Solutions of the two-body Salpeter equation under the Coulomb and exponential potential for any l state with Laplace approach

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Abstract: In this paper we have solved the two-body spinless-Salpeter (SS) equation under the Coulomb and exponential type potentials. We have applied an approximation for the centrifugal term in our calculations. The energy eigenvalues and the corresponding eigenfunctions are reported by using the Laplace transform approach for any n, ℓ states.

Key words: spinless-Salpeter equation, Coulomb and exponential potential, Laplace approach

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1 Introduction

The problem of solving the non-relativistic and relativistic wave equations has been attracting much intention in the literature. In relativistic quantum mechanics, the solutions of the Klein-Gordon (KG), Dirac and SS equations with various physical potentials play an important role in nuclear physics and other related areas. Because of the presence of the centrifugal term, the exact solution of the above equations is only possible for some potentials. To solve the SS, KG and Dirac equations many methods have been applied for example, super symmetry quantum mechanics (SUSY) [1], the Nikiforov-Uvarov (NU) method [2], the quantization rules [3], series expansion [4], ansatz method [5, 6], etc.

In the present work we have considered the SS equation which describes the spinless particles. A few authors have considered this equation, for instance in Ref. [7] the SS equation is analytically solved for Hulthén potential after using appropriate approximations with the NU method. To deal with this equation, we have obtained the energy eigenvalues and the corresponding eigenfunctions of the system under the Coulomb and exponential type potentials by using the powerful Laplace transform approach [8, 9].

The Laplace transform is a widely used integral transform with many applications in physics and engineering. Applying this method allows us obtain the exact solutions of central and non-central potentials [8]. In this approach the SS equation reduces in to a first- order differential equation meaning those solutions may be obtained easily. The organization of this paper is as follows: In Section 2 we review the two-body Hamiltonian of the SS equation. The eigenvalues and eigenfunctions of the SS equation under the Coulomb and exponential potentials with Laplace transform are brought in at Sub sections 2.1 and 2.2. and finally, our conclusion is given in Section 3.

2 The two-body- Hamiltonian

In the center of a mass system the SS equation under a spherically symmetric potential has the form [10]

$$\left[\sum_{i=1,2}\sqrt{m_i^2-\Delta}+V(r)-E_{n,l}-m_1-m_2\right]\chi(\vec{r})=0. \quad (1)$$

Where

$$\chi(\vec{r})=R_{nl}(r)Y_{lm}(\theta,\phi), \quad (2)$$

where m_1 and m_2 are the masses of particles, $\Delta=\nabla^2$ and $V(r)$ are the interaction potentials and $E_{n,l}$ is the total energy of the system. In the case of heavy interacting particles, Eq. (1) can be written as

$$\left[-\frac{1}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+1)}{r^2}\right)-\frac{\mu^2}{2\eta^3}(V(r)-E_{n,l})^2 +(V(r)-E_{n,l})\right]R_{n,l}(r)=0. \quad (3)$$

Applying the well-known transformation

$$R_{n,l}(r)=\frac{\psi_{n,l}(r)}{r}. \quad (4)$$

Eq. (3) reads as

$$\left[-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+W_{n,l}(r)-\frac{W_{n,l}^2(r)}{2\tilde{m}}\right]\psi_{n,l}(r)=0, \quad (5)$$

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where

$$W_{nl}(r)=V(r)-E_{n,l},\quad \mu=\frac{m_1m_2}{m_1+m_2},$$
$$\eta=\mu\left(\frac{m_1m_2}{m_1m_2-3\mu^2}\right)^{1/3},\quad \tilde{m}=\frac{\eta^3}{\mu^2}. \tag{6}$$

In the next sections, we consider the SS equation under the Coulomb and exponential type potentials.

