# Massive self-duality solution associated with invariant one-forms

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**Abstract:** A massive self-duality solution associated with invariant 1-forms is presented. At the zero mass limit the massive self-dual theory of the $SO(3)$ gauge group on 4 dimensions cannot be reduced to that of massless self-duality. In such a case the self-dual connection turns to the flat connection and one cannot obtain a massless theory in such an approach.

**Keywords:** massive gauge fields, self-duality, invariant one-forms

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## 1 Introduction

Massive [1] and massless [2] Euclidean Yang-Mills fields [3] have been handled by some authors in the sense of Feynman diagrams [4] in the quantum field theories. In their results, a massless gauge theory cannot be obtained as a limit case of a finite massive theory [5]. However, the self-duality of massive Yang-Mills gauge fields is not extensively discussed today. General opinion about self-dual Yang-Mills fields [6], [7] is that they are nontrivial solutions to the vacuum Yang-Mills equation. In such a case, the masses of the gauge fields mostly seem unimportant.

However, together with the investigation of the symmetry breaking mechanism [8], [9], [10], in which the gauge bosons gain mass, the gauge group $SU(2)$ of the weak nuclear force becomes important for massive gauge fields. We know that this group has covering groups such as $SO(4)=SU(2)\times SU(2)$ and has an isomorphism $SU(2)\cong SO(3)$. Therefore, since there exist two identical quotients $\frac{SO(4)}{SU(2)}=\frac{SO(4)}{SO(3)}$, we can handle a self-dual connection $A\in\Lambda^1(\mathfrak{so}(3),\mathbb{R}^4)$ together with its curvature $F\in\Lambda^2(\mathfrak{so}(3),\mathbb{R}^4)$. Thus, in this paper we purpose a self-duality solution to a massive gauge theory of the gauge group $SO(3)$ on 4 dimensions.

## 2 Self-duality

Let $P$ be a principal $G$ bundle of a Lie group $G$ with Lie algebra $\mathfrak{g}$ on a 4-dimensional Euclidean manifold $M$ with local coordinates $\{x^\mu\}\in\mathbb{R}^4$. We show by $\Lambda^r(\mathfrak{g})$ the bundle of $\mathfrak{g}$-valued $r$-forms. Let $\nabla:\Lambda^0(P)\to\Lambda^1(P)$ be a connection on this bundle together with covariant derivative $\nabla=\mathrm{d}+A$, where $A=(A^A_B)_\mu\mathrm{d}x^\mu=A^A_B\in\Lambda^1(\mathfrak{g})$ is a connection 1-form. The curvature of this connection is

$$F^A_B=\mathrm{d}A^A_B+A^A_C\wedge A^C_B=(F^A_B)_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\in\Lambda^2(\mathfrak{g}). \quad (1)$$

In the sense of Hodge duality we write the self-duality equations for the curvature of a $SO(3)$-connection on 4 dimensions as follows:

$$(F^1_2)_{12}=(F^1_2)_{34}=q_1, \quad (2)$$
$$(F^2_3)_{13}=-(F^2_3)_{24}=q_2, \quad (3)$$
$$(F^1_3)_{14}=(F^A_B)_{23}=q_3, \quad (4)$$

where $q_1,q_2,q_3\in\mathcal{C}^\infty(M)$. The components of the curvature matrix become

$$F^1_2 = q_1 I_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=q_1(\mathrm{d}x^1\wedge\mathrm{d}x^2+\mathrm{d}x^3\wedge\mathrm{d}x^4), \quad (5)$$
$$F^2_3 = q_2 J_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=q_2(\mathrm{d}x^1\wedge\mathrm{d}x^3-\mathrm{d}x^2\wedge\mathrm{d}x^4), \quad (6)$$
$$F^1_3 = q_3 K_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=q_3(\mathrm{d}x^1\wedge\mathrm{d}x^4+\mathrm{d}x^2\wedge\mathrm{d}x^3). \quad (7)$$

where

$$I=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}, J=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$
$$K=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}. \quad (8)$$

It is easily seen that

$$L_i^2=-\mathbb{I}_{4\times4},\ L_iL_j=-L_jL_i,\ i,j=1,2,3, \quad (9)$$

where $L_i=I,J,K$. This means that the triplet $(I,J,K)$ presents a quaternionic structure on $\mathbb{R}^4$. We know already that some solutions to the $SU(2)$ Yang-Mills the-

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ory have quaternionic structures [11] and that the instanton equation on 4 dimensions presents a quaternion valued connection in the sense of division algebras [12].

