Analytical formula of free electron laser exponential gain for a non-resonant electron beam *

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Abstract: The free electron laser (FEL) gain formulas for a non-resonant case are studied, and some new rigorous analytical formulas are given explicitly. For the mono-energetic and non-resonant electron beam, the exact expression of the solution of the FEL characteristic cubic equation is obtained with a form much more simple than that in the literatures, and the gain length as the function of the detuning parameter is explicitly given. Then the gain for different detuning parameters and from low to high can be easily calculated. A simplified approximation formula is also given for the exponential gain calculation in the non-resonant case. For the case of the electron beam with an energy spread, the solution of the characteristic cubic equation is given explicitly for rectangular energy distribution and Lorentz distribution, respectively. Moreover the explicit expression also can be used for the solution of the characteristic cubic equation including the impact of the space charge. The transition from the low gain to the high gain is analyzed. The variations of the gain bandwidth and of the detuning parameter for the maximum gain are demonstrated. The applicable ranges of the small signal gain formula and the exponential gain formula are analyzed.

Key words: free electron laser, exponential gain, non-resonant

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1 Introduction

The optical field gain is one of most important parameters in a free electron laser (FEL), which is defined as the ratio of the optical power increment to the initial power. In the linear regime, the FEL optical field is under saturation; usually it has a low gain per pass in the oscillator mode and a high gain in the single-amplifying mode. The level of the optical field gain of FEL is determined by the intensity of the electron current density and the length of the undulator. The effects of the two factors can be characterized by the number of gain lengths in the length of the undulator. The gain length is defined as the e-folding length of the optical power increased by a factor of $e{=}2.718\cdots$.

Though analytical gain formulas are convenient for calculation and analysis compared with the numerical method, they exist only for special cases of FEL gain calculation. Typically there are the small signal gain formula in the low gain regime and the exponential gain formula in the high gain regime. The former is the initial gain in the multi-amplifying of oscillator FEL and the latter is the gain in single-amplifying high gain FEL. The existing exponential gain formula is only for the electron beam at the resonant energy. In this paper we consider the exponential gain for the non-resonant case, including the mono-energetic electron beam but non-resonant and the electron beam with an energy spread. We try to give some analytical expressions of the gain, and we also analyze the transition from the range of the small signal gain to the range of the exponential gain.

The paper is organized as follows: in the second section we give a brief review of the existing analytical theory of the exponential gain; then in the third section the general non-resonant case will be analyzed; the case of the electron beam with an initial energy spread will be considered in the fourth section; and at last we give an analysis on the relation between the different gain formulas.

2 The existing analytic theory of exponential gain [1–5]

The optical field in the linear regime for the mono-energetic electron beam is given as

$$\begin{aligned}\tilde{a}_{\mathrm{s}}(\hat{z}) &= \mathrm{i}\sum_{\substack{m=1\\ m\neq l,k}}^{3}\frac{\tilde{a}_{\mathrm{s}0}\mathrm{e}^{\hat{\mu}_m\hat{z}}}{\hat{\mu}_m(\hat{\mu}_m-\hat{\mu}_l)(\hat{\mu}_m-\hat{\mu}_k)}\\ &= \tilde{a}_{\mathrm{s}0}\sum_{m=1}^{3}\frac{\hat{\mu}_m+\mathrm{i}\hat{\eta}}{3\hat{\mu}_m+\mathrm{i}\hat{\eta}}\mathrm{e}^{\hat{\mu}_m\hat{z}},\end{aligned}\tag{1}$$

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where $\tilde{a}_{s0}$ is the initial dimensionless vector potential of the optical field, $\hat{z}=2k_{\mathrm{u}}\rho z$ is the scaled position along the undulator, k_{u} is the wave number of the undulator, ρ is FEL parameter, $\hat{\eta}=\delta\gamma/\gamma\rho$ is the scaled detuning parameter, $\delta\gamma/\gamma$ is the beam energy deviation from the resonance, $\hat{\mu}_m$ (m=1, 2, 3) is the roots of the well-known characteristic cubic equation:

$$\hat{\mu}(\hat{\mu}+\mathrm{i}\hat{\eta})^2=\mathrm{i}. \tag{2}$$

When $\hat{\eta}<3/\sqrt[3]{4}=1.89$, the cubic equation has one imaginary root corresponding to the oscillatory mode of the optical field, and two complex roots with the real parts of equal magnitude but opposite sign, corresponding to the exponential amplifying mode and the exponential decaying mode, respectively (and if $\hat{\eta}>1.89$, the cubic equation will have three imaginary roots). For a sufficiently long undulator the leading role is the exponential growth term corresponding to the root with a positive real part.

