Hawking radiation and entropy of a black hole in Lovelock-Born-Infeld gravity from the quantum tunneling approach *

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Abstract: The tunneling radiation of particles from black holes in Lovelock-Born-Infeld (LBI) gravity is studied by using the Parikh-Wilczek (PW) method, and the emission rate of a particle is calculated. It is shown that the emission spectrum deviates from the purely thermal spectrum but is consistent with an underlying unitary theory. Compared to the conventional tunneling rate related to the increment of black hole entropy, the entropy of the black hole in LBI gravity is obtained. The entropy does not obey the area law unless all the Lovelock coefficients equal zero, but it satisfies the first law of thermodynamics and is in accordance with earlier results. It is distinctly shown that the PW tunneling framework is related to the thermodynamic laws of the black hole.

Keywords: Tunneling radiation, Lovelock-Born-Infeld gravity, first law of thermodynamics, black hole

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1 Introduction

Forty years ago, it was proved by Hawking [1] that black holes can radiate thermally. All research on black hole radiation, such as Refs. [2–6], aimed to prove that the energy spectrum is precisely thermal until 1995, when a method to describe Hawking radiation as a tunneling process where a particle moves in dynamic geometry was developed by Kraus and Wilczek [7] and elaborated upon by Parikh and Wilczek [8–9]. They think that the barrier is created by the outgoing particle itself and their key insight is to find a coordinate system which is well behaved at the horizon. Taking the self-interaction effect into account and considering energy conservation, they calculated the corrected emission spectra of spherically symmetric black holes, such as Schwarzschild black holes and Reissner-Norström black holes. In 2005, Hawking changed his opinion and argued that information can indeed get out of the black hole [10], which maybe partly be based on the PW work. Since then, the PW method has been used to calculate the emission rate of particles from various black holes [11–24] and to obtain modified spectra. In this paper, we would like to extend the quantum tunneling approach to black holes in LBI gravity to calculate the corrected emission spectrum of particles from their event horizons and their entropies, so as to explore the influence on the thermodynamic properties due to the higher derivative gravity. Because the metric contains a hypergeometric function, the analysis becomes complicated and it is nontrivial to calculate the thermodynamics completely.

2 Radial motion equation of particles

Among the higher curvature gravity theories, the so-called Lovelock gravity [25] is quite special. Its Lagrangian consists of the dimensionally extended Euler densities. In this gravity theory, the field equation is only second order and the quantization of the linearized Lovelock theory is free of ghosts [26]. Thus, it is natural to study the effects of higher curvature terms on the properties and thermodynamics of black holes. The black hole solutions and their thermodynamics in Lovelock gravity have been widely studied [27–54]. Moreover, it is natural to consider the nonlinear terms on the matter side of the action while accepting the nonlinear terms of the invariants constructed by the Riemann tensor on the gravity action. Thus, in the presence of an electromagnetic field, it is worth to apply the Born-Infeld action [55] instead of the Maxwell action. Motivated by this, Ref. [27] presented topological black hole solutions in Lovelock-Born-Infeld gravity.

The action of third order Lovelock gravity with nonlinear Born-Infeld electromagnetic field is [27]

$$I_G=\frac{1}{16\pi}\int \mathrm{d}^{n+1}x\sqrt{-g}(-2\Lambda+L_1+\alpha_2L_2+\alpha_3L_3+L(F)), \tag{1}$$

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where Λ is the cosmological constant, α_2 and α_3 are the second and third order Lovelock coefficients, $L_1 = R$ is the Einstein-Hilbert Lagrangian, $L_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet Lagrangian,

$$\begin{aligned}L_3 = &2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}_{\mu\nu} + 8R^{\mu\nu}_{\sigma\rho}R^{\sigma\kappa}_{\nu\tau}R^{\rho\tau}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}_{\mu}\\ &+ 3RR^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 1624R^{\mu\nu}R_{\nu\sigma}R^{\sigma}_{\mu}\\ &- 12RR^{\mu\nu}R_{\mu\nu} + R^3\end{aligned} \tag{2}$$

is the third order Lovelock Lagrangian, and $L(F)$ is the Born-Infeld Lagrangian given as

$$L(F) = 4\beta^2\left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}}\right). \tag{3}$$

When the Born-Infeld parameter β goes to infinity, $L(F)$ reduces to the standard Maxwell form $L(F) = -F^2$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A_μ is the electromagnetic vector field given as

$$A_\mu = \sqrt{\frac{n-1}{2n-4}}\frac{q}{r^{n-2}}F(\eta)\delta^0_\mu. \tag{4}$$

When $\eta \to 0$ ($\beta \to \infty$) , $F(\eta) \to 1$, and therefore the vector potential (4) reduces to the gauge potential of the Maxwell field.