2.1 The SS equation under the Coulomb potential

We consider the Coulomb potential as

$$V(r)=-\frac{C}{r}, \tag{7}$$

where C is a constant, with substitution Coulomb potential into Eq. (3), we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\left(D+E'_{n,l}r+Fr^2\right)\right]R_{n,l}(r)=0, \tag{8}$$

where

$$D=\frac{\mu C^2}{\hbar^2\tilde{m}}-\ell(\ell+1),\quad E'_{n,l}=\frac{2\mu C}{\hbar^2}+\frac{2\mu CE_{n,l}}{\hbar^2\tilde{m}},$$
$$F=\frac{2\mu E_{n,l}}{\hbar^2}+\frac{\mu E^2_{n,l}}{\hbar^2\tilde{m}}. \tag{9}$$

By choosing

$$R_{n,l}(r)=r^Af_{n,l}(r), \tag{10}$$

where A is a constant parameter. Substituting Eq. (10) into Eq. (8) one can find

$$\left[\frac{\mathrm{d}^2f_{n,l}(r)}{\mathrm{d}r^2}+\frac{2A+1}{r}\frac{\mathrm{d}f_{n,l}(r)}{\mathrm{d}r}+\frac{1}{r^2}\left[Fr^2-E'_{n,l}r+(A^2+A+D)\right]f_{n,l}(r)\right]=0. \tag{11}$$

By considering the following condition

$$A^2+A+D=0\rightarrow A=\frac{-1-\sqrt{1-4D}}{2}, \tag{12}$$

we obtain the confluent hypergeometric equation and the value of A.

By using Eq. (12), Eq. (11) is modified as

$$\left[r\frac{\mathrm{d}^2f_{n,l}(r)}{\mathrm{d}r^2}+(2A+1)\frac{\mathrm{d}f_{n,l}(r)}{\mathrm{d}r}+\frac{1}{r}\left[Fr^2-E'_{n,l}r\right]f_{n,l}(r)\right]=0. \tag{13}$$

Applying the Laplace transform defined as [11]

$$L\{f(r)\}=f(t)=\int_0^\infty \mathrm{d}r\mathrm{e}^{-tr}f(r), \tag{14}$$

we arrive at a first-order differential equation from Eq. (13) as

$$(t^2+F)\frac{\mathrm{d}f_{n,l}(t)}{\mathrm{d}t}-\left[(2A-1)t-E'_{n,l}\right]f_{n,l}(t)=0, \tag{15}$$

where $f_{n,l}(t)$ is the inverse Laplace of $f_{n,l}(r)$. By solving Eq. (15) we have

$$f_{n,l}(t)=(t-F^{1/2})^{2A-1}\left(\frac{t+F^{1/2}}{t-F^{1/2}}\right)^{\frac{E}{2F^{1/2}}+A-\frac{1}{2}}. \tag{16}$$

To obtain a single-valued wave function, we impose the following condition

$$n=\frac{E'_{n,l}}{2F^{1/2}}+A-\frac{1}{2}. \tag{17}$$

From Eqs. (9) and (12), Eq. (17) is simplified as

$$n=\left(\frac{\frac{2\mu C}{\hbar^2}+\frac{2\mu CE_{n,l}}{\hbar^2\tilde{m}}}{2\left(\frac{2\mu E_{n,l}}{\hbar^2}+\frac{\mu E^2_{n,l}}{\hbar^2\tilde{m}}\right)^{\frac{1}{2}}}\right)-\frac{1}{2}\left(1-\frac{4\mu C^2}{\hbar^2\tilde{m}}+4\ell(\ell+1)\right)^{\frac{1}{2}}-1. \tag{18}$$

Considering the above condition and applying a simple series expansion to Eq. (16), we obtain

$$f_{n,l}(t)\approx\sum_{m=0}^{n}\frac{n!}{(n-m)!m!}(t-F^{1/2})^{2A-1-m}(2F^{1/2})^m. \tag{19}$$

Using the inverse Laplace transformation into Eq. (19) we obtain [11]

$$f_{n,l}(r)=N_{n,l}r^{1-2A}\mathrm{e}^{F^{1/2}r}\sum_{m=0}^{n}\frac{(-1)^mn!}{(n-m)!m!}\frac{\Gamma(-2A+2)}{\Gamma(m-2A+2)}\times(-2F^{1/2}r)^m, \tag{20}$$

where $N_{n,l}$ is the normalization constant. From the following definition of hypergeometric function

$${}_1F_1(-n,\sigma,x)=\sum_{m=0}^{n}\frac{(-1)^mn!}{(n-m)!m!}\frac{\Gamma(\sigma)}{\Gamma(\sigma+m)}x^m. \tag{21}$$