The maximum gain rate occurs at the resonant energy. For this case, the detuning parameter $\hat{\eta}$=0, the three roots are $\hat{\mu}_{1,2,3}=(\mathrm{i}\pm\sqrt{3})/2$, $-\mathrm{i}$, then the gain formula can be given analytically

$$\begin{aligned}\frac{P}{P_0} &= \frac{1}{9}\left\{1+4\mathrm{ch}\left(\frac{\sqrt{3}}{2}\hat{z}\right)\left[\mathrm{ch}\left(\frac{\sqrt{3}}{2}\hat{z}\right)+\cos\left(\frac{3}{2}\hat{z}\right)\right]\right\}\\ &\approx \frac{1}{9}\mathrm{e}^{\frac{z}{L_{\mathrm{g}}}},\end{aligned} \tag{3}$$

where P_0 is the input optical power (seeding) for amplifier FEL, and for self-amplified spontaneous emission (SASE) it is the effective noise power. In the above equation, the last approximation of exponential gain is valid for a large rate of L/L_{g} (see Section 5), L is the length of the undulator, $L_{\mathrm{g}}=1/2k_{\mathrm{u}}\sqrt{3}\rho$ is the power gain length.

For the case of non-resonance, if the detuning is small $|\hat{\eta}|\ll 1$, namely near the resonance, the standard approach (e.g. in Refs. [1–3]) expands the cubic equation near $\hat{\eta}$=0 to the second order in $\hat{\eta}$ and has

$$\hat{\mu}=\hat{\mu}_0-\mathrm{i}\frac{2}{3}\hat{\eta}-\frac{1}{9\hat{\mu}_0}\hat{\eta}^2, \tag{4}$$

then the root corresponding to the exponential growth term is

$$\hat{\mu}_1=\frac{\sqrt{3}}{2}\left[1-\frac{1}{9}\hat{\eta}^2\right]+\frac{\mathrm{i}}{2}\left[1-\frac{4}{3}\hat{\eta}+\frac{1}{9}\hat{\eta}^2\right], \tag{5}$$

and the exponential gain of the optical field is

$$\tilde{a}_{\mathrm{s}}/\tilde{a}_{\mathrm{s}0}\propto\mathrm{e}^{\sqrt{3}\hat{z}}\mathrm{e}^{\mathrm{i}\left[\frac{1}{2}-\frac{2}{3}\hat{\eta}\right]\hat{z}}\mathrm{e}^{-\frac{\hat{\eta}^2}{4\sigma_\eta^2}\left[1-\frac{\mathrm{i}}{\sqrt{3}}\right]}, \tag{6}$$

where $\sigma_\eta^2=3\sqrt{3}/2\hat{z}$ is the gain bandwidth, following from which the radiation bandwidth can be given as $\sigma_\omega/\omega=2\rho\sigma_\eta$. It decreases with the increase of z and goes to ρ when the optical field tends to saturation, which happens where the length of the undulator equals about twenty gain length for SASE.

It should be pointed out that actually besides the exponential factor in Eq. (6), the pre-exponential factor is also relevant to the detuning parameter.

3 The general non-resonant case

For the mono-energetic and non-resonant electron beam, the exact solution of the cubic equation (Eq. (2)) was given, but in a very complicated form (for example as in Refs. [1, 6]). By careful reduction we get the exact solution with a much more simple form as follows:

$$\begin{aligned}\hat{\mu}_{1,2} &= \pm(p-q)+\mathrm{i}\left(\frac{p+q}{\sqrt{3}}-\frac{2\hat{\eta}}{3}\right),\\ \hat{\mu}_3 &= -\mathrm{i}2\left(\frac{p+q}{\sqrt{3}}+\frac{\hat{\eta}}{3}\right),\\ p,q &= \frac{\sqrt{3}}{2\sqrt[3]{4}}\left\{1\pm\sqrt{1-4\left(\frac{\hat{\eta}}{3}\right)^3}\right\}^{\frac{2}{3}},\left(\hat{\eta}=\frac{\delta\gamma}{\gamma_{\mathrm{r}}\rho}<\frac{3}{\sqrt[3]{4}}\right).\end{aligned} \tag{7}$$