Considering the case

$$\alpha_2 = \frac{\alpha}{(n-2)(n-3)}, \qquad \alpha_3 = \frac{\alpha^2}{72\begin{pmatrix} n-2 \\ 4 \end{pmatrix}}, \tag{5}$$

Ref. [27] derived the $(n+1)$-dimensional static solution. For the $k = 1$ case corresponding to spherical topology, the metric of the black hole reads

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2 \tag{6}$$

where

$$\begin{aligned}f(r) &= 1 + \frac{r^2}{\alpha}\left(1 - \sqrt[3]{g(r)}\right),\\ g(r) &= 1 + \frac{3\alpha m}{r^n} - \frac{12\alpha\beta^2}{n(n-1)}\\ &\quad\left[1 - \sqrt{1+\eta} - \frac{\Lambda}{2\beta^2} + \frac{n-1}{n-2}\eta F(\eta)\right],\\ \mathrm{d}\Omega^2 &= \mathrm{d}\theta_1^2 + \sum_{i=2}^{n-1}\prod_{j=1}^{i-1}\sin^2\theta_j\mathrm{d}\theta_i^2.\end{aligned} \tag{7}$$

$\mathrm{d}\Omega^2$ denotes the line element of a $(n-1)$ dimensional hypersurface with constant curvature $(n-1)(n-2)$, and $F(\eta)$ is a hypergeometric function of the form

$$F(\eta) = {}_2F_1\left(\frac{1}{2}, \frac{n-2}{2n-2}; \frac{3n-4}{2n-2}; -\eta\right) \tag{8}$$

where

$$\eta = \frac{(n-1)(n-2)q^2}{2\beta^2 r^{2n-2}}. \tag{9}$$

The ADM (Arnowitt-Deser-Misner) mass and electric charge of the black hole in terms of the parameters m and q are respectively given as [27]

$$M = \frac{(n-1)V_{n-1}}{16\pi}m, \qquad Q = \frac{V_{n-1}}{4\pi}\sqrt{\frac{(n-1)(n-2)}{2}}q \tag{10}$$

where V_{n-1} is the volume of the $(n-1)$-dimensional hypersurface mentioned above.

Solving the equation $f(r) = 0$, we get the largest positive root and denote it by r_+, from which we can determine the event horizon radius. In terms of r_+, the ADM mass and Hawking temperature of the black hole can be written as

$$\begin{aligned}M = \frac{(n-1)V_{n-1}r_+^n}{48\pi\alpha}&\left\{-1 + \left(1 + \frac{\alpha}{r_+^2}\right)^3 + \frac{12\alpha\beta^2}{n(n-1)}\right.\\ &\left.\left[1 - \frac{\Lambda}{2\beta^2} - \sqrt{1+\eta_+} + \frac{n-1}{n-2}F(\eta_+)\eta_+\right]\right\},\end{aligned} \tag{11}$$

$$T = \frac{f'(r)}{4\pi} = \frac{(n-1)[3(n-2)r_+^4 + 3(n-4)\alpha r_+^2 + (n-6)\alpha^2] + 12r_+^6\beta^2(1 - \sqrt{1+\eta_+}) - 6\Lambda r_+^6}{12\pi(n-1)r_+(r_+^2+\alpha)^2} \tag{12}$$

where $\eta_+ = \dfrac{(n-1)(n-2)q^2}{2\beta^2 r_+^{2n-2}}$ and we have used the equality

$$(n-2)F(\eta_+) - 2(n-1)\eta_+F'(\eta_+) = \frac{n-2}{\sqrt{1+\eta_+}}. \tag{13}$$