Eq. (20) can be written as

$$f_{n,l}(r)=N_{n,l}r^{1-2A}\mathrm{e}^{D^{1/2}r}{}_1F_1(-n,2-2A,2F^{1/2}r). \tag{22}$$

From the relation of Laguerre polynomials and confluent hypergeometric [12]

$$L_n^\eta(x)=\frac{\Gamma(n+\eta+1)}{n!\Gamma(\eta+1)}{}_1F_1(-n,\eta+1,x), \tag{23}$$

we obtain

$$f_{n,l}(r)=\frac{n!\Gamma(2-2A)}{\Gamma(n+2-2A)}N_{n,l}r^{1-2A}\mathrm{e}^{F^{1/2}r}L_n^{1-2A}(2F^{1/2}r). \tag{24}$$

Therefore the wave function of the system is

$$R_{n,l}(r)=N_{n,l}\frac{n!\Gamma(2-2A)}{\Gamma(n+2-2A)}r^{1-A}\mathrm{e}^{F^{1/2}r}L_n^{1-2A}(2F^{1/2}r). \tag{25}$$

2.2 The SS equation under the exponential potential

The exponential type potential that we have considered, has the following form [10]

$$V(r)=\nu_0 \mathrm{e}^{-\alpha(r-r_0)}. \tag{26}$$

Substitution of Eq. (26) into Eq. (5) gives

$$\left[-\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+\left(\nu_0\mathrm{e}^{-\alpha(r-r_0)}-E_{n,l}\right)-\frac{(\nu_0^2\mathrm{e}^{-2\alpha(r-r_0)}+E_{n,l}^2-2\nu_0\mathrm{e}^{-\alpha(r-r_0)}E_{n,l})}{2\tilde{m}}\right]\psi_{n,l}(r)=0. \tag{27}$$

To proceed on, we make use of an approximation of the form

$$\frac{1}{r^2}\approx(c_0+c_1\mathrm{e}^{-\alpha x}+c_2\mathrm{e}^{-2\alpha x}), \tag{28}$$

where

$$x=\frac{r-r_0}{r_0},\quad \alpha=ar_0,\quad c_0=\frac{1}{r_0^2}\left(1-\frac{3}{\alpha}+\frac{3}{\alpha^2}\right),$$
$$c_1=\frac{1}{r_0^2}\left(\frac{4}{\alpha}-\frac{6}{\alpha^2}\right),\quad c_2=\frac{1}{r_0^2}\left(-\frac{1}{\alpha}+\frac{3}{\alpha^2}\right). \tag{29}$$

By introducing a new variable of the form

$$z=\mathrm{e}^{-\alpha x}. \tag{30}$$

And using Eqs. (28) and (30), Eq. (27) comes in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}-\frac{1}{z^2}\left[\frac{1}{\alpha^2}(Bz+Cz^2+D)\right]\right)\psi_{n,l}(z)=0, \tag{31}$$

where

$$B=r_0^2\left(c_1\ell(\ell+1)+\frac{2\mu\nu_0}{\hbar^2}+\frac{2\nu_0E_{nl}\mu}{\tilde{m}\hbar^2}\right),$$
$$C=r_0^2\left(-\frac{\mu\nu_0^2}{\tilde{m}\hbar^2}+c_2\ell(\ell+1)\right), \tag{32}$$
$$D=r_0^2\left(c_0\ell(\ell+1)-\frac{2\mu E_{nl}}{\hbar^2}-\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right).$$

To obtain the confluent hypergeometric equation, we apply a transformation of the form

$$\psi_{n,l}(z)=z^A f_{n,l}(z). \tag{33}$$

Inserting Eq. (33) in Eq. (31), we have

$$\left\{\frac{\mathrm{d}^2f_{n,l}(z)}{\mathrm{d}z^2}+\frac{2A+1}{z}\frac{\mathrm{d}f_{n,l}(z)}{\mathrm{d}z}-\frac{1}{z^2}[B'z+C'z^2+(D'-A^2)]f_{n,l}(z)\right\}=0, \tag{34}$$

where

$$B'=\frac{B}{\alpha^2},C'=\frac{C}{\alpha^2},D'=\frac{D}{\alpha^2}, \tag{35}$$

and

$$D'-A^2=0\rightarrow A=\pm\sqrt{D'}\rightarrow A=-\sqrt{D'}. \tag{36}$$

We can write Eq. (34) as

$$\left\{z\frac{\mathrm{d}^2f_{n,l}(z)}{\mathrm{d}z^2}+(2A+1)\frac{\mathrm{d}f_{n,l}(z)}{\mathrm{d}z}-\frac{1}{z}[B'z+C'z^2]f_{n,l}(z)\right\}=0. \tag{37}$$