From the root $\hat{\mu}_1$ we obtain the rigorous gain length formula explicitly as:

$$\begin{aligned}\frac{L_{\mathrm{g}}}{L_{\mathrm{g}0}} = \sqrt[3]{4}\Bigg/\Bigg\{&\left[1+\sqrt{1-4\left(\frac{\hat{\eta}}{3}\right)^3}\right]^{\frac{2}{3}}\\ &-\left[1-\sqrt{1-4\left(\frac{\hat{\eta}}{3}\right)^3}\right]^{\frac{2}{3}}\Bigg\},\end{aligned} \tag{8}$$

where $L_{\mathrm{g}0}$ is the gain length for the resonant case. From Eq. (8) it is obvious to the requirement of $\hat{\eta}<3/\sqrt[3]{4}=1.89$. Variation of the gain length with the detuning parameter (i.e. the electron energy) is shown as Fig. 1, the solid curve is exactly the same as in the literatures (e.g. in Refs. [1, 2, 7]) where it was given numerically, but here it is given by the formula. Therefore the gain length of the general case can be calculated conveniently. At the resonance the detuning parameter is equal to zero, and the gain length is the shortest, the growth rate reaches its maximum. The gain length increases sharply for the detuning parameter larger than zero and slowly for the detuning parameter smaller than zero. The result given by Eq. (5) is also plotted in Fig. 1. It can be seen that it is only applicable for the case of $|\hat{\eta}|<1$.

By using Eq. (7) and Eq. (1), the variation of the gain with the detuning parameter can be calculated. The result for the case near the saturation ($L\sim 20L_{\mathrm{g}}$) is shown in Fig. 2. It is seen that for the high gain, the gain bandwidth $\Delta\hat{\eta}_{\mathrm{FWHM}}\sim 1$, this agrees with the result from Eq. (6): $\sigma_{\hat{\eta}}\sim 1/2$. From Fig. 2 one can also see that the maximum gain is not at the resonance $\hat{\eta}=0$ as is the maximum growth rate (Fig. 1), but is at $\hat{\eta}\approx 0.12$, and for the detuning parameter smaller than zero, the gain does not vary slowly as the gain length does (Fig. 1). This is because, for the non-resonant case, the non-exponential factor in the optical field expression (Eq. (1)) is also related to the detuning as we previously pointed out.

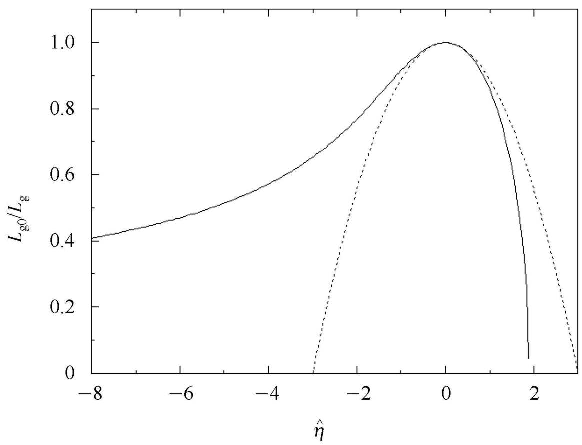

Fig. 1. The gain length vs. the detuning parameter (Eq. (8)). The dashed line is given by Eq. (5).

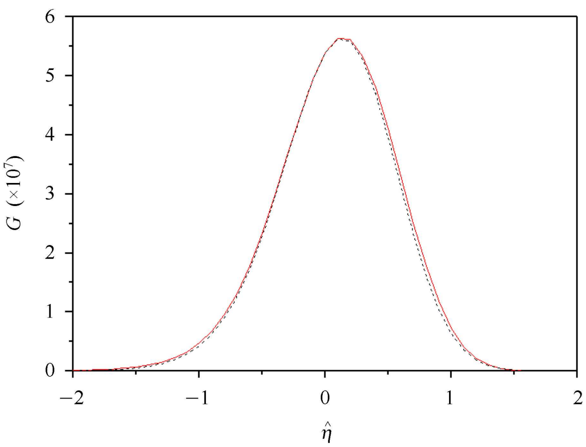

Fig. 2. The variation of the gain with the detuning parameter (z=20L_g). The solid line: Eq. (7) and (1); the dashed line: Eq. (9).

