

# Implications of equalities among the elements of CKM and PMNS matrices<sup>\*</sup>

Hong-Wei Ke (柯红卫)<sup>1;1)</sup> Song-Xue Zhao (赵松雪)<sup>1</sup> Xue-Qian Li (李学潜)<sup>2;2)</sup>

<sup>1</sup> School of Science, Tianjin University, Tianjin 300072, China

<sup>2</sup> School of Physics, Nankai University, Tianjin 300071, China

**Abstract:** Investigating the CKM matrix in different parameterization schemes, it is noticed that those schemes can be divided into a few groups where the sine values of the  $CP$  phase for each group are approximately equal i.e. there exist several relations among the  $CP$  phases. Using those relations, several approximate equalities among the elements of CKM matrix are established. The case can also be generalized to the PMNS matrix for the lepton sector. Assuming them to be exact, there are infinite numbers of solutions and by choosing special values for the free parameters in those solutions, several textures presented in the literature are obtained. Other authors have derived several mixing textures by using presumed symmetries; amazingly, some, though not all, of their forms are the same as those we obtained. This hints at the existence of a hidden symmetry which is broken in the practical world. Nature makes its own selection of the underlying symmetry and the way to break it, while we just guess what it is.

**Keywords:** CKM, PMNS, symmetry, parameterization scheme

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## 1 Introduction

Due to the mismatch between the eigenstates of the weak interaction and those of mass, the  $3 \times 3$  unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] is introduced to mix the three generations of quarks [3–5]. The CKM matrix is determined by three independent mixing angles and one  $CP$ -phase. The CKM matrix can be parametrized in different schemes and there are nine schemes proposed in the literature. Generally, the values of the three angles and the  $CP$  phase can be different for various parameterization schemes. By closely investigating the matrix, it is noticed that there exist some relations [6] among the  $CP$  phases in these schemes. For convenience let us label the nine schemes with subscripts  $a$  through  $i$ . Namely, we may divide the nine parameterization schemes into a few groups and determine corresponding equalities among those  $\sin \delta_n$  i.e.  $\sin \delta_a \approx \sin \delta_d \approx \sin \delta_e$ ,  $\sin \delta_b \approx \sin \delta_c$ ,  $\sin \delta_f \approx \sin \delta_h \approx \sin \delta_i$ . Then considering the constraint of the Jarlskog invariant [7–9], the above relations lead to several approximate equalities among the CKM matrix elements  $|U_{jk}|$  which are measured in experiments. These equalities are indeed

approximate, but independent of any concrete parameterization scheme.

These equalities tempt us to guess that there should exist underlying symmetries to determine them [6]. Our discussion on the implications of these equalities is based on observation and phenomenological. In parallel, an alternative route was also suggested that these equalities can be deduced by rephasing the invariants of the quark mixing matrix [10] as long as the mixing angles among quarks are small. In order to clarify the physical picture, we further study these equalities.

In analog to the quark sector, the Pontecorvo-Maki-Nakawaga-Sakata (PMNS) matrix [11, 12] relates the lepton flavor eigenstates with the mass eigenstates. Thus it is natural to extend the relations for the CKM matrix to the PMNS case. Unsurprisingly, we find that all the equalities also hold for the lepton sector, even though the accuracy is not as high as for the quark sector. The explanation based on only rephasing [10] is incomplete because it cannot explain why these equalities also hold for neutrino mixing where at least two mixing angles are large.

Since these equalities are respected by both CKM and

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1) E-mail: khw020056@hotmail.com

2) E-mail: lixq@nankai.edu.cn



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PMNS matrices, it is tempting to conjecture that there might be an underlying symmetry which results in symmetric forms for both CKM and PMNS matrices, which is broken in the practical world. Based on group theory, Lam showed a possibility that the mixing matrices originate from a higher symmetry [13, 14] which then breaks differently for quark and lepton sectors. The existence of quark-lepton complementarity and self-complementarity [15–26] also hints a higher symmetry. All the progress in this area inspires a trend of searching for whether such a high symmetry indeed exists and moreover investigation of its phenomenological implications is also needed.

