Critical temperature and condensed fraction of Bose-Einstein condensation in an external potential

ZHANG Zhan-Jun(张占军)[1] MA Qing-Shan(马青山)[2,1] ZHANG Jun(张军)[1;1)]

[1] College of Physics Science and Technology, Xinjiang University, Urumqi 830046, China
[2] Department of Physics, Xi'an Jiaotong University, Xi'an 710049, China

Abstract: Based on Bose-Einstein condensation at minimized momentum state, we get the expressions for the critical temperature and condensed fraction of Bose-Einstein condensation (BEC) in an external potential in the three-dimensional (3D) case. For the 1D and 2D cases, we present not only the critical temperature and corresponding particles but also the condition of BEC occurrence.

Key words: Bose-Einstein condensate (BEC), critical temperature, minimum momentum state, ideal Bose gas

PACS: 05.30.Jp, 65.90.+i **DOI:** 10.1088/1674-1137/35/8/005

1 Introduction

In 1924, Bose and Einstein predicted the theory of Bose-Einstein condensation phenomena (BEC). Since then, a lot of theoretical and experimental research on Bose-Einstein Condensation (BEC) has been carried out [1–7]. Alkali Metal BEC was first achieved in 1995 [8–11]. In the experiment [12–15], BEC was achieved often with external potentials.

First, Vanderlei Bangnato [16] presented the theoretical results for the critical temperature for BEC, condensate fraction, and heat capacity of a gas of Bose particles that were confined by a generic power-law potential trap. All three of these quantities were found to vary clearly with the shape of the potential. Later, Mo Ying [17] showed the theoretical results for the critical temperature for BEC that were different from the published results in the literature. These results show the relationship between the critical temperature and the external potentials, and they are based on the minimized momentum state. By comparison, it is significant to study the relationship between the external potential and the critical temperature using the method of the paper [17]. Therefore, we take advantage of the method to study the critical temperature and condensed fraction of BEC in different dimensions with external potential field.

2 Critical temperature in general potential field

Considering the ideal Bose gas placed in a potential $V(r)$, with the local density approximation (LDA) [18], the density of states in this case can be expressed as

$$\begin{aligned} D(\varepsilon) &= \frac{2\pi}{h^3}(2M)^{3/2}(\varepsilon - V(r))^{1/2}, \\ n_\varepsilon &= g_\varepsilon \frac{1}{\mathrm{e}^{\frac{\varepsilon-\mu}{k_\mathrm{B}T}}-1}, \end{aligned} \tag{1}$$

where n_ε denotes the occupation of particles with energy ε, g_ε degeneracy, μ the chemical potential, h the planck constant, T the temperature of the Bose gas system, and k the Boltzmann constant. Therefore, the total particle number can be expressed as

$$\begin{aligned} N &= N_0 + \int_{V(r)}^{\infty} D(\varepsilon) n_\varepsilon \mathrm{d}\varepsilon \\ &= N_0 + \frac{(2\pi M)^{3/2}}{h^3} g_\varepsilon (k_\mathrm{B}T)^{3/2} \sum_{l=1}^{\infty} l^{-3/2} \mathrm{e}^{-l\frac{V(r)-\mu}{k_\mathrm{B}T}}, \end{aligned} \tag{2}$$

with $V(r) = \varepsilon_1 \left|\frac{x}{a}\right|^p + \varepsilon_2 \left|\frac{y}{b}\right|^l + \varepsilon_3 \left|\frac{z}{c}\right|^q$, then Eq. (2) becomes

$$N = N_0 + \frac{(2\pi M)^{3/2}}{h^3} g_\varepsilon \frac{abc}{plq} \frac{(k_\mathrm{B}T)^{\eta+1}}{\varepsilon_1^{1/P} \varepsilon_2^{1/l} \varepsilon_3^{1/q}}$$

Received 25 October 2010, Revised 5 November 2010
1) Corresponding author. E-mail: zhj@xju.edu.cn

$$\times\Gamma(1/p)\Gamma(1/l)\Gamma(1/q)\times\left(\sum_{l=1}^{\infty}l^{-\eta-1}\mathrm{e}^{l\frac{\mu}{k_{\mathrm{B}}T}}\right), \quad (3)$$

where

$$\Gamma(1/k)=\int_0^{\infty}\theta^{\frac{1}{k}-1}\mathrm{e}^{-\theta}\mathrm{d}\theta, \quad k=(p,l,q),\ \eta=\frac{1}{2}+\frac{1}{p}+\frac{1}{l}+\frac{1}{q}. \quad (4)$$

