\nu, \tag{4}$$

where $|\epsilon|\ll 1$. Under the coordinate transformation (3) and (4), to first order in $|\epsilon|$, we obtain the expansion of the Finsler structure,

$$\bar{F}(\bar{x},\bar{y})=\bar{F}(x,y)+\epsilon V^\mu\frac{\partial F}{\partial x^\mu}+\epsilon y^\nu\frac{\partial V^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu}, \tag{5}$$

where $\bar{F}(\bar{x},\bar{y})$ should equal $F(x,y)$. Under the transformation (3) and (4), a Finsler structure is called isometry if and only if

$$F(x,y)=\bar{F}(x,y). \tag{6}$$

Then, deducing from the expression (5), we obtain the Killing equation in Finsler space,

$$V^\mu\frac{\partial F}{\partial x^\mu}+y^\nu\frac{\partial V^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu}=0. \tag{7}$$

3 Symmetries of VSR

The VSR preserves the law of energy-momentum conservation [6]. It implies that the translation invariance should be contained in the symmetries of the VSR. The left symmetries of the VSR include four possible subgroups of the Lorentz group. We introduce the notation $T_1=(K_x+J_y)/\sqrt{2}$ and $T_2=(K_y-J_x)/\sqrt{2}$, where J and K are the generators of rotations and boosts, respectively. The four subgroups of the Lorentz group are given as [22]

i) $T(2)$, the Abelian subgroup of the Lorentz group, generated by T_1 and T_2, with the structure

$$[T_1,T_2]=0; \tag{8}$$

ii) $E(2)$, the group of two-dimensional Euclidean motion, generated by T_1, T_2 and J_z, with the structure

$$[T_1,T_2]=0,\ [J_z,T_1]=-\mathrm{i}T_2,\ [J_z,T_2]=\mathrm{i}T_1; \tag{9}$$

iii) $HOM(2)$, the group of orientation-preserving similarity transformations, generated by T_1, T_2 and K_z, with the structure

$$[T_1,T_2]=0,\ [T_1,K_z]=\mathrm{i}T_1,\ [T_2,K_z]=\mathrm{i}T_2; \tag{10}$$

iv) $SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by T_1, T_2, J_z and K_z, with the structure

$$\begin{aligned}&[T_1,T_2]=0,\quad [T_1,K_z]=\mathrm{i}T_1,\quad [T_2,K_z]=\mathrm{i}T_2,\\&[J_z,K_z]=0,\quad [J_z,T_1]=-\mathrm{i}T_2,\quad [J_z,T_2]=\mathrm{i}T_1.\end{aligned} \tag{11}$$

We will show that there is a relation between the isometric group of the Finsler structure [10],

$$F=(\eta_{\mu\nu}y^\mu y^\nu)^{(1-n)/2}(b_\rho y^\rho)^n\,, \tag{12}$$

and symmetries of the VSR. Here, n is an arbitrary constant, $\eta_{\mu\nu}$ is the Minkowskian metric and

$b_\rho=\eta_{\mu\rho}b^\mu$ is a constant vector. It is referred to as the VSR metric. By making use of the Killing equation (7), we obtain the Killing equation for the VSR metric,

$$y^\nu\frac{\partial V^\mu}{\partial x^\nu}\left(\frac{(1-n)y_\mu(b_\rho y^\rho)^n+n(\eta_{\alpha\beta}y^\alpha y^\beta)^{1/2}b_\mu(b_\rho y^\rho)^{n-1}}{(\eta_{\alpha\beta}y^\alpha y^\beta)^{(1+n)/2}}\right)=0. \tag{13}$$

Eq. (13) has such solutions,

$$V^\mu = Q^\mu{}_\nu x^\nu+C^\mu, \tag{14}$$

$$b_\mu Q^\mu{}_\nu = 0, \tag{15}$$

where $Q_{\mu\nu}$ is a constant skew-symmetric matrix and C^μ is an arbitrary constant vector. If one requires that the transformation group for the vectors is no other than the Lorentz group or a subgroup of the Lorentz group, formula (14) together with the constraint (15) is the only solution of the Killing equation (7) for the VSR metric.

