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An approximate solution of the DKP equation under the Hulthén vector potential

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Abstract: Using the analytical NU technique as well as an acceptable physical approximation to the centrifugal term, the bound-state solutions of the Duffin-Kemmer-Petiau equation are obtained for arbitrary quantum numbers. The solutions appear in terms of the Jacobi Polynomials. Various explanatory figures and tables are included to complete the study.

Key words: Duffin-Kemmer-Petiau equation, Hulthén potential, NU technique

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1 Introduction

Calculation of eigenfunctions and the corresponding eigenvalues in many physical sciences is the first step in performing research. So many studies have been carried out on the wave equations of quantum mechanics including the non-relativistic Schrödinger equation and, relativistic Dirac and Klein-Gordon (KG) equations. On the contrary, the Duffin-Kemmer-Petiau (DKP) equation, which is capable of investigating either spin-zero or spin-one particles in a unified basis, has been studied only by a few authors. This equation has been successful in describing high energy interactions of hadrons with nuclei and other branches of physics [1–4]. Nevertheless, we are not sure about the equivalence or nonequivalence of the DKP equation with its counterparts, i.e. KG of Proca equations [5–11]. There are interesting papers which discuss various aspects of the equation and solve it under a variety of potentials including the Coulomb, linear, Harmonic, Hulthén, et al. [12–34]. Here, we intend to work on the Hulthén potential which yields notable results in particle, nuclear, atomic and condensed matter physics [35–38].

2 The DKP equation

The DKP Hamiltonian for scalar and vector interactions is

$$(\beta\cdot\vec{p}c+mc^2+U_{\rm s}+\beta^0U_{\rm v}^{\rm o})\psi(\vec{r})=\beta^0E\psi(\vec{r}), \tag{1}$$

where

$$\psi(\vec{r})\ =\begin{pmatrix}\psi_{\rm upper}\\ {\rm i}\psi_{\rm lower}\end{pmatrix}, \tag{2}$$

$$\beta^0\ =\begin{pmatrix}\theta & \tilde{0}\\ \bar{0}_{\rm T} & 0\end{pmatrix},\ \beta^i=\begin{pmatrix}\tilde{0} & \rho^i\\ -\rho^i_{\rm T} & 0\end{pmatrix},$$

with $\tilde{0}$, $\bar{0}$ and 0 respectively being 2×2, 2×3, 3×3 zero matrices

$$\theta\ =\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \rho^1=\begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \rho^2=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\rho^3\ =\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}.$$

The upper and lower components respectively have the form

$$\psi_{\rm upper}\equiv\begin{pmatrix}\phi\\ \varphi\end{pmatrix}, \tag{3}$$

$$\psi_{\rm lower}\equiv\begin{pmatrix}A_1\\ A_2\\ A_3\end{pmatrix}, \tag{4}$$

where $U_{\rm s}$, $U_{\rm v}^{\rm o}$ respectively represent the scalar and vector interactions. The equation, in (3+0)-dimensions, is written as [1–4]

$$\begin{aligned}(mc^2+U_{\rm s})\phi&=(E-U_{\rm v}^{\rm o})\varphi+\hbar c\vec{\nabla}\cdot\vec{A},\\ \vec{\nabla}\phi&=(mc^2+U_{\rm s})\vec{A},\\ (mc^2+U_{\rm s})\varphi&=(E-U_{\rm v}^{\rm o})\phi,\end{aligned} \tag{5}$$

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where $\vec{A}=(A_1,\ A_2,\ A_3)$. In Eq. (3) ψ is a simultaneous eigenfunction of J^2 and J_3, i.e.

$$J^2\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}=\begin{pmatrix}L^2\psi_{\text{upper}}\\ (L+S)^2\psi_{\text{lower}}\end{pmatrix}=J(J+1)\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},$$
$$J_3\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix}=\begin{pmatrix}L_3\psi_{\text{upper}}\\ (L_3+s_3)\psi_{\text{lower}}\end{pmatrix}=M\begin{pmatrix}\psi_{\text{upper}}\\ \psi_{\text{lower}}\end{pmatrix},\tag{6}$$

and the general solution is considered as

$$\psi_{JM}(r)=\begin{pmatrix}f_{nJ}(r)Y_{JM}(\Omega)\\ g_{nJ}(r)Y_{JM}(\Omega)\\ \mathrm{i}\sum_L h_{nJL}(r)Y^M_{JL1}(\Omega)\end{pmatrix},\tag{7}$$

where spherical harmonics $Y_{JM}(\Omega)$ are of order J, $Y^M_{JL1}(\Omega)$ represent the normalized vector spherical harmonics and f_{nJ}, g_{nJ} and h_{nJL} stand for the radial wavefunctions. What is mentioned above leads to [26]