Furthermore, we give a simplified approximation formula for the exponential gain calculation in the non-resonant case as follows:

$$\frac{P}{P_0}\approx\frac{1}{9}\left(1+\frac{\hat{\eta}}{3}\right)^2 \mathrm{e}^{z/L_g},\ (-3<\hat{\eta}<2, z>5L_g), \qquad (9)$$

$$L_g\approx L_{g0}\Bigg/\left[1-\left(\frac{\hat{\eta}}{3}\right)^2-\frac{2}{3}\left(\frac{\hat{\eta}}{3}\right)^3\right]. \qquad (10)$$

It agrees well with the exact result from Eq. (7, 8) and Eq. (1) (Fig. 2). If the cubic term in Eq. (10) is neglected, it becomes the result of Eq. (5).

4 Influence of beam energy spread

For the electron beam having an initial energy spread, the situation is more complex, the characteristic equation (the dispersion relation) now has the form

$$\hat{\mu}-\mathrm{i}\int\frac{f_0}{(\hat{\mu}+\mathrm{i}\hat{\eta})^2}\mathrm{d}\hat{\eta}=0 \qquad (11)$$

where f_0 is normalized distribution of the initial detuning parameter. Generally, the dispersion relation has to be solved numerically for a given initial detuning distribution; the analytic solutions exist only for a few types of distributions. Here we consider a simple case: the rectangular distribution with a half-width $\Delta\hat{\eta}=\delta$

$$f(\hat{\eta})=1/2\delta, \quad \hat{\eta}_m-\delta<\hat{\eta}<\hat{\eta}_m+\delta.$$

Then the cubic equation becomes

$$\hat{\mu}[(\hat{\mu}+\mathrm{i}\widehat{\eta}_m)^2+\delta^2]=\mathrm{i}. \qquad (12)$$

The corresponding optical field expression Eq. (1) becomes

$$\begin{aligned}\tilde{a}_s(z) &= \tilde{a}_{s0}\sum_{j=1}^{3}\mathrm{e}^{\hat{\mu}_j\hat{z}}\frac{(\hat{\mu}_j+\mathrm{i}\widehat{\eta}_m)^2+\delta^2}{(3\hat{\mu}_j+\mathrm{i}\widehat{\eta}_m)(\hat{\mu}_j+\mathrm{i}\widehat{\eta}_m)+\delta^2}\\ &= \tilde{a}_{s0}\sum_{j=1}^{3}\frac{\mathrm{i}\mathrm{e}^{\hat{\mu}_j\hat{z}}}{\hat{\mu}_j[(3\hat{\mu}_j+\mathrm{i}\widehat{\eta}_m)(\hat{\mu}_j+\mathrm{i}\widehat{\eta}_m)+\delta^2]}. \qquad (13)\end{aligned}$$

Solving Eq. (12), we give its solution which has the same form as Eq. (7) but the p, q in it are replaced with

$$p,q = \frac{1}{\sqrt[3]{2}L_{g0}}\left\{1-2\left(\frac{\widehat{\eta}_m}{3}\right)^3+\frac{2}{3}\widehat{\eta}_m\delta^2\pm\sqrt{\left[1-2\left(\frac{\widehat{\eta}_m}{3}\right)^3+\frac{2}{3}\widehat{\eta}_m\delta^2\right]^2-4\left[\left(\frac{\widehat{\eta}_m}{3}\right)^2+\delta^2\right]^3}\right\}^{\frac{1}{3}}. \qquad (14)$$

From the root corresponding to the growing mode, the gain length as the function of the detuning can be given. In Fig. 3 the gain length versus $\widehat{\eta}_m$ is plotted by the formulas for different values of the energy spread δ. The curves are the same as that numerically given in Ref. [1]. It shows that as the energy spread increases, the gain length increases, and the detuning parameter corresponding to the shortest gain length also increases.

The optical field gain for a rectangular distribution of the beam energy can be calculated by Eq. (13). Fig. 4 shows the influence of the energy spread on the gain bandwidth near the saturation. As the energy spread increases, the detuning parameter corresponding to the maximum gain also increases, and the gain bandwidth becomes a little bit narrower. The optical power evolutions for different energy spreads are calculated and shown in Fig. 5. We can see that compared with the mono-energetic beam case, achieved optical power dropped about two orders for an energy spread of δ=0.5.