For the three-dimensional (3D) optical lattice potential well [19, 20]

$$V(r)=V_0(\sin^2 kx+\sin^2 ky+\sin^2 kz). \quad (5)$$

Here it can approximate that [19, 20]

$$V(r)\approx V_0((kx)^2+(ky)^2+(kz)^2),\ k=\frac{2\pi}{\lambda}, \quad (6)$$

where k is the wave vector and the total atom number is

$$N = N_0+\frac{(2\pi M)^{3/2}}{h^3}g_{\varepsilon}(k_{\mathrm{B}}T)^{3/2}\times\sum_{l=1}^{\infty}l^{-3/2}\left(\frac{\sqrt{\pi}}{2\alpha^{1/2}}\right)^3\mathrm{e}^{l\mu/k_{\mathrm{B}}T},\ \alpha=\frac{lV_0k^2}{k_{\mathrm{B}}T}. \quad (7)$$

The integral still diverges for $\mu>0$, but it is the maximum for $\mu=0$. In the ideal bose gas, by taking $\mu=0$ and $N_0=0$, we get an equation to determine the critical temperature and the condensation fraction respectively

$$N = \frac{(2\pi M)^{3/2}}{h^3}g_{\varepsilon}(k_{\mathrm{B}}T)^{3/2}\sum_{l=1}^{\infty}l^{-3/2}\left(\frac{\sqrt{\pi}}{2\alpha^{1/2}}\right)^3, \quad (8)$$

$$T_{\mathrm{C}} = \frac{1}{k_{\mathrm{B}}}\left[\frac{Nh^3(V_0k^2)^{3/2}}{(2\pi M)^{3/2}g_{\varepsilon}\left(\frac{\sqrt{\pi}}{2}\right)^3\sum_{l=1}^{\infty}l^{-3}}\right]^{1/3}. \quad (9)$$

BEC under low dimension.

In a free potential field, BEC does not occur in the 1D and 2D cases, but it may occur to BEC in the external potential field. The following discussion is that BEC might happen in the 1D and 2D potential field of power function.

The 1D Bose gas density of quantum states can be expressed as [21]

$$D(\varepsilon)=\frac{\sqrt{2M}}{h}\int_{-l(\varepsilon)}^{l(\varepsilon)}\frac{\mathrm{d}x}{\sqrt{\varepsilon-V(r)}},\ l(\varepsilon)=L\left(\frac{\varepsilon}{V_0}\right)^{1/\eta}. \quad (10)$$

Suppose the external potential field $V(x)=V_0\left(\frac{x}{L}\right)^{\eta}$, then we can state that

$$D(\varepsilon)=\frac{\sqrt{2M}}{h}L\left(\frac{\varepsilon^{1/\eta-1/2}}{V_0^{1/\eta}}\right)F(\eta), \quad F(\eta)=\int_{-1}^{1}\frac{y^{\frac{1-\eta}{\eta}}}{\sqrt{1-y}}\mathrm{d}y, y=\frac{V_0}{\varepsilon}\left(\frac{x}{L}\right)^{\eta}. \quad (11)$$

The logarithm of the giant partition function can be expressed as

$$\begin{aligned}\ln\Xi &= -\sum\omega_i\ln[1-\mathrm{e}^{\frac{\mu-\varepsilon_i}{k_{\mathrm{B}}T}}]\\ &= -\int_{V(x)}^{\infty}D(\varepsilon)\ln[1-\mathrm{e}^{\frac{\mu-\varepsilon_i}{k_{\mathrm{B}}T}}]\mathrm{d}\varepsilon\\ &= -\frac{\sqrt{2M}}{h}L\frac{F(\eta)(K_{\mathrm{B}}T)^{1/\eta-1/2}}{V_0^{1/\eta}}\\ &\quad\times\int_r^{\infty}x'^{1/\eta-1/2}\ln[1-z\mathrm{e}^{-x'}]\mathrm{d}x',\end{aligned} \quad (12)$$

where

$$r=\frac{V(x)}{K_{\mathrm{B}}T},\ x'=\frac{\varepsilon}{K_{\mathrm{B}}T},\ z=\mathrm{e}^{\frac{\mu}{K_{\mathrm{B}}T}}. \quad (13)$$