Taking the light cone coordinate [7] $\eta_{\alpha\beta}y^\alpha y^\beta=2y^+y^--y^iy^i$ (with i ranging over the values 1 and 2) and supposing $b_\mu=\{0,0,0,b_-\}(b_-=1)$, we know that in general $Q^-{}_\mu\neq 0$. This means that the Killing vectors of the VSR metric (12) do not have non-trivial components Q_{+-} and Q_{+i}. The isometric group of a Finsler space is a Lie group [20]. The non-trivial Lie algebra corresponding to the Killing vectors (14), which satisfies the constraint (15), is given as

$$\begin{aligned}&[J_z,T^i]=\mathrm{i}\epsilon_{ij}T^j,\ [J_z,P^i]=\mathrm{i}\epsilon_{ij}P^j,\\&[T_i,P^-]=-\mathrm{i}P_i,\quad [T_i,P^j]=-\mathrm{i}\delta_{ij}P^+,\end{aligned} \tag{16}$$

where $\epsilon_{12}=-\epsilon_{12}=1,\epsilon_{11}=\epsilon_{22}=0$ and $P^\pm=(P_0\pm P_z)/\sqrt{2}$. It is obvious that the generators of the isometric group of the VSR metric are generators of $E(2)$ and four spacetime translation generators. This result induces the $E(2)$ scenario of VSR from the VSR metric (12). The $HOM(2)$ scenario of VSR could be induced in the same approach, for $HOM(2)$ algebra isomorphic to $E(2)$ algebra.

The above investigations are under the premise that the direction of spacetime is arbitrary or the transformation group for the vectors is none other than the Lorentz group or subgroups of the Lorentz group. This means that no preferred direction exists in spacetime. If the spacetime does have a special direction, the Killing equation (7) for the VSR metric will have a special solution. The VSR metric was first suggested by Bogoslovsky [23]. He assumed that spacetime has a preferred direction. Following the assumption and taking the null direction to be the preferred direction, we obtain the solution of the Killing equation (13),

$$V^\mu=(Q^\mu_\nu+\delta^\mu_\nu)x^\nu+C^\mu, \tag{17}$$

where Q^μ_ν is an antisymmetrical matrix and satisfies the constraint

$$Q_{+-}n^-=-n^-. \tag{18}$$

Here, n^- is a null direction. One can check that the Killing vectors (17) do not have non-trivial components Q_{+i}. This implies that the null direction is invariant under the transformation

$$\Lambda^-_-n^-\equiv\left(\delta^-_-+\epsilon(n\delta^-_-+Q^-_-)\right)n^-=(1+\epsilon(n-1))\,n^-. \tag{19}$$

Here, Λ^μ_ν denotes the counterpart of the Lorentz transformation. Therefore, if the spacetime has a preferred direction in the null direction, the symmetry corresponding to Q_{+-} is restored. One can see that the Killing vectors (17) have a non-trivial component $\delta^\mu_\nu x^\nu$. This represents the dilations. Thus, we know that the transformation group for the VSR metric (12) contains dilations, while the null direction is a preferred direction. One could obtain the Lie algebra for such a transformation group. In fact, the non-trivial Lie algebra is just the algebra of the $DISIM(2)$ group proposed by Gibbons et al. [10],

$$\begin{aligned}&[K_z,P^\pm]=-\mathrm{i}(b\pm1)P^\pm,\ \ [K_z,P^i]=-\mathrm{i}bP^i,\\&[K_z,T_i]=-\mathrm{i}T_i,\qquad [J_z,T^i]=\mathrm{i}\epsilon_{ij}T^j,\\&[J_z,P^i]=\mathrm{i}\epsilon_{ij}P^j,\qquad [T_i,P^-]=-\mathrm{i}P_i,\\&[T_i,P^j]=-\mathrm{i}\delta_{ij}P^+.\end{aligned} \tag{20}$$