The most classical self-dual potential (or connection in the geometrical sense) concepts in the Yang-Mills theories are the BPST [6] and 't Hooft [7] solutions on 4 dimensions of the Euclidean case. We choose an $\mathfrak{so}(N)$-valued connection 1-form as follows

$$A_B^A = h(r)\frac{x^\alpha}{r}(\varepsilon_B^A)_{\mu\alpha}\mathrm{d}x^\mu = h(r)(\omega_B^A)_\mu \mathrm{d}x^\mu, \tag{10}$$

where $h \in \mathcal{C}^\infty(M)$ is a smooth scalar, $\omega_B^A = (g^{-1}\mathrm{d}g)_B^A$ is the Maurer-Cartan 1-form for $g \in SO(N)$, and we choose an auxiliary constant tensor $(\varepsilon_B^A)_{\mu\alpha}$ which is skew-symmetric with respect to each of the indices $A$ and $B$ of the group and $\mu$ and $\nu$ of coordinates. The tensorial properties of the coordinates with respect to the metric tensor $\eta_{\mu\nu}$ on the base manifold are

$$r^2 = \eta_{\mu\nu}x^\mu x^\nu,\ x_\mu = \eta_{\mu\nu}x^\nu,\ \eta^{\mu\sigma}\eta_{\sigma\nu} = \delta_\nu^\mu. \tag{11}$$

Therefore, the curvature of this connection is written as

$$\begin{aligned}(F_B^A)_{\mu\nu} = &-\frac{2}{r}(h-h^2)(\varepsilon_B^A)_{\mu\nu} + \frac{1}{r^2}(h^\cdot + 2r(h-h^2)) \\ &\times\left[(\varepsilon_B^A)_{\nu\alpha}x_\mu - (\varepsilon_B^A)_{\mu\alpha}x_\nu\right]x^\alpha,\end{aligned} \tag{12}$$

where $h^\cdot = \partial h/\partial r$. In order to preserve the tensorial structure of the curvature it must be

$$h^\cdot + 2r(h-h^2) = 0. \tag{13}$$

This indicates that the connection is self-dual like the BPST instanton [6]. The solution to this equation is as follows:

$$h(r) = \frac{r^2}{r^2+r_0^2}, \quad 0 \leqslant r_0 \leqslant r. \tag{14}$$

Then the self-dual connection is obtained such that

$$(\omega_B^A)_\mu = \frac{1}{2}\frac{(r^2+r_0^2)^2}{r_0^2 r^2}(F_B^A)_{\mu\nu}x^\nu, \tag{15}$$

$$(A_B^A)_\mu = h(r)(\omega_B^A)_\mu = \frac{1}{2}\frac{(r^2+r_0^2)}{r_0^2}(F_B^A)_{\mu\nu}x^\nu. \tag{16}$$

Hence, considering the components of the curvature in Eqs. (5), (6) and (7), the components of our self dual connection are obtained as

$$\begin{aligned}A_2^1 &= \frac{q_1}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu I_{\mu\nu}\mathrm{d}x^\mu \\ &= \frac{q_1}{2}\frac{(r^2+r_0^2)}{r_0^2}(x^2\mathrm{d}x^1 - x^1\mathrm{d}x^2 + x^4\mathrm{d}x^3 - x^3\mathrm{d}x^4),\end{aligned} \tag{17}$$

$$\begin{aligned}A_3^2 &= \frac{q_2}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu J_{\mu\nu}\mathrm{d}x^\mu \\ &= \frac{q_2}{2}\frac{(r^2+r_0^2)}{r_0^2}(x^3\mathrm{d}x^1 - x^4\mathrm{d}x^2 - x^1\mathrm{d}x^3 + x^2\mathrm{d}x^4),\end{aligned} \tag{18}$$

$$\begin{aligned}A_3^1 &= \frac{q_3}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu K_{\mu\nu}\mathrm{d}x^\mu \\ &= \frac{q_3}{2}\frac{(r^2+r_0^2)}{r_0^2}(x^4\mathrm{d}x^1 + x^3\mathrm{d}x^2 - x^2\mathrm{d}x^3 - x^1\mathrm{d}x^4),\end{aligned} \tag{19}$$

