Analytical determination of optimal luminosity for τ mass scan *

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Abstract: Resorting to Hessian matrix, the analytical formula is obtained to determine the optimal luminosity proportion for the experiment of τ mass scan. Comparison of numerical results indicate the consistency between the present analytical evaluation and the previous computation based on the sampling technique.

Key words: τ mass, statistical optimization, luminosity determination

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1 Introduction

The mass of τ lepton (m_τ) is one of the elementary parameters in the Standard Model (SM), and its accuracy is of great importance for testing and improving SM. However, the accuracy of m_τ is merely at the level of 10^{-4} for the time being, which is about four orders of magnitude lower than that of the electron and muon. Therefore, the precise measurement of m_τ is an important work for τ physics. Two methods are usually employed to measure m_τ: the pseudomass technique [1, 2] and the threshold scan method. The former depends on the reconstruction of the invariant mass and energy of the hadronic τ decay final states, while the latter depends on the good understanding of the production cross section near threshold.

The most accurate measurement of m_τ was first obtained by BES Collaboration [3–5] two decades ago by adopting the threshold scan method. With the upgraded detector BESIII [6], large τ data are expected and a more precise measurement of m_τ could be achieved with the development of experimental techniques [7] and theoretical calculations [8–11].

For the given data, the taking time (equivalently the limited total luminosity), the data taking points and the luminosity for each point should be optimized before the data taking process. Conventionally, the Monte Carlo simulation and sampling technique are used to simulate various data taking cases and luminosity distribution [12, 13]. This method can acquire a comparatively optimal scheme, but at the same time it consumes a large amount of computing time. In this paper, an analytical formula is obtained to determine the optimal proportion of luminosity for a certain energy point distribution.

2 Methodology

Suppose in the scan experiment, a total of $N_{\rm pt}$ points need to be taken in the vicinity of m_τ threshold. To analyze the scan data, the following χ^2 function is constructed:

$$\chi^2=\sum_{k=1}^{N_{\rm pt}}\left(\frac{N_k-\mu_k}{\Delta N_k}\right)^2, \tag{1}$$

where N_k is the number of events determined by experiment at the k-th energy point, μ_k is the expectation number of events determined by theory, ΔN_k is the experimental error. In the following analysis, only statistical error is concerned, which means $\Delta N_k=\sqrt{N_k}$.

In the scan experiment of m_τ, suppose only eμ final state is concerned. The expectation of event number μ_k is given by

$$\mu_k=[\varepsilon\cdot B_{\rm e\mu}\cdot\sigma_{\rm obs}(m_\tau,E^k_{\rm cm})+\sigma_{\rm BG}]\cdot L_k, \tag{2}$$

where ε is the event selection efficiency for eμ final state, $B_{\rm e\mu}$ is the branching ratio of the final state, $\sigma_{\rm obs}$ is the observed cross section for τ pair production at k-th energy point with energy $E^k_{\rm cm}$, which could be calculated by Voloshin's fomulas [8], $\sigma_{\rm BG}$ is the total cross section of background channels, and L_k is the luminosity at k-th energy point.

In Eq. (2), some parameters (m_τ, ε, $\sigma_{\rm BG}$) need to be determined using experimental data. This could be achieved by minimizing the χ^2 function in Eq. (1). For

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the fitting results of all relevant parameters, what we mostly care about is the uncertainty of m_τ. To obtain the smallest uncertainty of m_τ, many studies have been performed [12–14], and the main conclusions are summarized as follows:

1) One energy point is sufficient for one parameter optimization.

2) The optimal position of the energy point can be determined by the sampling technique [12, 13] or the analytical method [14].

3) The optimal proportion of luminosity can be determined by the sampling technique [13].

In this paper, the first two conclusions are taken for granted, which means if there are n parameters, the energy points to be taken are also n, that is $N_{\rm pt}=n$. Then we will obtain an analytical formula to determine the optimal proportion of luminosity.

3 Luminosity distribution

In this section the summation of index is always from 1 to n. Suppose the total luminosity is L_0, $L_k=x_k\cdot L_0$ is the luminosity at k-th energy point, and x_k denotes the corresponding fraction of luminosity. Now define

$$\sigma_k^* = N_k/L_k, \tag{3}$$

$$\sigma_k = \mu_k/L_k=\varepsilon\cdot B_{\rm e\mu}\cdot\sigma_{\rm obs}(m_\tau,E_{\rm cm}^k)+\sigma_{\rm BG}, \tag{4}$$

as the effective cross sections for experiment and theory, then the χ^2 in Eq. (1) is recast as

$$\chi^2=L_0\sum_{k=1}^{n}x_k\cdot\frac{(\sigma_k^*-\sigma_k)^2}{\sigma_k^*}. \tag{5}$$

The minimization of χ^2 leads to the optimal values of relevant variables. Now we consider the Hessian matrix at the minimization point of χ^2. The element of the Hessian matrix is

$$H_{ij} = \frac{\partial^2\chi^2}{\partial\theta_i\partial\theta_j} \tag{6}$$

$$= 2L_0\sum_{k=1}^{n}\frac{x_k}{\sigma_k}\left[\left(\frac{\partial\sigma_k}{\partial\theta_i}\right)\left(\frac{\partial\sigma_k}{\partial\theta_j}\right) -(\sigma_k^*-\sigma_k)\left(\frac{\partial^2\sigma_k}{\partial\theta_i\partial\theta_j}\right)\right]. \tag{7}$$