Following this idea, we assume that the equalities are exact to compose equations, and solving these equations, the solutions might offer hints towards the unknown symmetry. To confirm or falsify the scenario, we further investigate the implications of these resultant matrices. It is found that these solutions coincide with the symmetrical CKM and PMNS textures. Moreover, some authors recently obtained some symmetric textures based on presumed symmetries, and it is found that some, but not all, of their resultant forms are the same as ours. We will further discuss the implications of this in the last section.

The paper is organized as follows. After the intro-

duction we review the equalities in Section 2. In Section 3, we present the solutions which satisfy those equalities (in fact, a few groups of solutions, each of which contains a free parameter) and their implications. In Section 4 we give a summary and discussion.

## 2 Relations among elements of the CKM matrices

Mixing among different flavors of quarks via the CKM matrix has been firmly recognized and the  $3 \times 3$  mixing matrix is written as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}. \quad (1)$$

Generally, for a  $3 \times 3$  unitary matrix there are four independent parameters, namely three mixing angles and one  $CP$ -phase. There can be various schemes to parameterize the matrix, but only nine schemes are independent. These are clearly listed in Ref. [24], but for readers' convenience, we summarise them in Table 1.

To be more clear, we present the explicit expressions of two typical parameterization schemes  $P_a$  and  $P_e$  as

$$V_{P_a} = \begin{pmatrix} c_{a1}c_{a3} & s_{a1}c_{a3} & s_{a3} \\ -c_{a1}s_{a2}s_{a3} - s_{a1}c_{a2}e^{-i\delta_a} & -s_{a1}s_{a2}s_{a3} + c_{a1}c_{a2}e^{-i\delta_a} & s_{a2}c_{a3} \\ -c_{a1}s_{a2}s_{a3} + s_{a1}s_{a2}e^{-i\delta_a} & -s_{a1}s_{a2}s_{a3} - c_{a1}c_{a2}e^{-i\delta_a} & c_{a2}c_{a3} \end{pmatrix}, \quad (2)$$

and

$$V_{P_e} = \begin{pmatrix} -s_{e1}s_{e2}s_{e3} + c_{e1}c_{e3}e^{-i\delta_e} & -c_{e1}s_{e2}s_{e3} - s_{e1}c_{e2}e^{-i\delta_e} & c_{e2}s_{e3} \\ s_{e1}c_{e2} & c_{e1}c_{e2} & s_{e2} \\ -s_{e1}s_{e2}c_{e3} - c_{e1}s_{e3}e^{-i\delta_e} & -c_{e1}s_{e2}c_{e3} + s_{e1}s_{e3}e^{i\delta_e} & c_{e2}c_{e3} \end{pmatrix}. \quad (3)$$

Table 1. Nine different parameterization schemes for the CKM matrix.

scheme	Jarlskog invariant	$CP$ phase
$P_a$	$J_a = s_{a1}s_{a2}s_{a3}c_{a1}c_{a2}c_{a3}^2 \sin \delta_a$	$\delta_a = \left( 69.10^{+2.02}_{-3.85} \right)^\circ$
$P_b$	$J_b = s_{b1}s_{b2}^2s_{b3}c_{b1}c_{b2}c_{b3} \sin \delta_b$	$\delta_b = \left( 89.69^{+2.29}_{-3.95} \right)^\circ$
$P_c$	$J_c = s_{c1}^2s_{c2}s_{c3}c_{c1}c_{c2}c_{c3} \sin \delta_c$	$\delta_c = \left( 89.29^{+3.99}_{-2.33} \right)^\circ$
$P_d$	$J_d = s_{d1}s_{d2}s_{d3}c_{d1}^2c_{d2}c_{d3} \sin \delta_d$	$\delta_d = \left( 111.95^{+3.82}_{-2.02} \right)^\circ$
$P_e$	$J_e = s_{e1}s_{e2}s_{e3}c_{e1}c_{e2}^2c_{e3} \sin \delta_e$	$\delta_e = \left( 110.94^{+3.85}_{-2.02} \right)^\circ$
$P_f$	$J_f = s_{f1}s_{f2}s_{f3}c_{f1}c_{f2}c_{f3}^2 \sin \delta_f$	$\delta_f = \left( 22.72^{+1.25}_{-1.18} \right)^\circ$
$P_g$	$J_g = s_{g1}^2s_{g2}s_{g3}c_{g1}c_{g2}c_{g3} \sin \delta_g$	$\delta_g = \left( 1.08^{+0.06}_{-0.06} \right)^\circ$
$P_h$	$J_h = s_{h1}s_{h2}s_{h3}c_{h1}c_{h2}^2c_{h3} \sin \delta_h$	$\delta_h = \left( 157.31^{+1.18}_{-1.25} \right)^\circ$
$P_i$	$J_i = s_{i1}s_{i2}s_{i3}c_{i1}^2c_{i2}c_{i3} \sin \delta_i$	$\delta_i = \left( 158.32^{+1.13}_{-1.20} \right)^\circ$