Therefore, from Eq. (15) we have the following gauge invariant quadratic terms

$$\mathfrak{tr}\|F_B^A\|^2 = -2(q_1^2+q_2^2+q_3^2)\mathrm{dVol} \tag{20}$$

$$\mathfrak{tr}\|\omega_B^A\|^2 = \frac{1}{4r_0^4h^2}\mathfrak{tr}\|F_B^A\|^2. \tag{21}$$

## 3 Massive theory

It is well known that to add a mass term into a gauge invariant Lagrangian is not easy, because the mass term tries to break the gauge invariance. However, the first attempt was the Proca action [13], which can be considered as a massive Abelian action. For a massive non-Abelian gauge theory, the mass term $\mu$ of the gauge potential, or connection 1-form $A$ is added into the Lagrangian by an invariant term such that $\mu^2 A\wedge * A$, similar to the Higgs mechanism [8]. Due to this mass term the gauge fields are acquired by a (spontaneous) symmetry breaking mechanism [14].

The Euclidean massive Yang-Mills theory is non-renormalizable, so it cannot be interpreted as a correct quantum field model to describe physical interactions, but the spontaneous symmetry breaking mechanism provides an adequate way to reach a renormalizable massive Yang-Mills theory. In contrast to this method, we intend to find a non-Abelian self-duality solution to this theory without symmetry breaking by adding a mass term to the Yang-Mills action after modifying the connection.

Any connection on a fiber bundle transforms as inhomogeneous under a local gauge transformation $g\in G$, so that $A' = g^{-1}Ag + g^{-1}\mathrm{d}g$, where the term violating the homogeneity is the Maurer-Cartan form

$$\omega = g^{-1}\mathrm{d}g \in \Lambda^1(\mathrm{g}) \tag{22}$$

and it satisfies the Maurer-Cartan equation

$$\mathrm{d}\omega + \omega\wedge\omega = 0. \tag{23}$$

In contrast to the connection, its curvature transforms as homogenous, that is $F \to g^{-1}Fg$, and it presents a gauge invariance quadratic term, i.e. $F\wedge *F$. The difference of two connections on a bundle is a tensor. Therefore, if we choose two connections on the same principal $G$-bundle, for example if $A$ is any connection and the Maurer-Cartan 1-form $\omega$ of the group $G$ is another, then we write

$$\varpi = A - \omega, \tag{24}$$

and so this 1-form has a homogeneous transformation rule under the local group transformation

$$\varpi \to g^{-1}\varpi g,\ g \in G. \tag{25}$$

Consequently the quadratic term $\mathfrak{tr}\|\varpi\|^2$ becomes a gauge invariance. Therefore the action integral for a massive Yang-Mills theory without symmetry breaking can be written as follows

$$|S_{\mu\neq 0}| = \int_M \left\{ \mathrm{g}\mathfrak{tr}\|F^A_B\|^2 + \mu^2 \mathfrak{tr}\|\varpi^A_B\|^2 \right\}, \tag{26}$$

where $\mu$ is the mass of the gauge field, g is the coupling constant, $\mathfrak{tr}$ is the trace operator and $\|F^A_B\|^2 = F^A_C \wedge * F^C_B$. Also, $||$ means that the metric signature of the base manifold is not considered and the action integral is positively defined in every case. This action gives the following field equation

$$\nabla * F^A_B = \mu^2 \varpi^A_B. \tag{27}$$

When the connection is self-dual, i.e $*F = F$, then we have $\nabla * F^A_B = 0$ because of the Bianchi identity $\nabla F^A_B = 0$. This case tells us that the gauge field is massless, $\mu = 0$, or a pure gauge (flat connection) $A = \omega = g^{-1}\mathrm{d}g$. However, we consider a gauge field which is massive and self-dual. On the other hand one can write

$$\mathfrak{tr}\|\varpi^A_B\|^2 = (h-1)^2 \mathfrak{tr}\|\omega^A_B\|^2 = \frac{1}{4h^2}\mathfrak{tr}\|F^A_B\|^2. \tag{28}$$

