Dirac oscillator in noncommutative space

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Abstract: We study the Dirac oscillator problem in the presence of the Aharonov-Bohm effect with the harmonic potential in commutative and noncommutative spaces in $S=V$ and $S=-V$ symmetry limits. We calculate exact energy levels and the corresponding eigenfunctions by the Nikiforov-Uvarov (NU) method and report the impact of the spin and the magnetic flux on the problem. Helpful numerical data is included.

Key words: Dirac oscillator, Aharonov-Bohm effect, harmonic potential, noncommutative space, Nikiforov-Uvarov (NU) method

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1 Introduction

The jargon Dirac oscillator (DO), which was first introduced by Moshinsky and Szczepaniak, is of interest in many fields of physics because it bears the effect of an external magnetic field [1]. It is well-known that the terminology corresponds to the transformation of the momentum operator $\vec{p}$ into $\vec{p}-\mathrm{i}m\omega\beta\vec{r}$, where $\vec{r}$, m and ω, respectively, show the position vector, the mass of the particle, and the frequency [2–4].

On the other hand, a very challenging point is the noncommutative (NC) formulation of quantum mechanics in which our common commutation relations are altered. The NC quantum mechanics, which originates from string theory [5–7], has oriented theoretical researches on many related topics, including: Matrix theory [8], quantum Hall-effect [9], quantum gravity [10] and quantization rules [11], Aharonov-Bohm effect [12], Aharonov-Casher effect [13] and Landau levels [14]. Invaluable comments on the interface of the subject with quantum field theory can be found in Refs. [15–19].

Within our study, bearing in mind the significance of Dirac oscillator, the NC version of quantum mechanics, harmonic interaction, and an external magnetic field, we investigate the Dirac oscillator with an Aharanov-Bohm field [20] in the presence of an external harmonic term in NC space. We proceed on an analytical framework, and report the energy spectrum and corresponding wavefunctions by the NU method. This work is organized as follows. Section 2 gives a brief review of the NC quantum mechanics. Sections 3 and 4 will review the most necessary formulae of Dirac equation in commutative space with an Aharonov-Bohm effect in the presence of a harmonic potential. Finally, Sections 5 and 6 will study the problem in NC space. The work ends with our concluding remarks and some useful numerical results.

2 Noncommutative quantum mechanics (NCQM)

High-energy physics, and in particular string theory, imply NC relations instead of our ordinary commutative relations. A number of interesting papers have discussed the problem and its various related aspects. In summary, our commutative relations, including the coordinates x^{i} and the momentum p^{i}, are replaced by $\hat{x}^{\mathrm{i}}$ and $\hat{p}^{\mathrm{i}}$, which obey the algebra [21–23]

$$\left[\hat{x}^{\mathrm{i}},\hat{x}^{\mathrm{j}}\right]=\mathrm{i}\theta^{\mathrm{ij}},\quad \left[\hat{p}^{\mathrm{i}},\hat{p}^{\mathrm{j}}\right]=0,\quad \left[\hat{x}^{\mathrm{i}},\hat{p}^{\mathrm{j}}\right]=\mathrm{i}\delta^{\mathrm{ij}}, \tag{1}$$

where $\theta^{\mathrm{ij}}=\theta\varepsilon^{\mathrm{ij}}$, $\varepsilon^{\mathrm{ij}}$ is an anti-symmetric tensor and θ is called the NC parameter. Thus, we must change x^{i} as

$$\hat{x}^{\mathrm{i}}\rightarrow x^{\mathrm{i}}-\frac{1}{2}\theta\varepsilon^{\mathrm{ij}}p^{\mathrm{j}}, \tag{2}$$

and the Moyal product is defined by $(f*g)(x) = \exp(\mathrm{i}\theta^{\mathrm{ij}}\partial_{x^{\mathrm{i}}}\partial_{x^{\mathrm{j}}})f(x^{\mathrm{i}})g(x^{\mathrm{j}})$, where f and g are arbitrary functions. Some physical systems were analyzed in NC phase-space (NCPS). The latter implies the commutation relations [24]

$$\left[\hat{x}^{\mathrm{i}},\hat{x}^{\mathrm{j}}\right]=\mathrm{i}\theta^{\mathrm{ij}},\quad \left[\hat{p}^{\mathrm{i}},\hat{p}^{\mathrm{j}}\right]=\mathrm{i}\bar{\theta}^{\mathrm{ij}},\quad \left[\hat{x}^{\mathrm{i}},\hat{p}^{\mathrm{j}}\right]=\mathrm{i}\delta^{\mathrm{ij}}, \tag{3}$$

where $\bar{\theta}^{\mathrm{ij}}$ is again an antisymmetric constant tensor and $\bar{\theta}^{\mathrm{ij}}=\bar{\theta}\varepsilon^{\mathrm{ij}}$. Thus, in order to map the NCPS on its com-

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mutative counterpart, we must change $\hat{x}$ and $\hat{p}$ as

$$\hat{x}^{\mathrm{i}}\rightarrow \Lambda x^{\mathrm{i}}-\frac{1}{2\Lambda}\theta\varepsilon^{\mathrm{ij}}p^{\mathrm{j}},\quad \hat{p}^{\mathrm{i}}\rightarrow \Lambda p^{\mathrm{i}}+\frac{1}{2\Lambda}\bar{\theta}\varepsilon^{\mathrm{ij}}x^{\mathrm{j}}, \tag{4}$$

where the constant Λ is a scaling factor and the parameters Λ, $\bar{\theta}$ and θ represent the NC nature of the phase-space.