$$\begin{aligned}&(E_{n,J}-U^0_{\text{v}})F_{n,J}(r)\\ &=(mc^2+U_{\text{s}})G_{n,J}(r),\\ &\left(\frac{\mathrm{d}F_{n,J}(r)}{\mathrm{d}r}-\frac{J+1}{r}F_{n,J}(r)\right)\\ &=-\frac{1}{\alpha_J}(mc^2+U_{\text{s}})H_{1,n,J}(r),\\ &\left(\frac{\mathrm{d}F_{n,J}(r)}{\mathrm{d}r}+\frac{J}{r}F_{n,J}(r)\right)\\ &=\frac{1}{\zeta_J}(mc^2+U_{\text{s}})H_{-1,n,J}(r),\\ &-\alpha_J\left(\frac{\mathrm{d}H_{1,n,J}(r)}{\mathrm{d}r}+\frac{J+1}{r}H_{1,n,J}(r)\right)\\ &+\zeta\left(\frac{\mathrm{d}H_{-1,n,J}(r)}{\mathrm{d}r}-\frac{J}{r}H_{-1,n,J}(r)\right)\\ &=\frac{1}{\hbar c}((mc^2+U_{\text{s}})F_{n,J}(r)-(E_{n,J}-U^0_{\text{v}})G_{n,J}(r)),\end{aligned}\tag{8}$$

which give

$$\begin{aligned}&\frac{\mathrm{d}^2F_{n,J}(r)}{\mathrm{d}r^2}\left[1+\frac{\zeta_J^2}{\alpha_J^2}\right]-\frac{\mathrm{d}F_{n,J}(r)}{\mathrm{d}r}\left[\frac{U'_{\text{s}}}{(m+U_{\text{s}})}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)\right]\\ &+F_{n,J}(r)\left[-\frac{J(J+1)}{r^2}\left(1+\frac{\zeta_J^2}{\alpha_J^2}\right)\right.\\ &+\frac{U'_{\text{s}}}{(m+U_{\text{s}})}\left(\frac{J+1}{r}-\frac{\zeta_J^2}{\alpha_J^2}\frac{J}{r}\right)\\ &\left.-\frac{1}{\alpha_J^2}((m+U_{\text{s}})^2-(E_{n,J}-U^0_{\text{v}})^2)\right]=0,\end{aligned}\tag{9}$$

where $\alpha_J=\sqrt{(J+1)/(2J+1)}$ and $\zeta_J=\sqrt{J/(2J+1)}$. When $U_{\text{s}}=0$, we arrive at [26]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{J(J+1)}{r^2}+(E_{n,J}-U^0_{\text{v}})^2-m^2\right)F_{n,J}(r)=0.\tag{10}$$

Substitution of the Hulthén interaction $-\dfrac{V_0}{\mathrm{e}^{\alpha r}-1}$ in Eq. (10) gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{J(J+1)}{r^2}+\left(E_{n,J}+\frac{V_0}{\mathrm{e}^{\alpha r}-1}\right)^2-m^2\right)F_{n,J}(r)=0.\tag{11}$$

This potential behaves like a Coulomb potential for small values of r and decreases exponentially for large values without having the problems of the former. Before proceeding further, it should be noted that the Hulthén potential appears in different notations. For example, in atomic physics, we represent the potential as $V(r)=-Ze^2\lambda\dfrac{\mathrm{e}^{-\lambda r}}{1-\mathrm{e}^{-\lambda r}}$, where λ is the screening parameter and Z is a constant identified with the atomic number. Now, substituting the approximation [39]

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(\mathrm{e}^{\alpha r}-1)^2},\tag{12}$$

Eq. (11) yields

$$\begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\alpha^2J(J+1)}{(\mathrm{e}^{\alpha r}-1)^2}+E^2_{n,J}+\frac{2V_0E_{n,J}}{\mathrm{e}^{\alpha r}-1}\right.\\ &\left.+\frac{V_0^2}{(\mathrm{e}^{\alpha r}-1)^2}-m^2\right)F_{n,J}(r)=0.\end{aligned}\tag{13}$$

Next, we apply the transformation $z=\mathrm{e}^{-\alpha r}$ to bring Eq. (13) into the form

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1-z}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{z^2(1-z)^2}\left\{z^2\left[\frac{V_0^2-\alpha^2J(J+1)}{\alpha^2}\right.\right.\\ &\left.-\frac{2E_{n,J}V_0}{\alpha^2}+\frac{E^2_{n,J}-m^2}{\alpha^2}\right]+z\left[\frac{2E_{n,J}V_0}{\alpha^2}-\frac{2(E^2_{n,J}-m^2)}{\alpha^2}\right]\\ &\left.+\left(\frac{E^2_{n,J}}{\alpha^2}-\frac{m^2}{\alpha^2}\right)\right\}F_{n,J}(z)=0.\end{aligned}\tag{14}$$