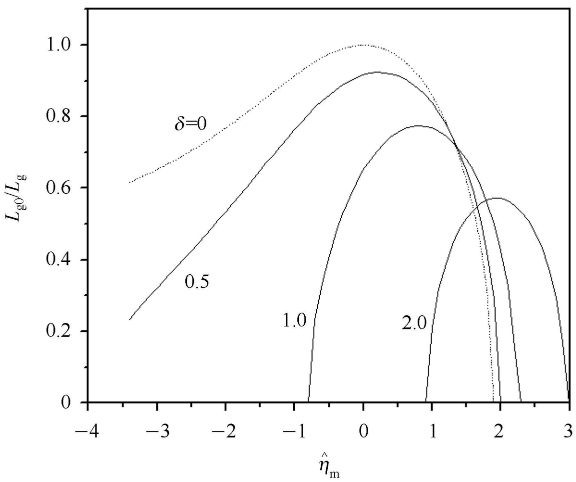

Fig. 3. Energy spread effect on the gain length for a rectangular energy distribution with different half-width δ.

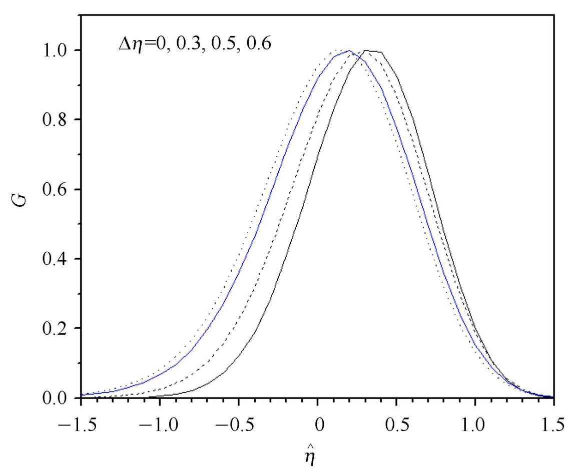

Fig. 4. Normalized gain vs. the detuning parameter for different energy spread (rectangular energy distribution).

In addition, we find that the characteristic cubic equation including the impact of the space charge for a mono-energetic beam

$$\hat{\mu}\left[(\hat{\mu}+\mathrm{i}\hat{\eta})^2+\widehat{k}_p^2\right]=\mathrm{i} \tag{15}$$

is the same as that for a beam with rectangular energy distribution (Eq. (12)), only the δ^2 in it is substituted by $\widehat{k}_\mathrm{p}^2=k_\mathrm{p}^2/(2k_\mathrm{u}\rho)^2$, $k_\mathrm{p}^2=\lambda_\mathrm{s}\omega_\mathrm{p}^2/\lambda_\mathrm{u}c^2\gamma$. Therefore the impact of the space charge on the gain also can be explicitly given by the corresponding substitution in Eqs. (13)–(14) and Fig. 3–5. We obtain the requirement of the practical physical quantities for neglecting the space charge field from $\widehat{k}_\mathrm{p}^2<1$

$$\rho<\frac{a_\mathrm{u}^2JJ^2}{2(1+a_\mathrm{u}^2)}, \tag{16}$$

where a_u is the dimensionless vector potential of the rms undulator field, JJ is the usual Bessel function factor.

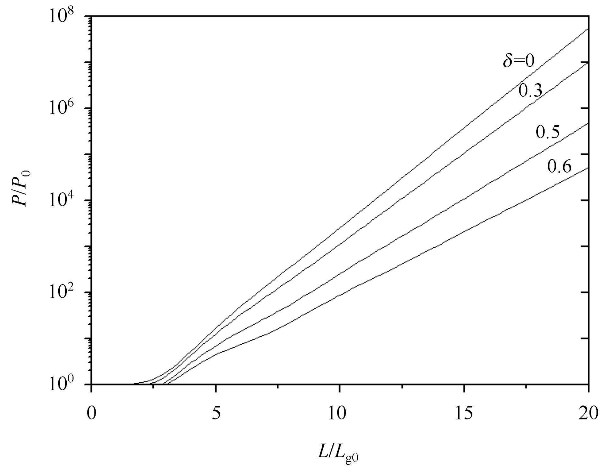

Fig. 5. The optical power evolution with different energy spread (rectangular energy distribution).

