Computing parallel/coincident phase D-brane superpotentials and Type-II/F -theory duality *

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Abstract: In this paper we study the parallel phase and the coincident phase of D-brane systems with the compactification of one closed modulus. D-brane systems with two phases are described by different 4-folds in terms of Type-II/F-theory duality, and the phase transitions are related by the blow-up from a 4-fold with singularities to a 4-fold without. In terms of gauge theory, the phase transition corresponds to the enhancement of gauge group $U(1)\times U(1)\to U(2)$ connecting the Coulomb branch and the Higgs branch. For the sextic and octic with two D-branes, using mirror symmetry and Type-II/F theory duality, A-model superpotentials are obtained from the B-model side for the two phases, and the $U(1)$ *Ooguri-Vafa invariants* for the parallel phase and $U(2)$ *Ooguri-Vafa invariants* for the coincident phase are extracted from the A-model superpotential. The difference between the invariants of the two phases is evidence of the phase transition between the Coulomb branch and the Higgs branch.

Keywords: Ooguri-Vafa invariants, TypeII/F-theory duality, phase transition

PACS: 02.40.Tt, 11.15.Tk, 11.25.Tq **DOI:** 10.1088/1674-1137/42/9/093104

1 Introduction

Closed-string mirror symmetry equipped with $N=2$ supersymmetry identifies the Kähler moduli space of the topological A-model and the complex structure moduli space of the topological B-model. The appearance of a D-brane breaks the supersymmetry to $N=1$, leading to open-string mirror symmetry. Then Type-II compactification theory is described by an effective $N=1$ supergravity action with non-trivial superpotential on the moduli space $\mathcal{M}$. Similar to the prepotential in the $N=2$ supersymmetric situation, the superpotential that determines the F-term of low-energy effective theory and the string vacuum structure is available from a B-model computation. For a space-filling D5-brane wrapped on the curve γ embedded in a divisor $\mathcal{D}$ of the Calabi-Yau 3-fold M_3, the effective D-brane superpotential is the relative period captured by:

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat{z})=\Pi_\gamma(z,\hat{z})=\int_\gamma \Omega^{(3,0)}(z,\hat{z}),\quad \gamma\in H_3(M_3,\mathcal{D}) \tag{1}$$

where the z and $\hat{z}$ indicate closed and open moduli respectively. Thus the four-dimensional effective superpotential has a unified expression as a general linear combination of the integral of the basis of relative period [1, 2]:

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat{z})=\sum N_\alpha\Pi_\alpha(z,\hat{z})=\mathcal{W}_{\text{open}}(z,\hat{z})+\mathcal{W}_{\text{closed}}(z), \tag{2}$$

where the $\{\alpha\}$ is a set of bases of $H_3(M_3,\mathcal{D})$. The coefficients N_α are determined by the topological charges of branes and background flux, and $\Pi_\alpha(z,\hat{z})$ are the relative period integrals in both open and closed sectors.

However, the superpotentials in Type-II string theory have a dual description in F-theory as the flux superpotentials equalizing the open and closed parameters as the complex structure moduli of the 4-fold for F-theory compactification [3]. This Type-II/F-theory duality suggests a way to obtain the superpotentials of Type-II string theory compactified on a Calabi-Yau 3-fold from the computation on a Calabi-Yau 4-fold compactification of F-theory [4]. The superpotential of the 4-form flux G_4 in F-theory compactified on the Calabi-Yau 4-fold M_4 is a section of the line bundle over the complex structure moduli space $\mathcal{M}_{\text{CS}}(M_4)$. It is called the Gukov-Vafa-Witten superpotential [5] which has general form [6]

$$\begin{aligned}\mathcal{W}_{\text{GVW}}(M_4)&=\int_{M_4} G_4\wedge\Omega^{(4,0)}\\&=\sum_\Sigma N_\Sigma(G_4)\Pi_\Sigma(z,\hat{z})+\mathcal{O}(g_\text{s})+\mathcal{O}(\text{e}^{-1/g_\text{s}}),\end{aligned} \tag{3}$$