Here  $s_{aj}$  and  $c_{aj}$  ( $s_{ej}$  and  $c_{ej}$ ) denote  $\sin \theta_{aj}$  and  $\cos \theta_{aj}$  ( $\sin \theta_{ej}$  and  $\cos \theta_{ej}$ ) with  $j = 1, 2, 3$ .  $\theta_{nj}$  and  $\delta_n$  are the mixing angles and  $CP$ -phase respectively. The

corresponding expressions in other schemes  $P_n$  can be found in Ref. [24].

From the data measured in various experiments, one can deduce values of the angles  $\theta_{nj}$  and  $CP$  phase  $\delta_n$  which are not the same for different parameterizations.

Close observation of the values of  $\delta_n$  in different schemes exhibits several approximate equalities

$$\sin \delta_a \approx \sin \delta_d \approx \sin \delta_e, \quad \sin \delta_b \approx \sin \delta_c, \quad \sin \delta_f \approx \sin \delta_h \approx \sin \delta_i. \quad (4)$$

Namely, the nine phase factors in the nine schemes are divided into a few groups and their sine values in each group are approximately equal. It is noted that there are nine parameterization schemes in total and  $P_g$  is a special one whose  $CP$  phase is very small. Since this scheme is indeed peculiar, in Eq. (4), the listed relations do not include  $\delta_g$  at all. The scenario with the  $P_g$  scheme was carefully investigated and discussed by the authors of Ref. [27]. If more complex relations are con-

sidered, we can involve the small  $\delta_g$  together with that in the other schemes. The Jarlskog invariant is scheme-independent, so using the above relations in Eq. (4) and substituting  $s_{nj}$  and  $c_{nj}$  with the ratios of modules of corresponding elements, one can deduce several interesting relations among the CKM elements, which are experimentally measured values and obviously free of parameterization schemes:

$$\begin{aligned} \frac{|V_{21}||V_{22}|}{1-|V_{23}|^2} - \frac{|V_{11}||V_{12}|}{|V_{23}|^2+|V_{33}|^2} &\approx 0 \\ \frac{|V_{11}||V_{12}||V_{21}|}{1-|V_{11}|^2} - \frac{|V_{23}||V_{32}||V_{33}|}{1-|V_{33}|^2} &\approx 0 \\ \frac{|V_{21}||V_{23}||V_{33}|}{1-|V_{23}|^2} - \frac{|V_{11}||V_{12}||V_{32}|}{|V_{22}|^2+|V_{32}|^2} &\approx 0 \\ \frac{|V_{12}||V_{22}|}{1-|V_{32}|^2} - \frac{|V_{11}||V_{21}|}{|V_{11}|^2+|V_{21}|^2} &\approx 0 \\ \frac{|V_{12}||V_{32}||V_{33}|}{|V_{12}|^2+|V_{22}|^2} - \frac{|V_{11}||V_{21}||V_{23}|}{1-|V_{21}|^2} &\approx 0. \end{aligned} \tag{5}$$

More details about these equalities can be found in Ref. [6].

### 3 Implication of the relations

#### 3.1 On these relations

Even though our allegation starts from a phenomenological observation, it is natural to attribute these equalities to an underlying symmetry. In parallel, it was argued that they can automatically emerge from a different ansatz which we briefly outline in the appendix.