Besides rectangular energy distribution, the analytic solution also exists for the Lorentz distribution

$$f(\hat{\eta})=\frac{1}{\pi}\frac{\delta}{(\hat{\eta}-\hat{\eta}_m)^2+\delta^2}, \tag{17}$$

here $\delta=\sigma_\gamma/\gamma_0\rho$ is normalized energy spread. The cubic characteristic equation now is

$$\hat{\mu}(\hat{\mu}+\mathrm{i}\hat{\eta}_m+\delta)^2=\mathrm{i}. \tag{18}$$

Comparing it to Eq. (2), we can give its explicit solution in a similar way, we only need to replace the $\hat{\eta}$ in Eq. (2) with $\hat{\eta}_m-\mathrm{i}\delta$.

5 From low gain to high gain

From the expressions of the optical field and the characteristic roots (Eq. (1) and Eq. (7)), not only the exponential gain, the gain for the general case from the low gain to the high gain can be given. The transition from the low gain to the high gain was involved in several publications (e.g. in Refs. [1, 2]); here we give an analysis of it in detail. The level of FEL gain is determined by the number of the gain lengths included in the undulator. In the low gain regime, the small signal gain formula is often used:

$$g_{\rm ss} = -\hat{z}^3 \frac{\partial}{\partial x} {\rm sinc}^2 \frac{x}{2}, \ (x = \hat{\eta}\hat{z}). \tag{19}$$

The comparison between it and the exact result is presented in Fig. 6. It shows when the length of the undulator is larger than two gain lengths, the deviation of the small signal gain formula becomes large.

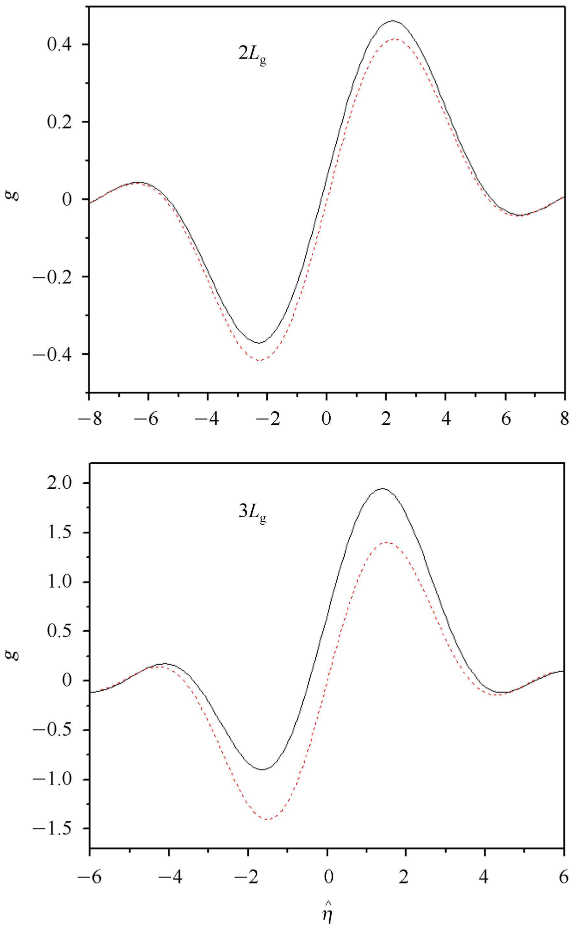

Fig. 6. Low gain, the exact result (from Eq. (7) and Eq. (1)) and the result of the small signal gain formula (Eq. (19), dashed line).

The variation of the gain with the detuning parameter for several different lengths of undulator is shown in Fig. 7. From Fig. 6 and Fig. 7, it can be seen that as the length of the undulator increases, the gain bandwidth decreases, from $\Delta\gamma/\gamma \approx 1/2N$ for the small signal gain to $\Delta\gamma/\gamma \approx \rho$ for the exponential gain, and the detuning parameter corresponding to the maximum gain decreases from $\hat{\eta} \approx 4.5 L_{\rm g}/L$ for the small signal gain to $\hat{\eta} \approx 0.12$ ($\sim 2.6 L_{\rm g}/L$) for the exponential gain.

The variation of the maximum gain with the undulator length is calculated and plotted in Fig. 8. For comparison, the gain at the resonance (Eq. (3)), the exponential gain (the approximation of Eq. (3)) and the small signal gain (Eq. (19)) are also plotted in Fig. 8. We can find that the small signal gain formula is more accurate for the case of $L < 2L_{\rm g}$, while the exponential gain formula is more accurate for the case of $L > 5L_{\rm g}$, the gain curves of two formulas cross at $L/L_{\rm g} \sim 3.2$.