3 The problem for equal scalar and vector interactions in commutative space

Let us consider the symmetry limit of a single neutral spin-half particle moving in an external magnetic field. In this limit, the Dirac Hamiltonian appears as

$$H_{\mathrm{D}}=\alpha\cdot\pi+(m+V)\beta. \tag{5}$$

In this section, we analyze the AB problem in commutative space in the $S=V$ limit which corresponds to

$$(\alpha\cdot\pi+(m+V)\beta)\psi_{n,\lambda}(\vec{r})=(\varepsilon_{n,\lambda}-V)\psi_{n,\lambda}(\vec{r}), \tag{6}$$

where $\pi=\vec{p}-e\vec{A}$ is the minimally coupled momentum and A is the vector potential. The potential$A$ in the Coulomb gauge is considered as [25–27]

$$e\vec{A}=\begin{cases}-\dfrac{\alpha}{r}\hat{u}_{\varphi} & r\succ R,\\ 0 & r\prec R\end{cases}. \tag{7}$$

Therefore, the Dirac oscillator equation takes the form

$$(\alpha\cdot(\pi-\mathrm{i}m\omega\beta\vec{r})+(m+V)\beta)\psi_{n,\lambda}(\vec{r})=(\varepsilon_{n,\lambda}-V)\psi_{n,\lambda}(\vec{r}). \tag{8}$$

We choose two quantum numbers n, λ and represent our two-component stationary spinor as

$$\psi_{n,\lambda}(\vec{r})=\begin{pmatrix}\Phi_{n,\lambda}(\vec{r})\\ X_{n,\lambda}(\vec{r})\end{pmatrix}, \tag{9}$$

and the matrices α and β are defined as

$$\beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad \alpha=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}, \tag{10a}$$

where I is a 2×2 unitary matrix and the spin matrices are

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{10b}$$

Eqs. (9) and (10) yield the coupled equations

$$(\sigma\cdot\pi)X_{n,\lambda}(\vec{r})+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})X_{n,\lambda}(\vec{r}) = (\varepsilon_{n,\lambda}-m-2V)\Phi_{n,\lambda}(\vec{r}), \tag{11a}$$

$$-(\sigma\cdot\pi)\Phi_{n,\lambda}(\vec{r})+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})\Phi_{n,\lambda}(\vec{r}) = -(\varepsilon_{n,\lambda}+m)X_{n,\lambda}(\vec{r}), \tag{11b}$$

which give

$$\{-(\vec{\sigma}\cdot\pi)(\vec{\sigma}\cdot\pi)-m^2\omega^2r^2+\mathrm{i}m\omega(\vec{\sigma}\cdot\pi)(\vec{\sigma}\cdot\vec{r})-\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\pi) +(\varepsilon_{n,\lambda}^2-m^2-2V(\varepsilon_{n,\lambda}+m))\}\Phi_{n,\lambda}(\vec{r})=0. \tag{12}$$

For the interaction term, we consider the harmonic potential,

$$V(r)=a_1r^2+a_2, \tag{13}$$

where a_1 and a_2 are the potential constants. Now, let us recall the useful relations

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B}+\mathrm{i}\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \tag{14a}$$

$$\vec{\sigma}\cdot\vec{p} = \vec{\sigma}\cdot\widehat{r}\left(\widehat{r}\cdot\vec{p}+\mathrm{i}\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{14b}$$

$$\vec{p} = -\mathrm{i}\hbar\vec{\nabla}, \tag{14c}$$

$$L_z = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{14d}$$

$$\beta = \gamma^0,\ \text{and}\ \alpha^{\mathrm{i}}=\beta\gamma^{\mathrm{i}}. \tag{14e}$$

We may now write Eq. (12) in cylindrical coordinates as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}+\mathrm{i}\alpha\right)^2-m^2\omega^2r^2 +2m\omega\left(1-s\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}+\alpha\right)\right)-\frac{2\alpha s}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\varepsilon_{n,\lambda}^2-m^2 -2a_1r^2(m+\varepsilon_{n,\lambda})-2a_2(\varepsilon_{n,\lambda}+m)\right\}\Phi_{n,\lambda}(\vec{r})=0, \tag{15}$$