These relations are proved to be exact equalities under the limit  $\theta_{a2} \rightarrow 0$  and  $\theta_{a3} \rightarrow 0$ , so they are the consequence of small  $\theta_{a2}$  and  $\theta_{a3}$  and the practical approximation indeed comes from being non-zero. For an illustration, anyone can check those relations for  $P_a$  parameterization and we present the details in Appendix A. There, since  $\theta_{a1} = (13.023_{-0.038}^{+0.038})^\circ$ ,  $\theta_{a2} = (2.360_{-0.038}^{+0.065})^\circ$ ,  $\theta_{a3} = (0.201_{-0.008}^{+0.010})^\circ$  [24], the picture seems to work almost perfectly.

Another way to obtain these equalities can be found by starting from rephasing the invariants of the quark mixing matrix. In Ref. [10] the authors pointed out that  $V_{i\alpha} V_{j\alpha}^* V_{j\beta} V_{i\beta}^*$  are invariants whose imaginary component is the traditional Jarlskog invariant. Since  $V_{i\alpha} V_{j\alpha}^* V_{j\beta} V_{i\beta}^*$  are invariants in different parameterizations one can use them to deduce relations among the physical matrix elements. For example by comparing the real parts and imaginary parts of the invariant  $V_{12} V_{22}^* V_{23} V_{13}^*$  in the  $P_a$  and  $P_e$  parameterizations,  $\sin\delta_a \approx \sin\delta_e$  can be deduced with the postulates of small  $\theta_{a2}$  and  $\theta_{a3}$ . Some details are presented in Appendix B.

The two ways are similar when the same condition of  $\theta_{a2}$  and  $\theta_{a3}$  being small is taken. If one just discusses the quark case the two ways seem to be parallel. However if one tries to extend these relations to the PMNS case, they need to be reconsidered more carefully because then the condition of small mixing angles no longer exists.

In fact, assuming those relations to be exact, solving the equations we obtain several independent solutions and each of them contains a free parameter to be fixed.

In the next section we will show that for the quark sector, the two ways correspond to special choices of the parameters in the solutions, but for the lepton sector they are different.

#### 3.2 Solutions of these relations

Now we replace the “ $\approx$ ” with an equality sign “=” in Eq. (5) to compose equations and obtain their solutions. Since these solutions are expected to correspond to the symmetrical textures for CKM and/or PMNS matrices, the normalization of the unitary matrix

$$|V_{11}|^2 + |V_{12}|^2 + |V_{13}|^2 = 1, |V_{11}|^2 + |V_{21}|^2 + |V_{31}|^2 = 1, \dots \tag{6}$$

should be retained.

It is noted that even though we establish the equalities from equating the  $CP$  phases of different parameterizations, in the latter procedures only the ratios of modules of the matrix elements are employed to build up equations, thus one cannot gain much information about the phases of the matrix elements from the normalization relations and Eq. (5). If one hopes to know the phases of the elements some new constraints must be further enforced, such as orthogonality between any two different rows or columns of the matrix. Now, the newly built equations are free of concrete parameterizations.

Satisfying all the requirements in Eq. (5), one can achieve several solutions. They are

$$\begin{aligned} |V_1| &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \\ |V_2| &= \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \end{pmatrix}, \\ |V_3| &= \begin{pmatrix} \sin\theta & \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} & \sin\theta & \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \sin\theta \end{pmatrix}, \end{aligned}$$

$$|V_4| = \begin{pmatrix} \sin\phi & \sin\phi & \sqrt{\cos 2\phi} \\ \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} & \sin\phi \\ \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} & \sin\phi \end{pmatrix},$$

$$|V_5| = \begin{pmatrix} \sin\phi & \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} \\ \sin\phi & \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} \\ \sqrt{\cos 2\phi} & \sin\phi & \sin\phi \end{pmatrix} \quad (7)$$

where  $\theta$  lies in the range  $0^\circ - 90^\circ$ ,  $\phi$  stays in the range  $0^\circ - 45^\circ$  and  $|V_a| (a = 1-5)$  represent the mixing matrices which only contain the moduli of matrix elements. Definitely, in such a way, the unitarity of the matrix does not manifest at all. Later, see below, when we discuss the practical CKM or PMNS matrices, we need to input phases by hand. As stated above, as other constraints involving the orthogonality among the elements are applied, the phases would be automatically taken in, but the procedure for obtaining solutions is much more complicated and tedious, so we will leave the task as the goal of our next work. One may notice that  $|V_1|$ ,  $|V_2|$  and  $|V_3|$  are just real symmetrical matrixes and  $|V_5|$  is just the transposed matrix of  $|V_4|$ .