3 The NU Technique

NU technique in its parametric form simply solves a differential equation of the form [40, 41]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\alpha_1-\alpha_2s}{s(1-\alpha_3s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{-\xi_1s^2+\xi_2s-\xi_3}{[s(1-\alpha_3s)]^2}\right]\psi(s)=0.\tag{15}$$

By comparing Eqs. (14) and (15), we obtain

$$\xi_1=-\frac{V_0^2-\alpha^2 J(J+1)}{\alpha^2}+\frac{2E_{n,J}V_0}{\alpha^2}-\frac{E_{n,J}^2-m^2}{\alpha^2}, \quad (16a)$$

$$\xi_2=\frac{2E_{n,J}V_0}{\alpha^2}-\frac{2(E_{n,J}^2-m^2)}{\alpha^2}, \quad (16b)$$

$$\xi_3=-\frac{E_{n,J}^2}{\alpha^2}+\frac{m^2}{\alpha^2}, \quad (16c)$$

$$\alpha_1=\alpha_2=\alpha_3=1. \quad (16d)$$

In the NU method, the energy eigenvalues satisfy

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3$$
$$+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \quad (17)$$

and the eigenfunctions are

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}p_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \quad (18)$$

where

$$\alpha_4=\frac{1}{2}(1-\alpha_1),\ \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),$$
$$\alpha_6=\alpha_5^2+\xi_1,\ \alpha_7=2\alpha_4\alpha_5-\xi_2, \quad (19a)$$
$$\alpha_8=\alpha_4^2+\xi_3, \quad (19b)$$
$$\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6, \quad (19c)$$
$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}, \quad (19d)$$
$$\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}), \quad (19e)$$
$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8}, \quad (19f)$$
$$\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}), \quad (19g)$$

and $p_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s)$ is the Jacobi polynomial.

4 The solution of the problem

Eqs. (14) and (15) immediately determine

$$\alpha_4=0,\ \alpha_5=-\frac{1}{2},\ \alpha_6=\frac{1}{4}+\xi_1,\ \alpha_7=-\xi_2,\ \alpha_8=\xi_3, \quad (20a)$$

$$\alpha_9=\frac{1}{4}+\xi_1+\xi_3-\xi_2,\ \alpha_{10}=1+2\sqrt{\xi_3}, \quad (20b)$$

$$\alpha_{11}=2+2\left(\sqrt{\frac{1}{4}+\xi_1+\xi_3-\xi_2}+\sqrt{\xi_3}\right),\ \alpha_{12}=\sqrt{\xi_3}, \quad (20c)$$

$$\alpha_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\xi_1+\xi_3-\xi_2}+\sqrt{\xi_3}\right). \quad (20d)$$

Thus, from Eqs. (17) and (18), the energy eigenvalues and eigenfunctions are written as

$$n+\frac{1}{2}(2n+1)+(2n+1)\left(\sqrt{\frac{1}{4}+\xi_1+\xi_3-\xi_2}+\sqrt{\xi_3}\right)$$
$$+n(n-1)-\xi_2+2\xi_3+2\sqrt{\xi_3\left(\frac{1}{4}+\xi_1+\xi_3-\xi_2\right)}=0, \quad (21)$$

$$F_{n,J}(r)=\exp(-\alpha\alpha_{12}r)(1-\exp(-\alpha r))^{-\alpha_{12}-\alpha_{13}}$$
$$\times P_n^{(\alpha_{10}-1,\alpha_{11}-\alpha_{10}-1)}(1-2\exp(-\alpha r)). \quad (22)$$

We have reported some numerical results in Table 1 for $m=10$, $V_0=0.05$, $\alpha=0.1$.

Table 1. Energy eigenvalues for different states.

$\vert n,J\rangle$	$E_{n,J}$	$\vert n,J\rangle$	$E_{n,J}$
$\vert 0,0\rangle$	7.096023618	$\vert 2,0\rangle$	9.830010003
$\vert 0,1\rangle$	9.687495014	$\vert 2,1\rangle$	9.939720311
$\vert 0,2\rangle$	9.867457411	$\vert 2,2\rangle$	9.966533865
$\vert 0,3\rangle$	9.927476105	$\vert 2,3\rangle$	9.979694256
$\vert 0,4\rangle$	9.954672859	$\vert 2,4\rangle$	9.987025400
$\vert 0,5\rangle$	9.969225011	$\vert 2,5\rangle$	9.991432557
$\vert 1,0\rangle$	9.511536512	$\vert 3,0\rangle$	9.922948020
$\vert 1,1\rangle$	9.874625818	$\vert 3,1\rangle$	9.969030744
$\vert 1,2\rangle$	9.935590908	$\vert 3,2\rangle$	9.982591989
$\vert 1,3\rangle$	9.961774549	$\vert 3,3\rangle$	9.989716791
$\vert 1,4\rangle$	9.975336341	$\vert 3,4\rangle$	9.993800843
$\vert 1,5\rangle$	9.983180686	$\vert 3,5\rangle$	9.996256663

In Figs. 1 and 2 we have plotted $E_{n,l}$ versus α and V_0 for some values of n, J respectively.