Eq. (15), via the change of variables: $z=r^2$ and $\Phi_{n,\lambda}(z)=\mathrm{e}^{\mathrm{i}\lambda\varphi}\Phi_{n,\lambda}^1(z)$, where λ is an integer number, for $r>R$ and $r<R$ regions is, respectively, written as [28–30]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{z}(1-s\alpha)\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2}\left[-\frac{1}{z^2}(\lambda+\alpha)^2/2 -\left(\frac{m^2\omega^2}{2}+a_1(m+\varepsilon_{n,\lambda})\right)+\frac{1}{z}((\varepsilon_{n,\lambda}^2-m^2)/2 +m\omega(1-s(\lambda+\alpha))-a_2(\varepsilon_{n,\lambda}+m))\right]\right\}\Phi_{n,\lambda}^1(z)=0,\quad r\succ R, \tag{16a}$$

and

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2}\left[-\frac{1}{z^2}(\lambda^2/2)-\left(\frac{m^2\omega^2}{2}+a_1(m+\varepsilon_{n,\lambda})\right)\right.\right.$$
$$+\frac{1}{z}((\varepsilon^2_{n,\lambda}-m^2)/2+m\omega(1-s\lambda)$$
$$\left.\left.-a_2(\varepsilon_{n,\lambda}+m))\right]\right\}\Phi^1_{n,\lambda}(z)=0,\quad r\prec R. \tag{16b}$$

To solve the above equation, we use NU method that is represented in the Appendix and write Eq. (16) as

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(1-s\alpha)}{z}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{z^2}(-\xi_1z^2+\xi_2z-\xi_3)\right\}\Phi^1_{n,\lambda}(z)=0, \tag{17}$$

where

$$\xi_1 = m^2\omega^2/4+a_1(m+\varepsilon_{n,\lambda})/2, \tag{18a}$$
$$\xi_2 = m\omega(1-s(\lambda+\alpha))/2-a_2(\varepsilon_{n,\lambda}+m)/2 +(\varepsilon^2_{n,\lambda}-m^2)/4, \tag{18b}$$
$$\xi_3 = (\lambda+\alpha)^2/4. \tag{18c}$$

A comparison of Eq. (17) with Eq. (A1) indicates the correspondence

$$\begin{aligned}
\alpha_1 &= 1-s\alpha,\ \ \alpha_2=0,\\
\alpha_3 &= 0,\ \ \alpha_4=s\alpha/2,\ \ \alpha_5=1/2(\alpha_2-2\alpha_3)=0,\\
\alpha_6 &= \alpha_5^2+\xi_1=\xi_1,\ \ \alpha_7=2\alpha_4\alpha_5-\xi_2=-\xi_2,\\
\alpha_8 &= \alpha_4^2+\xi_3=s^2\alpha^2/4+\xi_3,\ \ \alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6=\xi_1,\\
\alpha_{10} &= \alpha_1+2\alpha_4+2\sqrt{\alpha_8}=1+2\sqrt{s^2\alpha^2/4+\xi_3},\\
\alpha_{11} &= \alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})=2\sqrt{\xi_1},\\
\alpha_{12} &= \alpha_4+\sqrt{\alpha_8}=s\alpha+\sqrt{s^2\alpha^2/4+\xi_3},\\
\alpha_{13} &= \alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})=-\sqrt{\xi_1},
\end{aligned} \tag{19}$$

from Eqs. (A10) and (A11) in Appendix, the eigenfunction is

$$\Phi^1_{n,\lambda}(r) = r^{-s\alpha-\sqrt{s^2\alpha^2+(\lambda+\alpha)^2}}\mathrm{e}^{-\frac{1}{2}\sqrt{m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda})}r^2} \times L_n^{-\sqrt{s^2\alpha^2+(\lambda+\alpha)^2}} \times\left(\sqrt{m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda})}r^2\right),\quad r\succ R \tag{20}$$

which for $\alpha=0$ becomes

$$\Phi^1_{n,\lambda}(r) = r^{\lambda}\mathrm{e}^{-\frac{1}{2}\sqrt{m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda})}r^2} \times L_n^{\lambda}\left(\sqrt{m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda})}r^2\right).\quad r\prec R \tag{21}$$

Using the boundary conditions at $r=R$, that is,

$$\Phi^1_{n,\lambda}(r<R)\big|_{r=R} = \Phi^1_{n,\lambda}(r>R)\big|_{r=R}, \tag{22a}$$
$$\left.\frac{\mathrm{d}\Phi^1_{n,\lambda}(r<R)}{\mathrm{d}r}\right|_{r=R} = \left.\frac{\mathrm{d}\Phi^1_{n,\lambda}(r>R)}{\mathrm{d}r}\right|_{r=R}, \tag{22b}$$

as well as the Eqs. (22a) and (22b), we find

$$\left.\frac{\Phi^1_{n,\lambda}(r<R)}{\left(\Phi^1_{n,\lambda}(r<R)\right)'}\right|_{r=R}=\left.\frac{\Phi^1_{n,\lambda}(r>R)}{\left(\Phi^1_{n,\lambda}(r>R)\right)'}\right|_{r=R}, \tag{23}$$

which determines the energy eigenvalues. We have reported the energy spectra or Landau levels in Table 1.