The $DISIM(2)$ group is a subgroup of the Weyl group. It contains a subgroup $E(2)$ together with a combination of a boost in the $+-$ direction and a dilation. It should be noted that the deformed generator K_z acts not only as a boost but also as a dilation. The transformation acting by K_z is given as

$$\bar{x}^\pm=(\exp(\phi))^{\pm1+b}x^\pm,\ \ \bar{x}^i=(\exp\phi)^bx^i, \tag{21}$$

where $\exp(\phi)=\sqrt{\dfrac{1+v/c}{1-v/c}}$. The transformations acting by other generators of the $DISIM(2)$ group are the same with the Lorentz group.

If b_μ in the VSR metric (12) has the form $b_\mu=\{0,b_x,0,b_-\}$ ($b_x=b_-=1$), solutions of the Killing equation (13) show that the Killing vectors just have non-trivial components Q_{-y} and C^μ. However, the corresponding Lie algebra does not exist. For the generators corresponding to Q_{-y} together with the generators of translations cannot form a subalgebra of the Poincare algebra. Consequently, we show that the investigation of the Killing equation for the VSR metric (12) could account for the $E(2)$, $HOM(2)$ and $SIM(2)$ ($DISIM(2)$) scenarios of the VSR.

The geodesic equation for the VSR metric is given as [21]

$$\frac{\mathrm{d}x^2}{\mathrm{d}\tau^2}=0. \tag{22}$$

It implies that the motion of the particle is a straight line with constant speed. The relativistic momentum could be defined as

$$p^\mu=m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau},$$

where m is the mass of the particle. In the four VSR scenarios, only the $SIM(2)$ scenario could preserve the CPT symmetry. Here, we just focus the $SIM(2)$ scenario, for it is easier to be tested than other scenario by high energy experiment.

The Lagrangian for VSR metric is given as

$$\mathcal{L}=mF=m(\eta_{\mu\nu}y^\mu y^\nu)^{(1-n)/2}(b_\rho y^\rho)^n. \tag{23}$$

The corresponding dispersion relation is of the form

$$\eta^{\mu\nu}p_\mu p_\nu=m^2(1-b^2)\left(\frac{b^\rho p_\rho}{m(1-b)}\right)^{2b/(1+b)}. \tag{24}$$

The dispersion relation (24) is not Lorentz-invariant, but it is invariant under the transformations of the $DISIM(2)$ group. Ref. [23] shows that the ether-drift experiment gives a constraint $|n|<10^{-10}$ for the parameter n of the VSR metric (12). In the $SIM(2)$ scenario, the b_μ is set to be $b_\mu=\{0,0,0,1\}$ in the light cone coordinate. Then, deducing from the equation $F=0$, we get $c=1$. Thus, the speed of light in the $SIM(2)$ ($DISIM(2)$) scenario of the VSR is the same with Einstein's special relativity.

4 Conclusion

In this paper, we have presented an explicit relation between the isometric group of a specific Finsler space and symmetries of the VSR proposed by Cohen and Glashow (12). If the spacetime does not have a preferred direction, we have shown that the generators induced by Killing vectors are just isomorphic to the $E(2)$ group or $HOM(2)$ group semidirect the spacetime translation. While one chose the null direction to be the preferred direction, it is shown that the symmetry of the space is isomorphic to the $DISIM(2)$ group proposed by Gibbons et al. [10]. Only the $SIM(2)$ scenario of VSR preserves the CPT symmetry. The kinematic in the $DISIM(2)$ group was investigated, and a corresponding dispersion relation was obtained.

We would like to thank Prof. C. J. Zhu for useful discussions.

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