Figure 3 represents the eigenfunctions for some values of n, J.

As a typical example, let us check the energy relation for $\eta_c(1S)$.

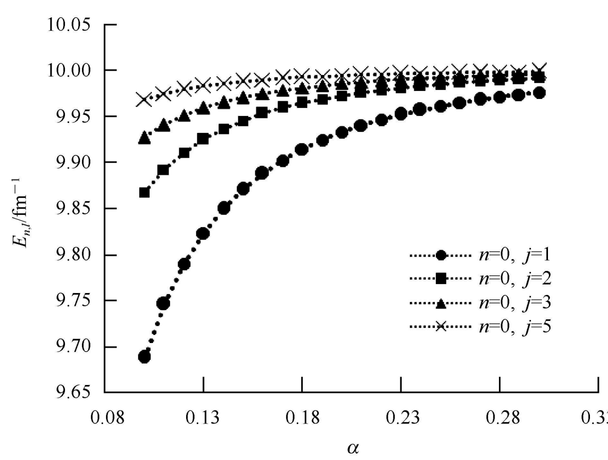

Fig. 1. $E_{n,l}$ Vs. α for $m=10$, $V_0=0.05$.

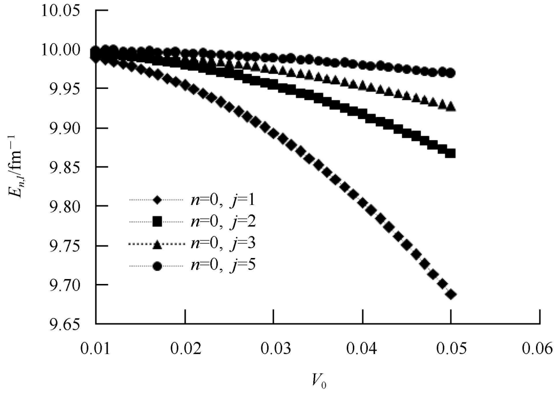

Fig. 2. $E_{n,l}$ Vs. V_0 for m=10, α=0.1.

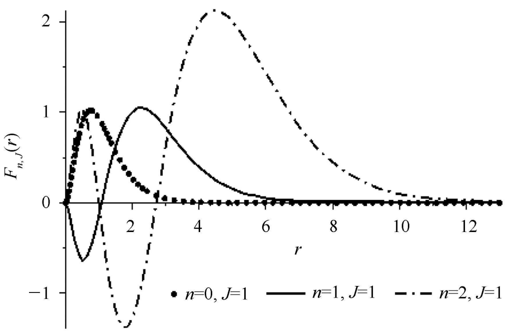

Fig. 3. The wavefunction of the system for some values of n, J, m=10, V_0=0.05, α=0.1.

Choosing $\alpha = \dfrac{1}{2.5}$ fm^{-1}, V_0 = 0.09 fm and m_c = 3 GeV, we obtain $m_{\text{theo.}}$=2988.13 MeV which is in acceptable agreement with its experimental value $m_{\text{exp.}}$= 2980.4 MeV. Nevertheless, we should bear in mind that the Hulthén potential does not include a confining term and therefore the difference looks logical. A simple glance at the energies reveals that the energy difference between the levels decreases for increasing principal quantum number, for example,

$$
\begin{aligned}
E_{0,5}-E_{0,4} &= 0.014552152 \text{ fm}^{-1},\\
E_{1,5}-E_{1,4} &= 0.007844345 \text{ fm}^{-1},\\
E_{2,5}-E_{2,4} &= 0.004407157 \text{ fm}^{-1},\\
E_{3,5}-E_{3,4} &= 0.00245582 \text{ fm}^{-1},
\end{aligned}
$$

which are consistent with our knowledge of quantum mechanics.

5 Conclusion

After using an approximation for the centrifugal term, we have solved the DKP equation under the vector Hulthén potential. We have reported the energy spectra for various quantum numbers and the behavior of the energy spectra versus some parameters is also represented. The results are definitely useful in a wide range of physical problems from meson spectroscopy to cosmology.

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