4 The problem in $S = -V$ symmetry limit in commutative space

In this section, we consider the case of $S=-V$ starting from Eq. (8), which yields the coupled equations

$$(\vec{\sigma}\cdot\pi)X_{n,\lambda}(\vec{r})+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})X_{n,\lambda}(\vec{r}) = (\varepsilon_{n,\lambda}-m)\Phi_{n,\lambda}(\vec{r}), \tag{24a}$$
$$-(\vec{\sigma}\cdot\pi)\Phi_{n,\lambda}(\vec{r})+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})\Phi_{n,\lambda}(\vec{r}) = -(\varepsilon_{n,\lambda}+m-2V)X_{n,\lambda}(\vec{r}). \tag{24b}$$

Eliminating $\Phi_{n,\lambda}$ in favor of $X_{n,\lambda}$ and using the same steps of the previous section, the solution $S=-V$ limit for our two regions $r>R$ and $r<R$ is, respectively, written as

$$X^1_{n,\lambda}(r) = r^{-s\alpha-\sqrt{s^2\alpha^2+(\lambda+\alpha)^2}}\mathrm{e}^{-\frac{1}{2}\sqrt{m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda})}r^2} \times L_n^{-\sqrt{s^2\alpha^2+(\lambda+\alpha)^2}}\left(\sqrt{m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda})}r^2\right),\quad r>R, \tag{25}$$

which, for $\alpha=0$ becomes

$$X^1_{n,\lambda}(r) = r^{-\lambda}\mathrm{e}^{-\frac{1}{2}\sqrt{m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda})}r^2} \times L_n^{-\lambda}\left(\sqrt{m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda})}r^2\right).\quad r<R \tag{26}$$

The corresponding energy is listed in Table 1.

Table 1. Energy spectra $\varepsilon_{n,\lambda}$ of the DOE for any arbitrary choices of n and λ quantum numbers with $a_1=-1$, $a_2=0.88$ and $\alpha=0.8$ in commutative space.

$\lvert n,\lambda\rangle$	$\varepsilon_{n,\lambda}(S(r)=V(r))$	$\varepsilon_{n,\lambda}(S(r)=-V(r))$
$\lvert 1,0\rangle$	−1.738507900	12.26149210
$\lvert 2,0\rangle$	−1.853344486	12.14665551
$\lvert 2,1\rangle$	−2.199226973	11.80077303
$\lvert 3,0\rangle$	−3.761890825	10.33330691
$\lvert 3,1\rangle$	−3.841833922	10.15816634
$\lvert 3,2\rangle$	−4.481967596	9.518032669

5 The problem for equal scalar and vector interactions in noncommutative space

In this section, we analyze the problem in NC space. Our starting equation in this space is

$$\{[(\vec{\sigma}\cdot\pi)+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})]\}*X^{(\mathrm{NC})}_{n,\lambda}(\vec{r})+\{2V(r)\}*\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$$
$$=\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}-m\right)\Phi^{(NC)}_{n,\lambda}(\vec{r}), \tag{27a}$$

$$\{-(\vec{\sigma}\cdot\pi)+\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})\}*\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$$
$$=-\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m\right)X^{(\mathrm{NC})}_{n,\lambda}(\vec{r}), \tag{27b}$$

where $\varepsilon^{(\mathrm{NC})}_{n,\lambda}$ denotes the eigenvalues in NC space. In NC formulation, the star product between two fields on NC space can be replaced by the generalized Bopp shift (2). To map NC space into commutative space, we use the Bopp shifts $\vec{r}\rightarrow\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}$. To solve the corresponding equation, that is,

$$\left[(\vec{\sigma}\cdot\pi)X^{(\mathrm{NC})}_{n,\lambda}(\vec{r})+\mathrm{i}m\omega\left(\vec{\sigma}\cdot\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\right]X^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$$
$$=\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}-m-2V\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r}), \tag{28a}$$

$$-(\vec{\sigma}\cdot\pi)\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})+\mathrm{i}m\omega\left(\vec{\sigma}\cdot\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$$
$$=-\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m\right)X^{(\mathrm{NC})}_{n,\lambda}(\vec{r}), \tag{28b}$$

we eliminate $\chi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$ in favor of $\phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})$ and obtain