By using the Laplace Transform into Eq. (37) we arrive at

$$(t^2-C')\frac{\mathrm{d}f_{n,l}(t)}{\mathrm{d}t}-[(2A+3)t-B']f_{n,l}(t)=0. \tag{38}$$

The solution of the above equation is

$$f_{n,l}(t)\approx(t+C'^{1/2})^{-(2D'^{1/2}-3)}\times\left(\frac{t-C'^{1/2}}{t+C'^{1/2}}\right)^{-B'/2(C'^{1/2})-\frac{1}{2}(2D'^{1/2}-3)}. \tag{39}$$

To obtain a single-valued wave function, we should have

$$n=-B'/2(C'^{1/2})-\frac{1}{2}(2D'^{1/2}-3). \tag{40}$$

From Eqs. (32) and (35), the energy eigenvalues satisfy the following equation

$$n=\frac{r_0\left(\frac{2\nu_0E_{nl}\mu}{\tilde{m}\hbar^2}+\frac{2\mu\nu_0}{\hbar^2}+c_1\ell(\ell+1)\right)}{2\left(-\frac{\mu\nu_0^2}{\tilde{m}\hbar^2}+c_2\ell(\ell+1)\right)^{\frac{1}{2}}}-\frac{r_0^2}{\alpha^2}\left(c_0\ell(\ell+1)-\frac{2\mu E_{nl}}{\hbar^2}-\frac{\mu E_{nl}^2}{\tilde{m}\hbar^2}\right)^{\frac{1}{2}}+\frac{3}{2}. \tag{41}$$

By applying a simple series expansion, we can write Eq. (39) as

$$f_{n,l}(t)\approx\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}(t+C'^{1/2})^{-(2D'^{1/2}-3)-m}(-2C'^{1/2})^m. \tag{42}$$

Using the inverse Laplace transformation in Eq. (42) we find [12]

$$f_{n,l}(z)\approx Nz^{(2D'^{1/2}-4)}\mathrm{e}^{-C'^{1/2}z}\sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!}\times\frac{\Gamma(2D'^{1/2}-3)}{\Gamma(2D'^{1/2}+m-3)}(2C'^{1/2}z)^m, \tag{43}$$

finally from Eq. (21) and Eq. (23) one can obtain

$$f_{n,l}(z)=Nz^{(2D'^{1/2}-4)}\mathrm{e}^{-C'^{1/2}z}{}_1F_1(-n,2D'^{1/2}-3,2C'^{1/2}z), \tag{44}$$

or

$$f_{n,l}(z) = \frac{n!\Gamma(2D'^{1/2}-3)}{\Gamma(n+2D'^{1/2}-3)} N z^{(2D'^{1/2}-4)}$$

$$\times \mathrm{e}^{-C'^{1/2}z} L_n^{2D'^{1/2}-4}(2C'^{1/2}z). \tag{45}$$

Inserting Eqs. (29), (30) and (45) into Eq. (33) gives

$$\psi_{n,l}(r) = N_{n,l} \frac{n!\Gamma(2D'^{1/2}-3)}{\Gamma(n+2D'^{1/2}-3)}$$

$$\times \exp\big(-a(2D'^{1/2}-4+A)(r-r_0)\big)$$

$$\times \exp\big(-C'^{1/2}\mathrm{e}^{-a(r-r_0)}\big) L_n^{2D'^{1/2}-4}$$

$$\times (2C'^{1/2}\mathrm{e}^{-a(r-r_0)}). \tag{46}$$

By using the eigenfunctions of a system we can study the static properties of the system.

3 Conclusion

We have obtained the solution of the SS equation under two types of potential, the Coulomb and exponential type potentials. We have reported the eigenfunctions of the system under these two types of potential by using the Laplace transformation. It is shown that the Laplace transformation is a useful and applicable method to obtain the energy spectra and the corresponding eigenfunctions of the system.

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