The second set of terms in the above equation can be ignored since it is simply the normalized residual [15]. Statistically, the normalized residuals should be small, and scattered randomly around zero in the vicinity of the minimization point of χ^2; hence, on being summed, these terms yield a negligible contribution to the Hessian. By throwing away the second order terms, the Hessian can be expressed compactly in matrix denotation, that is

$$H=2L_0AXA^{\rm T}, \tag{8}$$

with

$$a_{ik}=\frac{\partial\sigma_k}{\partial\theta_i}\ ,\ \ a_{kj}^{\rm T}=a_{jk}\ ,\ \ x_{ij}=\frac{x_i}{\sigma_j}\delta_{ij}\ . \tag{9}$$

The superscript T indicates the matrix transposition, and the δ_{ij} indicates that the matrix X is actually diagonal. The inverse of the Hessian matrix is connected with the error matrix. If suppressing the irrelevant constant, we focus on

$$U=(AXA^{\rm T})^{-1}=(A^{-1})^{\rm T}X^{-1}A^{-1}. \tag{10}$$

If we are solely interested in the error of a special variable, say the first variable x_1, the square of the equivalent error[1)] of variable x_1 is

$$u_{11}=\sum_k\frac{\alpha_{k1}^2\sigma_k}{x_k}, \tag{11}$$

where α_{k1} is the element of matrix A^{-1}, and u_{11} obviously depends on the fraction of luminosity x_i. Noticing the constraint $\sum_i x_i=1$, in order to acquire the optimal values of x_i, we adopt the Lagrange method of multipliers, introduce a new parameter λ, and construct the Lagrange function as follows

$$\mathcal{L}(x_k,\lambda)=u_{11}+\lambda\left(\sum_i x_i\right). \tag{12}$$

The minimization requirement of $\partial\mathcal{L}/\partial x_k=0$ immediately leads to $x_k^2=\alpha_{k1}^2\sigma_k/\lambda$, which readily yields the following relation

$$x_1{:}x_2{:}\cdots{:}x_n=(\alpha_{11}\sqrt{\sigma_1}){:}(\alpha_{21}\sqrt{\sigma_2}){:}\cdots{:}(\alpha_{n1}\sqrt{\sigma_n})\ . \tag{13}$$

Applying our conclusion to the τ mass scan, three indexes 1, 2, 3 correspond to the variables m_τ, ε, $\sigma_{\rm BG}$, respectively; three energy points are [13] E_1=3.5538 GeV, E_2=3.595 GeV, and E_3=3.50 GeV. The optimal fractions of luminosity at these points are x_1=70.0%, x_2=21.8%, and x_3=8.2%. Comparing with the results by the sampling technique [13] x_1=67.5%, x_2=22.5% and x_3=10.0%, two sets of results are consistent with each other fairly well.

In the above calculation, some relevant values are adopted from Ref. [13]: m_τ=1.77699 GeV, $B_{\rm e\mu}$=0.06194, ε=14.2% and $\sigma_{\rm BG}$=0.024 pb.

1) Since U is different from the error matrix only by a constant, the element of U is also different from the error of variable merely by a constant, and we call the element of U the equivalent error.

4 Discussion

The above method is developed by using the least square method. For the maximum likelihood fit, the likelihood function is constructed as

$$\mathrm{LF}=\prod_{k=1}^{n}\frac{\mu_k^{N_k}\mathrm{e}^{-\mu_k}}{N_k!}. \tag{14}$$

Finding the maximum of the likelihood function equals to finding the minimum of the function f defined as

$$f=-\ln\mathrm{LF}=-\sum_{k=1}^{n}\ln\left(\frac{\mu_k^{N_k}\mathrm{e}^{-\mu_k}}{N_k!}\right). \tag{15}$$

Define

$$f_k=\frac{\mu_k^{N_k}\mathrm{e}^{-\mu_k}}{N_k!},$$

the second order derivative of function f is

$$\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}=\sum_{k=1}^{n}\left[\left(\frac{1}{f_k}\frac{\partial f_k}{\partial\theta_i}\right)\left(\frac{1}{f_k}\frac{\partial f_k}{\partial\theta_j}\right)-\frac{1}{f_k}\frac{\partial^2 f_k}{\partial\theta_i\partial\theta_j}\right]. \tag{16}$$

According to the linearization assumption in the Gauss-Newton method [16], the second item in Eq. (16) can be ignored. Since

$$\frac{1}{f_k}\frac{\partial f_k}{\partial\theta_i}=\left(\frac{N_k}{\mu_k}-1\right)\frac{\partial\mu_k}{\partial\theta_i}=\sqrt{\frac{x_kL_0}{\sigma_k}}\frac{\partial\sigma_k}{\partial\theta_i}, \tag{17}$$

then Eq. (16) becomes

$$\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}=L_0\sum_{k=1}^{n}\frac{x_k}{\sigma_k}\left(\frac{\partial\sigma_k}{\partial\theta_i}\right)\left(\frac{\partial\sigma_k}{\partial\theta_j}\right). \tag{18}$$

The Hessian matrix can be expressed as $H=L_0AXA^{\mathrm{T}}$, which is exactly the one we obtained in Eq. (8) except for a constant factor 2. Therefore, the maximum likelihood function is equivalent to the χ^2 function defined in Eq. (1) and all deductions presented in the previous sections are applicable for likelihood maximization.

5 Summary

For multi-parameter optimization fitting, if we aim at the minimization of one parameter error, that is τ mass, for τ scan measurement, the optimal proportion of luminosity can be determined by an analytical formula, which is obtained by virtue of the Hessian matrix. The numerical results from the analytical evaluation agree fairly well with those in the previous paper, which are obtained based on the sampling technique.

Such an analytical result is applicable for both chi-square minimization and likelihood maximization methods. Moreover, the method developed in this paper is more robust, efficient, and time-saving compared with the simulation and sampling technique used before.

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