$$\Bigg\{-(\vec{\sigma}\cdot\pi)(\vec{\sigma}\cdot\pi)-m^2\omega^2r^2+\mathrm{i}m\omega(\vec{\sigma}\cdot\pi)(\vec{\sigma}\cdot\vec{r})$$
$$-\mathrm{i}m\omega(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\pi)+\frac{\mathrm{i}m\omega}{2\hbar}(\vec{\sigma}\cdot\pi)\left(\vec{\sigma}\cdot(\vec{\theta}\times\vec{p})\right)$$
$$-\frac{\mathrm{i}m\omega}{2\hbar}\left(\vec{\sigma}\cdot(\vec{\theta}\times\vec{p})\right)(\vec{\sigma}\cdot\pi)-\frac{m^2\omega^2}{2\hbar}(\vec{\sigma}\cdot\vec{r})\left(\vec{\sigma}\cdot(\vec{\theta}\times\vec{p})\right)$$
$$-\frac{m^2\omega^2}{2\hbar}\left(\vec{\sigma}\cdot(\vec{\theta}\times\vec{p})\right)(\vec{\sigma}\cdot\vec{r})-\frac{m^2\omega^2\theta^2}{4\hbar^2}p^2+\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}\right)^2-m^2$$
$$-\left(2a_1r^2+\frac{a_1p^2\theta^2}{2\hbar^2}+\frac{2\theta a_1L_z}{\hbar}+2a_2\right)\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m\right)\Bigg\}$$
$$\times\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})=0. \tag{29}$$

The latter, after some algebra, appears as

$$\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+\frac{1}{k}\Bigg[-\frac{2\alpha s}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\partial}{\partial\varphi}\frac{\mathrm{i}\alpha}{r^2}$$
$$+\frac{\mathrm{i}\alpha}{r^2}\frac{\partial}{\partial\varphi}+\left(\varepsilon^{(\mathrm{NC})}_{n,\lambda}\right)^2-m^2+2m\omega\left(1-s\left(\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}+\alpha\right)\right)$$
$$-\left[2a_1(\varepsilon_{n,l}+m)\frac{\theta}{\hbar}+\frac{m^2\omega^2\theta}{\hbar}-\frac{ms\omega\alpha\theta}{r^2}\right]L_z$$
$$-r^2(2a_1m+2a_1\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m^2\omega^2)-\frac{1}{r^2}\alpha^2$$
$$-2a_2(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m)+m^2\omega^2\theta s\Bigg]\Bigg\}\Phi^{(\mathrm{NC})}_{n,\lambda}(\vec{r})=0, \tag{30}$$

where

$$k=1+2a_1(\varepsilon_{n,l}+m)\frac{\theta^2}{4\hbar^2}+\frac{m^2\omega^2\theta^2}{4\hbar^2}+\frac{2ms\omega\theta}{\hbar}. \tag{31}$$

Now, using the change of variables $z=r^2$ and $\Phi^{(\mathrm{NC})}_{n,\lambda}(z)=\mathrm{e}^{\mathrm{i}\lambda\varphi}\Phi^{1(\mathrm{NC})}_{n,\lambda}(z)$, Eq. (29) is written as

$$\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(1-s\alpha/k)}{z}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{2k}\Bigg[\frac{1}{z^2}(-(ms\omega\alpha\theta L_z+k\lambda^2+\alpha^2)/2$$
$$-\lambda\alpha-\left(\frac{m^2\omega^2}{2}+a_1(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m)\right)+\frac{1}{z}\Bigg(\frac{1}{2}((\varepsilon^{(\mathrm{NC})}_{n,\lambda})^2-m^2)$$
$$-a_2(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m)+m\omega(1-s(\lambda+\alpha))+\frac{m^2\omega^2\theta}{2}\left(s-\frac{L_z}{\hbar}\right)$$
$$+2a_1(\varepsilon_{n,l}+m)\frac{\theta^2}{4\hbar^2}L_z\Bigg)\Bigg]\Bigg\}\Phi^{1(\mathrm{NC})}_{n,\lambda}(z)=0, \tag{32}$$

or, more neatly,

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{(1-s\alpha/k)}{z}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{1}{z^2}(-\xi_1z^2+\xi_2z-\xi_3)\right\}\Phi^{1(\mathrm{NC})}_{n,\lambda}(z)=0, \tag{33}$$

where

$$\xi_1=\frac{1}{k}(m^2\omega^2/4+a_1(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+a_1m)/2), \tag{34a}$$

$$\xi_2=\frac{1}{k}(((\varepsilon^{(\mathrm{NC})}_{n,\lambda})^2-m^2)/4-a_2(\varepsilon^{(\mathrm{NC})}_{n,\lambda}+m)/2$$
$$+m\omega(1-s(\lambda+\alpha))/2+m^2\omega^2\theta\left(s-\frac{L_z}{\hbar}\right)\Big/4$$
$$+2a_1(\varepsilon_{n,l}+m)\theta), \tag{34b}$$

$$\xi_3=\frac{1}{k}((ms\omega\alpha\theta L_z+k\lambda^2+\alpha^2)/4+\lambda\alpha/2). \tag{34c}$$