### 3.3 Issues related to the CKM matrix

As  $\theta$  in  $|V_2|$  and  $|V_3|$  is set to be  $90^\circ$ , one immediately obtains

$$|V_2| = |V_3| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (8)$$

which is just the the CKM matrix under the limits of  $\theta_{a2} \rightarrow 0$  and  $\theta_{a3} \rightarrow 0$ . At this moment one may be convinced that the way to understand the equalities discussed in subsection A is indeed practical. Actually, it has been noticed for a long time that the CKM matrix is close to being a unit matrix, and in Ref. [28] the authors suggested to transform a unit matrix to the practical CKM matrix by introducing a new D quark.

### 3.4 Issues related to some symmetrical PMNS patterns

Next, let us explore whether these solutions can be related to the symmetrical PMNS textures.

If  $\phi = 45^\circ$  in  $|V_4|$  we get

$$|V_4| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (9)$$

which is nothing more than the modula of the bimaximal mixing pattern [29–31]. This is not surprising because the proposed PMNS textures satisfy the equations exactly due to the existence of a hidden symmetry.

Cabibbo [32] and Wolfenstein [33] proposed a symmetrical PMNS matrix as

$$V_{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (10)$$

where  $\omega = e^{i2\pi/3}$ . It is found that if only the moduli of the matrix elements are concerned, the  $|V_{CW}|$  (i.e. as one only keeps the moduli of elements) is just our solution  $|V_1|$ . In the  $A_4$ [34, 35] or  $S_4$  [36, 37] models, the charged lepton mass matrix is diagonalized by the unitary  $V_{CW}$  and the Majorana mass matrix of neutrinos is diagonalized by  $V_\nu$ , which is written as

$$V_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (11)$$

where  $|V_\nu|$  is equal to our  $|V_2|$  by setting  $\theta = \frac{\pi}{4}$ . As we introduce phases in  $|V_2|$  to make it  $V_2$ , then moving further one can obtain the tribimaximal texture ( $V_{TB}$ ) which is the product  $V_{CW}V_\nu$ [38, 39],

$$V_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (12)$$

In Ref. [40] the authors constructed a new mixing pattern for neutrinos based on the  $\mu-\tau$  interchange symmetry, the trimaximal mixing in  $\nu_2$  and the self-complementarity relation. The mixing matrix is

$$V_{QM} = \begin{pmatrix} \frac{\sqrt{2}+1}{3} & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}-1}{3} \\ -\frac{\sqrt{2}+1}{6} \mp i \frac{\sqrt{6}-\sqrt{3}}{6} & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}-1}{6} \mp i \frac{\sqrt{6}+\sqrt{3}}{6} \\ \frac{\sqrt{2}+1}{6} \mp i \frac{\sqrt{6}-\sqrt{3}}{6} & -\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}-1}{6} \mp i \frac{\sqrt{6}+\sqrt{3}}{6} \end{pmatrix}. \quad (13)$$

In analog to the procedure of obtaining the tribimaximal mixing pattern we can derive  $V_{QM}$  from our solutions  $V_1$  and  $V_2$ , while proper phases are set by hand. Namely, as one sets

$$V_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ a & 1 & a^* \\ -a^* & -1 & -a \end{pmatrix}, V_2 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \quad (14)$$

the product  $V_1 V_2$  just gives the mixing matrix with  $a = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ . All the relations between the solutions with the proposed PMNS were unexpected before.

In Ref. [41] the authors assumed that neutrinos are Dirac particles, then they derived lepton mixing matrices from the flavor  $SU(3)$ . We notice that the solutions in Eq. (8), Eq.(9) and Eq. (10) can also be produced from the  $SU(3)$  group.