To apply the Parikh-Wilczek method, we make the transformation

$$\mathrm{d}T = \mathrm{d}t + \frac{1}{f(r)}\sqrt{\frac{r_+}{r}}\mathrm{d}r. \tag{14}$$

Then the line element (6) becomes

$$\mathrm{d}s^2 = -f(r)\mathrm{d}T^2 + 2\sqrt{\frac{r_+}{r}}\mathrm{d}T\mathrm{d}r + \frac{r-r_+}{r}\frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{15}$$

Obviously, the line element (15) is well-behaved at the horizon position, which is necessary to describe tunneling. The radial outgoing null geodesic is given by

$$\dot{r}=f(r)\left(1+\sqrt{\frac{r_+}{r}}\right)^{-1} \tag{16}$$

where the dot denotes differentiation with respect to T.

3 Emission rate and entropy

The total energy of a stationary space-time should be conserved during the emission. When particle's self-gravitation is taken into account and a particle of energy ω is emitted, the black hole mass will become $M-\omega$ and all above and following relevant equations should be used with $M\to M-\omega$. Since the metric is of spherical symmetry, regarding the outgoing particle as an s-wave, i.e. a shell of energy, is reasonable.

The imaginary part of the action for a particle crossing the horizon outwards from the initial radius $r_{\rm i}$ to the final radius $r_{\rm f}$ can be expressed as

$$\mathrm{Im}Z=\mathrm{Im}\int_{T_{\rm i}}^{T_{\rm f}}L\mathrm{d}T=\mathrm{Im}\int_{r_{\rm i}}^{r_{\rm f}}p_r\mathrm{d}r=\mathrm{Im}\int_{r_{\rm i}}^{r_{\rm f}}\int_0^{p_r}\mathrm{d}p_r\mathrm{d}r \tag{17}$$

where p_r is the canonical momentum conjugate to r. Taking the Hamiltonian equation into account, we can obtain

$$\dot{r}=\left.\frac{\mathrm{d}H}{\mathrm{d}p_r}\right|_r \tag{18}$$

where $(dH)_r=dM$. Changing the variable from the momentum to the energy and switching the order of integration, we have

$$\begin{aligned}\mathrm{Im}Z&=\mathrm{Im}\int_{M_{\rm i}}^{M_{\rm f}}\int_{r_{\rm i}}^{r_{\rm f}}\frac{\mathrm{d}r}{\dot{r}}\mathrm{d}M\\&=\mathrm{Im}\int_{M_{\rm i}}^{M_{\rm f}}\int_{r_{\rm i}}^{r_{\rm f}}\frac{1+\sqrt{r_+/r}}{f(r)}\mathrm{d}r\mathrm{d}M\end{aligned} \tag{19}$$

where $M_{\rm i}=M; M_{\rm f}=M-\omega$.

It is easy to find that the integrand is singular at the point $r=r_+$. The integral can be evaluated by deforming the contour around the pole. Doing the r integral, we have

$$\mathrm{Im}Z=-2\pi\int_{M_{\rm i}}^{M_{\rm f}}\frac{1}{f'(r_{\rm H})}\mathrm{d}M. \tag{20}$$

From Eq. (11), we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}r_+}=\frac{(n-1)[3(n-2)r_+^4+3(n-4)\alpha r_+^2+(n-6)\alpha^2]+12r_+^6\beta^2(1-\sqrt{1+\eta_+})-6\Lambda r_+^6}{48\pi}V_{n-1}r_+^{n-7}. \tag{21}$$

Substituting Eqs. (12) and (21) into Eq. (20), we have

$$\begin{aligned}\mathrm{Im}Z&=-\frac{n-1}{8}V_{n-1}\int_{r_{\rm i}}^{r_{\rm f}}(r_+^{n-2}+2\alpha r_+^{n-4}+\alpha^2r_+^{n-6})\mathrm{d}r_+\\&=\frac{n-1}{8}V_{n-1}\left[\left(\frac{r_{\rm i}^{n-1}}{n-1}+\frac{2\alpha r_{\rm i}^{n-3}}{n-3}+\frac{\alpha^2r_{\rm i}^{n-5}}{n-5}\right)\right.\\&\left.\quad-\left(\frac{r_{\rm f}^{n-1}}{n-1}+\frac{2\alpha r_{\rm f}^{n-3}}{n-3}+\frac{\alpha^2r_{\rm f}^{n-5}}{n-5}\right)\right].\end{aligned} \tag{22}$$