Thus, the required parameters in the NU technique are

$$\begin{aligned} \alpha_1 &= 1-s\alpha/k,\ \ \alpha_2=0,\ \ \alpha_3=0,\ \ \alpha_4=s\alpha/2k, \\ \alpha_5 &= 1/2(\alpha_2-2\alpha_3)=0,\ \ \alpha_6=\alpha_5^2+\xi_1=\xi_1, \\ \alpha_7 &= 2\alpha_4\alpha_5-\xi_2=-\xi_2,\ \ \alpha_8=\alpha_4^2+\xi_3=s^2\alpha^2/4k^2+\xi_3, \\ \alpha_9 &= \alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6=\xi_1, \\ \alpha_{10} &= \alpha_1+2\alpha_4+2\sqrt{\alpha_8}=1+2\sqrt{s^2\alpha^2/4k^2+\xi_3}, \\ \alpha_{11} &= \alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})=2\sqrt{\xi_1}, \\ \alpha_{12} &= \alpha_4+\sqrt{\alpha_8}=s\alpha/2k+\sqrt{s^2\alpha^2/4k^2+\xi_3}, \\ \alpha_{13} &= \alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})=-\sqrt{\xi_1}. \end{aligned} \tag{35}$$

Therefore, the eigenfunctions in the two regions are

$$\begin{aligned} \Phi_{n,\lambda}^{1(\mathrm{NC})}(r) &= r^{-\frac{s\alpha}{k}-\sqrt{\frac{1}{k}\left(\frac{s^2\alpha^2}{k}+(ms\omega\alpha\theta+2\alpha)\lambda+\lambda^2k+\alpha^2\right)}} \\ &\times \mathrm{e}^{\left(-\frac{1}{2}\sqrt{\frac{1}{k}(m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda}^{(\mathrm{NC})}))}r^2\right)} \\ &\times L_n^{-\sqrt{\frac{1}{k}\left(\frac{s^2\alpha^2}{k}+(ms\omega\alpha\theta+2\alpha)\lambda+\lambda^2k+\alpha^2\right)}} \\ &\times\left(\sqrt{\frac{1}{k}\left(m^2\omega^2+2a_1\left(m+\varepsilon_{n,\lambda}^{(\mathrm{NC})}\right)\right)}r^2\right).\ r\succ R \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\Phi_{n,\lambda}^{1(\mathrm{NC})}(r) \\ &= r^{\lambda}\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{1}{k}(m^2\omega^2+2a_1(m+\varepsilon_{n,\lambda}^{(\mathrm{NC})}))}r^2} \\ &\times L_n^{\lambda}\left(\sqrt{\frac{1}{k}\left(m^2\omega^2+2a_1\left(m+\varepsilon_{n,\lambda}^{(\mathrm{NC})}\right)\right)}r^2\right).\ \ r\prec R \end{aligned} \tag{37}$$

The corresponding energies are listed in Table 2. If θ=0, we recover the results of the commutative case (see Table 3).

Table 2. Energy spectra $\varepsilon_{n,\lambda}$ of the DOE for any arbitrary choices of n and λ quantum numbers with $\hbar$=1 and θ=0.05 in NC space.

$\lvert n,\lambda\rangle$	$\varepsilon_{n,\lambda}^{(\mathrm{NC})}(S(r)=V(r))$	$\varepsilon_{n,\lambda}^{(\mathrm{NC})}(S(r)=-V(r))$
	s=1	
$\lvert 1,0\rangle$	−5.281608016	12.10063321
$\lvert 2,0\rangle$	−5.248117852	11.73849188
$\lvert 2,1\rangle$	−5.154218341	8.850202525
$\lvert 3,0\rangle$	−4.243452888	7.612606747
$\lvert 3,1\rangle$	−3.703964883	7.122945433
$\lvert 3,2\rangle$	−1.476158319	5.766724107
	s=−1	
$\lvert 1,0\rangle$	−1.108717094	12.89128291
$\lvert 2,0\rangle$	−2.027681444	11.97231833
$\lvert 2,1\rangle$	−2.469393583	9.879645576
$\lvert 3,0\rangle$	−4.276190633	8.752550932
$\lvert 3,1\rangle$	−4.453492694	7.144148435
$\lvert 3,2\rangle$	−5.389981156	5.881886658

Table 3. Energy spectra $\varepsilon_{n,\lambda}$ of the DOE for any arbitrary choices of n and λ quantum numbers in the absence NC parameter θ=0.

$\lvert n,\lambda\rangle$	$\varepsilon_{n,\lambda}^{(\mathrm{NC})}(S(r)=V(r))$	$\varepsilon_{n,\lambda}^{(\mathrm{NC})}(S(r)=-V(r))$
$\lvert 1,0\rangle$	−1.738507900	12.26149210
$\lvert 2,0\rangle$	−1.853344486	12.14665551
$\lvert 2,1\rangle$	−2.199226973	11.80077303
$\lvert 3,0\rangle$	−3.761890825	10.33330691
$\lvert 3,1\rangle$	−3.841833922	10.15816634
$\lvert 3,2\rangle$	−4.481967596	9.518032669