It is noted that  $|V_1|$  and  $|V_2|$  are solutions of the equalities but  $|V_{TB}|$  and  $|V_{QM}|$  are not; they deviate from the solutions slightly. In Table 2 we calculate and list the deviations of the corresponding quantities in  $|V_{TB}|$  and  $|V_{QM}|$  from the left-hand sides of Eq. (5). There are five equalities in Eq. (5), and a few of the five are satisfied, while the others differ slightly.

Table 2. The values of the left-hand sides in Eq. (5). The labels No. 1 to No. 5 refer to the first to fifth equations in the equation group (5).

	No.1	No.2	No.3	No.4	No.5
$ V_{TB} $	0	0	0	0.1	0.0707
$ V_{QM} $	0.0020	-0.0023	0.0017	0.0900	0.0630

### 4 Summary and discussion

Based on the observed relations  $\sin\delta_a \approx \sin\delta_d \approx \sin\delta_e, \sin\delta_b \approx \sin\delta_c$  and  $\sin\delta_f \approx \sin\delta_h \approx \sin\delta_i$  among the  $CP$  phases in the nine parameterization schemes which have been widely discussed in the literature, it is conjectured that they originate from a higher symmetry which later breaks by some mechanism. Even though we so far do not know what the symmetry is and what

breaks it, one can be convinced by those equalities of their existence. Further assuming those relations to hold exactly, we are able to establish several scheme-independent equalities [26]. These relations which we obtained by exploring the CKM matrix also work for the PMNS matrix. How to understand these relations is one of the tasks of our work.

This probably corresponds to Lam’s suggestion [13, 14] that a generic potential is invariant under  $U(1) \times SO(3)$  and the potential causes a breakdown into three phases. Phase I has an  $A_4$  symmetry which is suitable for leptonic mixing whereas the other two phases have symmetries  $SO(2)$  and  $Z_2 \times Z_2$ . The  $SO(2)$  phase is ruled out by phenomenology and the  $Z_2 \times Z_2$  is for the quark mixing. We derive similar results from solving the equalities, i.e. as we showed in subsections III-C and III-D,  $|V_2|(\theta = 90^\circ)$  and  $|V_3|(\theta = 90^\circ)$  correspond to the quark mixing and  $|V_4|(\phi = 45^\circ)$  is related to leptonic mixing, and  $|V_2|$   $|V_3|$  and  $|V_4|$  are all solutions of Eq. (5). So far, we have derived the relations and got some symmetric textures from phenomenology and have not yet associated the results with the underlying symmetry, as discussed above, but we will in our later works.

It is possible to derive similar relations from different starting points. It has been conjectured that the equalities we derived above can just be the consequences of the small mixing angles between quarks and are irrelevant to any symmetry. Even though these equalities can be deduced by enforcing certain rephasing invariants on the quark mixing matrix with the condition of small mixing angles, the fact that these equalities also hold for the lepton sector, with two large mixing angles, obviously does not fit the arguments.

We obtain the solutions when the “ $\approx$ ” sign is set as “=” for those equalities. There is an infinite number of solutions and each of them has one free parameter. We note that the unit matrix is also one of the solutions, which is just the limit case of the CKM matrix under the condition  $\theta_2 \rightarrow 0$  and  $\theta_3 \rightarrow 0$  in any parameterization scheme. It implies that these equalities are indeed non-trivial after all.

We extend the relations to the lepton case, namely one can immediately relate some of the obtained solutions to the symmetric textures for the PMNS matrices proposed in the literature, such as the bimaximal and tri-bimaximal mixing patterns. Concretely, the bimaxi-

mal texture corresponds to one solution whereas the tri-bimaximal texture can be related to two solutions.

A more complex mixing texture which was suggested in Ref. [40] can be constructed from two of our solutions. The relations seem to weave a net to include many unexpected phenomena; all these may indicate that these equalities reflect the existence of a definite symmetry. These equalities may hold initially at high energy scales, such as the see-saw or GUT scale, then the symme-

try is distorted or broken by some mechanism, and the equalities become approximate for the CKM and PMNS matrices at practical energy scales. Further studies on these relations should lead to eventually understanding the symmetry and breaking mechanism.