For convenience, we only discuss the above infernal part,

$$\begin{aligned}&\int_r^{\infty}x'^{1/\eta-1/2}\ln[1-z\mathrm{e}^{x'}]\mathrm{d}x'\\ &= \frac{1}{1/\eta+1/2}\left[\ln[1-z\mathrm{e}^{-x'}]x'^{1/\eta+1/2}|_r^{\infty}-\int_r^{\infty}x'^{1/\eta+1/2}\mathrm{d}(\ln[1-z\mathrm{e}^{-x'}])\right]\\ &= \frac{1}{1/\eta+1/2}\left[-\int_r^{\infty}\frac{x'^{1/\eta+1/2}z\mathrm{e}^{-x'}}{1-z\mathrm{e}^{-x'}}\right]\mathrm{d}x',\end{aligned} \quad (14)$$

when $x'=r$ and if and only if $V(x)\gg k_{\mathrm{B}}T$, this part is equal to zero. For the ideal Bose gas, we get $0<z\leqslant 1$ because of $\mu\leqslant 0$, consequently,

$$\frac{z\mathrm{e}^{-x'}}{1-z\mathrm{e}^{-x'}}=\sum_{l=1}^{\infty}(z\mathrm{e}^{-x'})^l. \quad (15)$$

Eq. (14) can be rewritten as

$$\begin{aligned}&\frac{1}{1/\eta+1/2}\left[-\int_r^{\infty}x'^{1/\eta+1/2}\sum_{l=1}^{\infty}(z\mathrm{e}^{-x'})^l\mathrm{d}x'\right]\\ &= \frac{1}{1/\eta+1/2}\sum_{l=1}^{\infty}\int_r^{\infty}\mathrm{e}^{-lx'}(lx')^{1/\eta+1/2}z^l\mathrm{d}(lx')\\ &\quad\times\frac{1}{l^{1/\eta+1/2+1}}=\frac{1}{1/\eta+1/2}\Gamma'(m+1)g_{m+1}(z).\end{aligned} \quad (16)$$

Let:

$$m=1/\eta+1/2,\ \sum_{l=1}^{\infty}\frac{z^l}{l^{1/\eta+1/2+1}}=g_{m+1}(z). \quad (17)$$

This function converges for $z<1$, for z=1

$$g_m(1)=\sum_{l=1}^{\infty}\frac{1}{l^m}=\xi(m),\ \frac{\partial g_{m+1}(z)}{\partial z}=\frac{1}{z}g_m(z). \quad (18)$$

Therefore for $T > T_c$,

$$N = -\frac{\partial \ln \Xi}{\partial \alpha} = \frac{\sqrt{2M}}{h} L \frac{F(\eta)(K_B T)^{1/\eta+1/2}}{V_0^{1/\eta}} \times \frac{1}{1/\eta+1/2} g_m(z) \Gamma'(m+1), \quad (19)$$

$$T_c = \frac{1}{K_B}\left[\frac{N(1/\eta+1/2)V_0^{1/\eta}}{\frac{\sqrt{2M}}{h} L F(\eta) g_m(z) \Gamma'(m+1)}\right]^{\frac{1}{1/\eta+1/2}}. \quad (20)$$

For $\mu=0$, z=1, that is to say $T < T_c$, $g_m(1)$ converges for $m > 1$ and N is a limited value. Therefore, BEC can happen. Calling for $m = \frac{1}{\eta} + \frac{1}{2} > 1$, namely $\eta < 2$.

The 2D Bose gas density of quantum states can be expressed as

$$D(\varepsilon) = \frac{2\pi M}{h^2} \int_0^{r'} 2\pi r \mathrm{d}r. \quad (21)$$

Suppose the external potential field

$$V = V_0 r^\eta, \ r' = \left(\frac{\varepsilon}{V_0}\right)^{1/\eta},$$

Eq. (21) can be rewritten as

$$D(\varepsilon) = \frac{2M\pi^2}{h^2}\left(\frac{\varepsilon}{V_0}\right)^{2/\eta}. \quad (22)$$