6 The problem in $S=-V$ symmetry limit in noncommutative space

In this case, we start from

$$\begin{aligned} &\left\{(\vec{\sigma}\cdot\pi)+\mathrm{i}m\omega\left(\vec{\sigma}\cdot\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\right\}\chi_{n,\lambda}^{(\mathrm{NC})}(\vec{r}) \\ &= \left(\varepsilon_{n,\lambda}^{(\mathrm{NC})}-m\right)\phi_{n,\lambda}^{(\mathrm{NC})}(\vec{r}), \end{aligned} \tag{38a}$$

$$\begin{aligned} &\left\{-(\vec{\sigma}\cdot\pi)+\mathrm{i}m\omega\left(\vec{\sigma}\cdot\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\right\}\phi_{n,\lambda}^{(\mathrm{NC})}(\vec{r}) \\ &= -\left(\varepsilon_{n,\lambda}^{(\mathrm{NC})}+m-2V\left(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\right)\right)\chi_{n,\lambda}^{(\mathrm{NC})}(\vec{r}), \end{aligned} \tag{38b}$$

which yields the solutions

$$\begin{aligned} &X_{n,\lambda}^{1(\mathrm{NC})}(r) \\ &= r^{-\frac{s\alpha}{k}-\sqrt{\frac{1}{k}\left(\frac{s^2\alpha^2}{k}+(ms\omega\alpha\theta+2\alpha)\lambda+\lambda^2k+\alpha^2\right)}} \\ &\times \mathrm{e}^{(-\frac{1}{2}\sqrt{\frac{1}{k}(m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda}^{(\mathrm{NC})}))}r^2)} \\ &\times L_n^{-\sqrt{\frac{1}{k}\left(\frac{s^2\alpha^2}{k}+(ms\omega\alpha\theta+2\alpha)\lambda+\lambda^2k+\alpha^2\right)}} \\ &\times\left(\sqrt{\frac{1}{k}\left(m^2\omega^2-2a_1\left(m-\varepsilon_{n,\lambda}^{(\mathrm{NC})}\right)\right)}r^2\right).\ r\succ R \end{aligned} \tag{39}$$

and

$$\begin{aligned} &X_{n,\lambda}^{1(\mathrm{NC})}(r) \\ &= r^{\lambda}\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{1}{k}(m^2\omega^2-2a_1(m-\varepsilon_{n,\lambda}^{(\mathrm{NC})}))}r^2} \\ &\times L_n^{\lambda}\left(\sqrt{\frac{1}{k}\left(m^2\omega^2-2a_1\left(m-\varepsilon_{n,\lambda}^{(\mathrm{NC})}\right)\right)}r^2\right).\ \ r\prec R \end{aligned} \tag{40}$$

The associated energy levels are given in Table 2. We can see that the effect of the NC parameter on the Landau levels is not negligible. It is also observed that when the quantum number change from n=1, λ=0 to n=3, λ=2, the energy eigenvalues decrease in the both spaces. It is easy to show that we recover the case of commutative space as a special case of NC space by taking θ=0 (Table 3).

7 Conclusion

We studied the Dirac oscillator problem in the presence of a harmonic interaction in both commutative and noncommutative cases. The motivation behind our study was the outstanding role of the Dirac oscillator, harmonic interaction, and the external magnetic field in particle and high-energy physics. We showed that the problem can be simply solved in an exact analytical manner by using proper transformations and the NU technique. As we expect, the results of the NC space, in the limit $\theta\to 0$ yield the commutative analogues. As the exact relations for the energies and the eigenfunctions are presented, the results can be immediately used to investigate the binding energies and many static properties.

Appendix A

Nikiforov-Uvarov method

Let us consider the following differential equation [31]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{(\alpha_1-\alpha_2 s)}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{\{-\xi_1 s^2+\xi_2 s-\xi_3\}}{[s(1-\alpha_3 s)]^2}\right\}\psi=0. \quad \text{(A1)}$$

According to the NU method, the eigenfunctions and energies are obtained from [32]

$$\psi_n(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \quad \text{(A2)}$$

where the Jacobi polynomial is

$$P_n^{(c,d)}(z)=2^{-n}\sum_{p=0}^{n}\binom{n+c}{P}\binom{n+d}{n-p}(1-z)^{n-p}(1+z)^p, \quad \text{(A3)}$$

$$P_n^{(c,d)}(z)=\frac{\Gamma(n+c+1)}{n!\Gamma(n+c+d+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+c+d+r+1)}{\Gamma(r+c+1)}\times\left(\frac{z-1}{2}\right)^r, \quad \text{(A4)}$$

with

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}, \quad \text{(A5)}$$

and

$$\alpha_2 n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \quad \text{(A6)}$$

where

$$\begin{aligned}
\alpha_4 &= \frac{1}{2}(1-\alpha_1),\ \ \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\ \ \alpha_6=\alpha_5^2+\xi_1,\\
\alpha_7 &= 2\alpha_4\alpha_5-\xi_2,\ \ \alpha_8=\alpha_4^2+\xi_3,\\
\alpha_9 &= \alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}, \quad \text{(A7)}\\
\alpha_{11} &= \alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\\
\alpha_{12} &= \alpha_4+\sqrt{\alpha_8},\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}).
\end{aligned}$$