Thus the action integral (26) for the massive self dual gauge field becomes

$$|S_{\mu\neq 0}| = \int_M \left\{ \left( \mathrm{g} + \frac{\mu^2}{4h^2} \right) \mathfrak{tr}\|F^A_B\|^2 \right\}. \tag{29}$$

This action integral presents the field equation $\nabla(h * F^A_B) = \mathrm{d}h \wedge F^A_B = 0$ with respect to the self-dual connection. Of course this equation cannot be interpreted as a field equation of a massive gauge field.

## 4 Conclusion

In order to get a field equation for the massive self dual gauge field, the action integral (29) is rewritten as

$$|S_{\mu\neq 0}| = 2\int_M (q_1^2 + q_2^2 + q_3^2)\left( \mathrm{g} + \frac{\mu^2}{4h^2} \right) \mathrm{d}Vol. \tag{30}$$

From the Bianchi identity $\mathrm{d}F + A \wedge F - F \wedge A = 0$ for the connection $A \in \Lambda^1(\mathfrak{so}(3)\mathbb{R}^4)$ and its curvature $F \in \Lambda^2(\mathfrak{so}(3), \mathbb{R}^4)$, one gets the following equations:

$$\frac{q_2 q_3}{\dot{q_1}} = \frac{q_1 q_3}{\dot{q_2}} = \frac{q_1 q_2}{\dot{q_3}} = \lambda, \tag{31}$$

$$q_1^2 + q_2^2 + q_3^2 = \frac{1}{2}(Q^2 - 2\lambda \dot{Q}), \tag{32}$$

where $\lambda \in \mathcal{C}^\infty(M)$ and

$$Q = q_1 + q_2 + q_3. \tag{33}$$

Therefore, the dynamical variables for the gauge invariant action integral (30) of the massive self-dual gauge field become $(Q, \dot{Q})$.

In order to compare the massive self-dual gauge field with the massless one, we present an action integral for the massless self-dual gauge field as follows

$$|S_{\mu=0}| = \int_M \mathrm{g}\mathfrak{tr}\|F^A_B\|^2 = 2\mathrm{g}\int_M (q_1^2 + q_2^2 + q_3^2)\mathrm{d}Vol. \tag{34}$$

Therefore the integrals of massive and massless gauge fields are rewritten with respect to the new dynamical variables $(Q, \dot{Q})$ such that

$$|S(Q,\dot{Q})_{\mu\neq 0}| = \int_M (Q^2 - 2\lambda \dot{Q})\left( \mathrm{g} + \frac{\mu^2}{4h^2} \right) f \mathrm{d}r, \tag{35}$$

$$|S(Q,\dot{Q})_{\mu=0}| = \int_M (Q^2 - 2\lambda \dot{Q}) f \mathrm{d}r, \tag{36}$$

where $f \in \mathcal{C}^\infty(M)$ and $\mathrm{d}Vol = f(r)\mathrm{d}r$. The field equations for the massive and massless self-dual gauge fields, respectively, are

$$Qf + \frac{\mu^2}{4\mathrm{g}}\left\{ (Q + \mathrm{g}\dot{\lambda})\frac{f}{h^2} + \lambda\left(\frac{f}{h^2}\right)^{\cdot} \right\} = 0, \tag{37}$$

$$Qf + (\lambda f)^{\cdot} = 0. \tag{38}$$

Then the term $Q$ for the massive case is

$$Q_{\text{massive}} = -\frac{\mu^2}{4}\frac{(\mathrm{g}\lambda + \ln(f/h^2))^{\cdot}}{\mathrm{g} + \dfrac{\mu^2}{4h^2}} \tag{39}$$