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## Appendix A

### Check of the relation under limits

The relations in Eq. (5) can be proved under some limits. As an example we check the first one in  $P_a$  parameterization. The left side is

$$\begin{aligned} & \lim_{a_2 \rightarrow 0, a_3 \rightarrow 0} \frac{|V_{21}||V_{22}|}{1 - |V_{23}|^2} \\ &= \lim_{a_2 \rightarrow 0, a_3 \rightarrow 0} \frac{|-C_{a1}s_{a2}s_{a3} - s_{a1}c_{a2}e^{-i\delta_a}| | -s_{a1}s_{a2}s_{a3} + C_{a1}c_{a2}e^{-i\delta_a}|}{1 - |s_{a2}c_{a3}|^2} \\ &= s_{a1}c_{a1} \end{aligned} \quad (\text{A1})$$

## Appendix B

### Deduction of the relations using the rephasing invariants of the quark mixing matrix

In  $P_a$  parameterization

$$V_{12}V_{22}^*V_{23}V_{13}^* = c_{a1}c_{a2}c_{a3}^2s_{a1}s_{a2}s_{a3}e^{i\delta_a} - c_{a3}^2s_{a1}^2s_{a2}^2s_{a3}^2. \quad (\text{B1})$$

In  $P_e$  parameterization

$$V_{12}V_{22}^*V_{23}V_{13}^* = -c_{e1}c_{e2}^2c_{e3}s_{e1}s_{e2}s_{e3}e^{-i\delta_e} - c_{e1}^2c_{e2}^2s_{e2}^2s_{e3}^2. \quad (\text{B2})$$

From Eq. (2) and Eq. (3) one has  $s_{a3} = c_{e2}s_{e3}$ ,  $s_{a2}c_{a3} = s_{e2}$ ,  $c_{a2}c_{a3} = c_{e2}c_{e3}$  so Eq. (B2) changes into

$$V_{12}V_{22}^*V_{23}V_{13}^* = -s_{e1}c_{e1}c_{a3}^2s_{a3}s_{a2}c_{a2}e^{-i\delta_e} - c_{e1}^2s_{a3}^2s_{a2}^2c_{a3}^2. \quad (\text{B3})$$

and the right side is

$$\begin{aligned} & \lim_{a_2 \rightarrow 0, a_3 \rightarrow 0} \frac{|V_{11}||V_{12}|}{|V_{23}|^2 + |V_{33}|^2} \\ &= \lim_{a_2 \rightarrow 0, a_3 \rightarrow 0} \frac{|c_{a1}c_{a3}||s_{a1}c_{a3}|}{|s_{a2}c_{a3}|^2 + |c_{a2}c_{a3}|^2} = s_{a1}c_{a1} \end{aligned} \quad (\text{A2})$$

so one obtains  $\frac{|V_{21}||V_{22}|}{1 - |V_{23}|^2} = \frac{|V_{11}||V_{12}|}{|V_{23}|^2 + |V_{33}|^2}$ .

Using the invariant  $V_{12}V_{22}^*V_{23}V_{13}^*$ , which is supposed to be free of parameterization schemes, one can obtain

$$\begin{aligned} & c_{a1}s_{a1}c_{a3}^2s_{a2}s_{a3}c_{a2}e^{i\delta_a} - c_{a3}^2s_{a1}^2s_{a2}^2s_{a3}^2 \\ &= -s_{e1}c_{e1}c_{a3}^2s_{a3}s_{a2}c_{a2}e^{-i\delta_e} - c_{e1}^2s_{a3}^2s_{a2}^2c_{a3}^2. \end{aligned} \quad (\text{B4})$$

Dividing it by  $c_{a3}^2s_{a2}s_{a3}c_{a2}$  gives

$$c_{a1}s_{a1}e^{i\delta_a} - \frac{s_{a1}s_{a2}s_{a3}}{c_{a2}} = -s_{e1}c_{e1}e^{-i\delta_e} - \frac{c_{e1}^2s_{a3}s_{a2}}{c_{a2}}. \quad (\text{B5})$$

Considering both the real and imaginary parts to be invariant, and supposing small angles  $\theta_{a2}$  and  $\theta_{a3}$ , one has  $\tan\delta_a = -\tan\delta_e$  then the result  $\sin\delta_a = \sin\delta_e$  can be deduced. That is the same as we have by phenomenology, which is directly related to experimental measurements.

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