In the special case of $\alpha_3=0$,

$$\lim_{\alpha_3\to 0}P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-\alpha_3 s)=L_n^{\alpha_{10}-1}(\alpha_{11}s), \quad \text{(A8)}$$

$$\lim_{\alpha_3\to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}=\mathrm{e}^{\alpha_{13}s} \quad \text{(A9)}$$

and,

$$\psi_n(s)=s^{\alpha_{12}}\mathrm{e}^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s). \quad \text{(A10)}$$

In some cases, one may need a second solution of Eq. (10). In this case, if the same procedure is followed by using

$$\psi_n(s)=s^{\alpha^*_{12}}(1-\alpha_3 s)^{-\alpha^*_{12}-\frac{\alpha^*_{13}}{\alpha_3}}P_n^{\left(\alpha^*_{10}-1,\frac{\alpha^*_{11}}{\alpha_3}-\alpha^*_{10}-1\right)}(1-2\alpha_3 s), \quad \text{(A11)}$$

and the energy spectrum is

$$\alpha_2 n-2n\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8-2\sqrt{\alpha_8\alpha_9}+\alpha_5=0, \quad \text{(A12)}$$

Pre-defined α parameters are:

$$\alpha^*_{10}=\alpha_1+2\alpha_4-2\sqrt{\alpha_8}, \quad \text{(A13)}$$

$$\alpha^*_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8}), \quad \text{(A14)}$$

$$\alpha^*_{12}=\alpha_4-\sqrt{\alpha_8}, \quad \text{(A15)}$$

$$\alpha^*_{13}=\alpha_5-(\sqrt{\alpha_9}-\alpha_3\sqrt{\alpha_8}). \quad \text{(A16)}$$

References

1 Moshinsky M, Szczepaniak A. J. Phys. A: Math. Gen, 1989, **22**: 817
2 Moshinsky M, Smirnov Y F. The Harmonic Oscillator in Modern Physics. Amsterdam: Harwood Academic Publishers, 1996
3 Rozmej P, Arvieu R. J. Phys. A, 1999, **32**: 5367
4 Pacheco M H, Landim R R, Almeida C A S. Phys. Lett. A, 2003, **311**: 93
5 Seiberg N, Witten E. JHEP, 1999, **09**: 032
6 Seiberg N, Susskind L, Toumbas N. JHEP, 2000, **06**: 044
7 Douglas M R, Nekrasov N A. Rev. Mod. Phys., 2001, **73**: 977
8 Connes A, Douglas M R, Schwarz A. JHEP, 1998, **02**: 003
9 Susskind L. arXiv: hepth/0101029
10 Moffat J W. Phys. Lett. B, 2000, **493**: 142
11 Duval C, Horvathy P A. Phys. Lett. B, 2000, **479**: 284
12 Chaichian M, Sheikh-Jabbari M M, Tureanu A. Phys. Rev. Lett., 2001, **86**: 2716
13 Mirza B, Zarei M. Eur. Phys. J. C, 2004, **32**: 583
14 Horvathy P A. Ann. Phys. (N.Y.), 2002, **299**: 128
15 Gol'dman I I et al. Problems in Quantum Mechanics. New York: Academic Press, 1960. 308
16 Seiberg N, Witten E. JHEP, 1999, **09**: 032
17 Douglas M R, Nekrasov N A. Rev. Mod. Phys., 2001, **73**: 977
18 Bertolami O et al. Phys. Rev. D, 2005, **72**: 025010
19 Araki T, Ito K, Ohtsuka A. Phys. Lett. B, 2003, **573**: 209
20 Aharonov Y, Bohm D. Phys. Rev., 1959, **115**: 485
21 Connes A, Douglas M, Schwarz A. JHEP, 1998, **9802**: 003
22 Cheung Y, Krogh M. Nucl. Phys. B, 1999, **528**: 185
23 Chu C, Ho P. Nucl. Phys. B, 1999, **550**: 151
24 Zhang J Z. Phys. Lett. B, 2004, **584**: 204
25 Ferkous N, Bounames A. Phys. Lett. A, 2004, **325**: 21
26 Hagen C R. Phys. Rev. Lett., 1990, **64**: 503
27 Hagen C R. Int. J. Mod. Phys. A, 1991, **6**: 3119
28 DONG S H, SUN G H , Lozada-Cassou M. Phys. Lett. A, 2004, **328**: 299
29 ZHANG M C, SUN G H, DONG S H. Phys. Lett. A, 2010, **374**: 704
30 Miranda M G, SUN G H, DONG S H. Int. J. Mod. Phys. E, 2010, **19**: 123
31 Nikiforov A F, Uvarov V B. Special Function of Mathematical Physics. Basel: Birkhauser, 1988
32 Tezcan C, R. Sever R. Int. J. Theor. Phys, 2009, **48**: 337