The logarithm of the giant partition function can be expressed as

$$\ln \Xi = -\sum \omega_i \ln[1 - z\mathrm{e}^{-\beta\mu}] = -\frac{2M\pi^2}{h^2}\int_V^\infty \left(\frac{\varepsilon}{V_0}\right)^{2/\eta} \ln[1 - z\mathrm{e}^{-\beta\mu}] \mathrm{d}\varepsilon, \quad (23)$$

where

$$\beta = \frac{1}{k_B T}, \ z = \mathrm{e}^{\beta\mu}. \quad (24)$$

We get an equation to determine the critical temperature and the condensation fraction respectively by (19) and (20).

$$N = \frac{2\pi M}{h^2} \frac{1}{(2/\eta)+1} \frac{1}{V_0^{2/\eta}} \frac{1}{\beta^{(2/\eta)+1}} g_m(z) \Gamma'(m+1), \quad (25)$$

$$T_c = \frac{1}{k_B}\left[\frac{N}{\frac{2\pi M}{h^2} \frac{1}{(2/\eta)+1} \frac{1}{V_0^{2/\eta}} g_m(z) \Gamma'(m+1)}\right]^{\frac{1}{\frac{2}{\eta}+1}}, \quad (26)$$

when $T < T_c$, for $\mu \to 0$, $z \to 1$,

$$g_{m+1}(z) = \xi_{m+1}(1) = \sum_{l=1}^{\infty} \frac{1}{l^{\frac{2}{\eta}+1+1}}, \ m = \frac{2}{\eta}+1. \quad (27)$$

The function is divergent for any $\eta > 0$, at the same time, N is a limited value. BEC can occur for any $\eta > 0$.

3 Conclusion

Based on the Bose-Einstein condensation at minimized momentum state, we get expressions for the critical temperature and condensed fraction of Bose-Einstein condensation (BEC) in an external potential, taking the optical lattice potential well, for example, we present the critical temperature and the condensation fraction for three-dimensional optical lattice potential. What is more, we get the condition for occurring to BEC in the low-dimension and critical temperature and condensed fraction of Bose-Einstein condensation in an external potential.

References

1 David E P. Phys. Rev. Lett.,1983, **51**(15): 1336–1339

2 Wolfgang K, Davis K B, Michanel A J et al. Phys. Rev. Lett., 1993, **70**(15): 2253–2256

3 YU K, Shlyapnikov G V, Walraven J M. Phys. Rev. Lett., 1996, **76**(15): 2670

4 Dos F P, Léonard J W. Phys. Rev. Lett., 2001, **86**(16): 3459–3462

5 ZHANG S M, YE F. Phys. Sin., 1999, **48**: 997

6 WANG D Z, LU X H, HUANG H et al. Phys. Sin., 1999, **48**: 997

7 YI H H. Phys. Sin., 1999, **48**: 995

8 Anderson B P, Kasevich M A. Maroscopic. Science, 1998, **282**(5394): 1686–1689

9 Cataliotti F S, Burger S, Fort C et al. Science, 2001, **293**: 843-847

10 Greiner M, Mandel O, T W, Hänsch T W et al. Nature, 2002, **419**: 51–54

11 Ditmire T, Gumbrell E T, Smith R A et al. Phys. Rev. Lett., 1996, **77**: 498–501

12 Alan L M, John V P, Philips W M et al. Phys. Rev. Lett., 1985, **54**(24): 2596–2599

13 Raab E L, Prentiss M, Alex C et al. Phys. Rev. Lett., 1987, **59**(23): 2631–2634

14 Monroe C, Swann W, Robinson H et al. Phys. Rev. Lett., 1990, **65**(13): 1571–1574

15 Ketterle W, Davis K B, Joffe M A et al. Phys. Rev. Lett., 1993, **70**(15): 2253–2256

16 Bangato V. Phys. Rev. A, 1987, **35**: 4254

17 YU X C, MO Y. Acta Phys. Sin., 2004, **53**: 4075

18 Oliva J. Phys. Rev. B., 1989, **39**(7): 4197–4203

19 CHEN B L, HUANG X B, KOU S P. Phys. Rev. A., 2008, **78**(4): 043603

20 YU X C, HUANG X YE Y T. Science in China G., 2007, **50**(2): 616–623

21 LI X, LIU W S, ZHANG G F. Acta Sinica Quantum Optica., 2003, **9**(1): 27–31