According to the WKB approximation, the relationship between the tunneling probability and the imaginary part of the action is described by [56]

$$\Gamma\sim\exp(-2\mathrm{Im}Z). \tag{23}$$

Then we obtain

$$\begin{aligned}\Gamma\sim\exp\left\{\frac{n-1}{4}V_{n-1}\left[\left(\frac{r_{\rm f}^{n-1}}{n-1}+\frac{2\alpha r_{\rm f}^{n-3}}{n-3}+\frac{\alpha^2r_{\rm f}^{n-5}}{n-5}\right)\right.\right.\\\left.\left.-\left(\frac{r_{\rm i}^{n-1}}{n-1}+\frac{2\alpha r_{\rm i}^{n-3}}{n-3}+\frac{\alpha^2r_{\rm i}^{n-5}}{n-5}\right)\right]\right\}.\end{aligned} \tag{24}$$

The emission spectrum obviously deviates from the pure thermal spectrum but is consistent with an underlying unitary theory. Compared with the conventional tunneling rate

$$\Gamma\sim\mathrm{e}^{\Delta S}\quad, \tag{25}$$

where $\Delta S=S(M-\omega)-S(M)$ is the difference in the black hole entropy before and after the emission, which is shown in all of the early references about tunneling radiation, we obtain the entropy of the black hole in LBI gravity as

$$S=\frac{n-1}{4}V_{n-1}\left(\frac{r_{\rm t}^{n-1}}{n-1}+\frac{2\alpha r_{\rm t}^{n-3}}{n-3}+\frac{\alpha^2r_{\rm t}^{n-5}}{n-5}\right). \tag{26}$$

The result is in accordance with that given in Refs. [27, 28]. It is obvious that the area formula of black hole entropy breaks down unless $\alpha=0$.

Utilizing the equality (13) and noticing the black hole charge is invariable, it is easy to verify from Eqs. (26) and (11-12) that the first law of thermodynamics is satisfied as

$$\mathrm{d}M=T\mathrm{d}S. \tag{27}$$

It should be noted that the first law of thermodynamics for a charged black hole should take the form

$$\mathrm{d}M=T\mathrm{d}S+\phi\mathrm{d}Q. \tag{28}$$

However, in the case where the emitted particles are neutral and uncharged, the charge of the black hole does not change ($\mathrm{d}Q=0$), so the charge's contribution to the first law of thermodynamics is absent and Eq. (28) is reduced to (27).

4 Discussion

Equation (27) is an incorporation of the energy conservation law $\mathrm{d}M=\mathrm{d}Q_h$ (where Q_h is the heat quantity and no forces, including the electric field force, do work) and the second law of thermodynamics $\mathrm{d}S=\mathrm{d}Q_h/T$. The energy conservation is suitable for any process, but the equation $\mathrm{d}S=\mathrm{d}Q_h/T$ is only valid for a reversible process ($\mathrm{d}S>\mathrm{d}Q_h/T$ for an irreversible process). That is, the emission process has been treated as a reversible one in the PW tunneling framework.

For further discussion, we expand ΔS in ω. The tunneling rate (24) can be rewritten as

$$\Gamma\sim\exp(\Delta S)\approx\exp(-\beta\omega+a_2\omega^2+O(\omega^2)), \quad (29)$$

where

$$a_2=\frac{1}{2}\frac{\mathrm{d}^2(\Delta S)}{\mathrm{d}\omega^2}\bigg|_{\omega=0}, \quad (30)$$

and $\beta=\dfrac{1}{T}$ is the inverse of the Hawking temperature. The leading term in Eq. (29) gives the familiar thermal Boltzmann factor for the emanating radiation, while the other terms, which can be calculated to any desired order in ω, represent corrections from the response of the background geometry to the emission of a quantum [16].

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