From Eq. (31), $q_1, q_2, q_3$ are found as follows

$$q_2 = \sqrt{a-b}\,q_1,\ q_3 = \sqrt{b}\,q_1,\ Q = \alpha q_1, \tag{40}$$

where $a$ and $b$ are constants and

$$\alpha = 1 + \sqrt{b} + \sqrt{a-b}. \tag{41}$$

On the other hand, from Eq. (32) one gets

$$\dot{q_1} = \frac{3\beta}{2\lambda}, \tag{42}$$

where

$$\beta = (1 + a - b) - 2\left( \sqrt{b} + \sqrt{a-b} + \sqrt{b(a-b)} \right). \tag{43}$$

Considering Eq. (39) together with Eq. (14), one finds the following solution

$$q_1 = -\frac{3\beta}{2\mathrm{g}}\left\{ 4\left[ \left( \frac{\mathrm{g}}{\mu^2} + 1 \right) r - \frac{r_0^2}{r} + C \right] + \ln\left( \frac{(r^2 + r_0^2)f}{r^2} \right) \right\}, \tag{44}$$

where $C$ is a constant. Hence, the components of the connection given as the self-duality solution in Eqs. (17, 18, 19) are re-obtained as the massive self-duality solution with respect to Eq. (44) such that

$$A_2^1 = \frac{q_1}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu I_{\mu\nu}\mathrm{d}x^\mu, \tag{45}$$

$$A_3^2 = \sqrt{b}\frac{q_1}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu J_{\mu\nu}\mathrm{d}x^\mu, \tag{46}$$

$$A_3^1 = \sqrt{a-b}\frac{q_1}{2}\frac{(r^2+r_0^2)}{r_0^2}x^\nu K_{\mu\nu}\mathrm{d}x^\mu. \tag{47}$$

It is easily seen that at the limit $\mu\to 0$ the field equation (37) of the massive self-duality case is not reduced to the field equation (38) of the massless self-duality case. Thus, if $\mu\to 0$, then $Q\to 0$, that is the curvature tends to vanish. This means that at the limit $\mu\to 0$ a massive self-dual $SO(3)$-connection on 4 dimensions turns into the flat connection. However, the opposite of this case is not necessarily true. In addition, the zero limit of the mass term in Eq. (44) does not give a massless case. Therefore, we cannot obtain a massless theory of self-dual gauge fields at such a limit. Our solution is a non-trivial self-duality solution to massive Euclidean Yang-Mills theory without spontaneous symmetry breaking.

## References


1 J. Reiff and M. Veltman, Nucl. Phys. B, **13**(3): 545–564 (1969)
2 G. 'tHooft, Nucl. Phys. B, **33**(1): 173–199 (1971)
3 C. N. Yang and R. L. Mills, Phys. Rev., **96**: 191–195 (1954)
4 L. Faddeev and V. Popov, Phys. Lett. B, **25**(1): 29–30 (1967)
5 H. van Dam and M. Veltman, Nucl. Phys. B, **22**(2): 397–411 (1970)
6 A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, Phys. Lett. B, **59**(1): 85–87 (1975)
7 G. 't Hooft, Phys. Rev. D, **14**: 3432–3450 (1976)
8 P. W. Higgs, Phys. Rev. Lett., **13**: 508–509 (1964)
9 F. Englert and R. Brout, Phys. Rev. Lett., **13**: 321–323 (1964)
10 T. W. B. Kibble, Phys. Rev., **155**(5): 1554–1561 (1967)
11 M. F. Atiyah, *Geometry of Yang - Mills Fields* (Accademia Nazionale Dei Lincei Scuola Normale Superiore, 1979)
12 J. Figueroa-O'Farrill, J. Geom. Phys., **32**(2): 227–240 (1999)
13 A. Proca, Journal de Physique et le Radium, **7**: 347–353 (1936)
14 K. Huang, Quarks, *Leptons and Gauge Fields* (World Scientific, 1992)