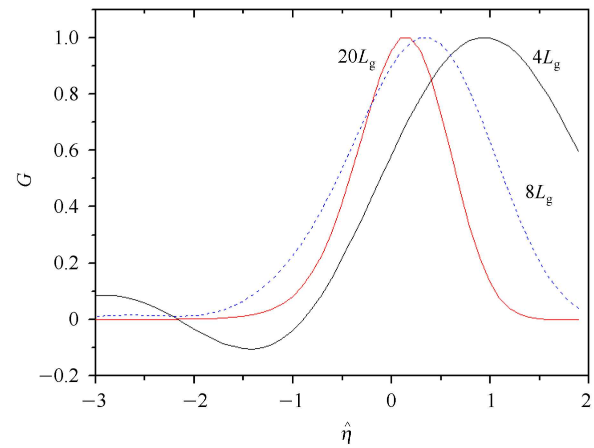

Fig. 7. The variation of the normalized gain with the detuning parameter for a different number of gain lengths.

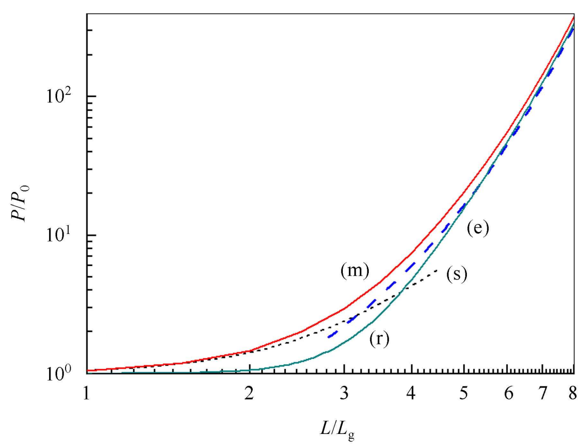

Fig. 8. The gain vs undulator length. The solid line (m): the maximum gain; The dotted line (s): the small signal gain formula (Eq. (19)); The solid line (r): the gain at the resonance (Eq. (3)); The dashed line (e): the exponential gain formula (the approximation of Eq. (3)). Notice that for the lines of the maximum gain and the small signal gain, the detunings are dependent on the longitudinal position.

6 Summary

We have studied the FEL gain formulas for the non-resonant case, and explicitly given some new rigorous analytical formulas. For the non-resonant electron beam, we give the explicit solution of the FEL characteristic cubic equation with a form much more simple than in the literatures [1, 6], and explicitly give the exact expression of the gain length as the function of the detuning parameter. We also give a simplified approximation formula for the exponential gain calculation in the non-resonant case. Then one can calculate the gain easily for different detuning parameters and from low to high, and calculate the gain length of the general case conveniently.

For the case of the electron beam having an initial energy spread, we give the solution of the characteristic cubic equation explicitly for the rectangular energy distribution and Lorentz distribution, respectively. Moreover we find that the explicit solution also can be used for the characteristic cubic equation including the impact of space charge, and give the requirement for the space charge field to be neglected.

We analyzed the transition of the gain from low to high. As the number of the gain length increased, the gain changes from the small signal gain to the exponential gain, the gain bandwidth decreases, from $\Delta\gamma/\gamma \approx 1/2N$ to $\Delta\gamma/\gamma \approx \rho$, and the detuning parameter corresponding to the maximum gain decreases from $\delta\gamma/\gamma \approx 2.6/2k_{\mathrm{u}}L$ to $\delta\gamma/\gamma \approx 0.12\rho$, though the maximum exponential growth rate is at the resonance $\delta\gamma/\gamma = 0$. The relations of the different gain formulas are revealed. It is shown that the small signal gain formula is more accurate in the case of $L < 2L_{\mathrm{g}}$, while the exponential gain formula is more accurate in the case of $L > 5L_{\mathrm{g}}$. Roughly, the small signal gain formula can be used for the undulator length smaller than about three gain lengths, and conversely, the exponential gain formula can be used.

The obtained analytical results provide convenience for the gain calculation, will help in the analysis and design of an FEL experiment, and will also help to develop insights into the FEL gain process.

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