where g_s is the string coupling strength and the leading

Received 2 April 2018, Published online 9 August 2018
∗ Supported by NSFC (11475178) and Y4JT01VJ01
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term on the right hand side is the D-brane superpotential $\mathcal{W}_{N=1}$ (2). The weak coupling limit $g_s \to 0$ brings us back to the D-brane superpotential $\mathcal{W}_{N=1}$ from the GVW superpotential W_{GVW} of F-theory as follows:

$$\lim_{g_s\to 0}\mathcal{W}_{\text{GVW}}(M_4)=\sum N_\Sigma(G_4)\Pi_\Sigma(z,\hat{z})=\mathcal{W}_{\mathcal{N}=1}(M_3,\mathcal{D}), \tag{4}$$

where most of the degrees of freedom decouple from the superpotential sector.

So far, most of the computations of superpotentials for the D-brane system with the compact target space are contributed by branes which are described by only one open-string deformation [4, 7–13]. We take the advantage of Type-II/F-theory duality, which packages all the open and closed moduli together as the complex structure moduli of the dual 4-fold M_4 to study the superpotential of the D-brane system more conveniently. In fact, using the Type-II/F-theory duality method, we have computed the superpotential and geometric invariants for many single D-brane models studied by subsystems integrated in the papers [4, 13], such as the mirror quintic, $P(1,1,1,6,9)$ and $P(1,1,2,2,2)$, and obtained completely the same results as in the above two papers. In the sextic and octic models, we introduce two D-branes into the system and calculate the superpotentials in the parallel phase (Coulomb branch) and coincident phase (Higgs branch) near the large radius limit point for the first time. The Ooguri-Vafa invariants are also extracted.

This paper is organized as follows. In Section 2, we construct the parallel phase and the coincident phase of the D-brane system which corresponds to the 4-folds for F-theory compactification, in terms of toric geometry. In Section 3, a brief review of the generalized GKZ system and its local solutions is given. These solutions are closely related to the mirror map and the potential functions of the D-brane system. In Section 4, we apply the formulism to the parallel phase and the coincident phase of two D-brane systems. The superpotentials for the D-brane geometries are obtained near the large complex structure point in terms of open-closed deforming space. The Ooguri-Vafa invariants in the parallel and coincident phases are extracted. The last section is a brief summary and further discussion.

2 Polyhedra construction for parallel/coincident phase

In this paper, we focus on the Calabi-Yau manifolds, which are defined by the hypersurface in ambient toric variety, and briefly give some necessary introduction and settings. More details and background can be found in [14–16]. The notations are as follows: (∇_4,Δ_4) is a pair of mutually reflective 4-dimensional polyhedrons leading to a pair of toric variety $(P_{\Sigma(\nabla_4)},P_{\Sigma(\Delta_4)})$ with fans $\Sigma(\nabla_4)$ and $\Sigma(\Delta_4)$ which are defined by the cones over the faces of ∇_4 and Δ_4 respectively. (W_3,M_3) are the hypersurfaces in ambient toric varieties $(P_{\Sigma(\nabla_4)},P_{\Sigma(\Delta_4)})$ which describe the pair of mirror 3-folds. For convenience, we also use (W_3,M_3) to stand for the pair of 3-folds. In the homogeneous coordinates x_j on the toric ambient space $P_{\Sigma(\Delta_4)}$, the hypersurface M_3 is defined by p integral vertices of ∇_4 as the zero locus of the polynomial P:

$$P=\sum_{i=0}^{p-1}a_i\prod_{v\in\Delta_4}x_j^{\langle v,v_i^*\rangle+1}, \tag{5}$$

where v_i^* are the integral points in ∇_4 while the v's are vertices of the dual polyhedron Δ_4. The coefficients a_i are complex parameters related to the complex structure of M_3.

The n parallel D-branes are defined by a reducible divisor:

$$\begin{aligned}Q(\mathcal{D})&=\prod_{m=0}^{n}(\phi_m a_0\prod_{v_j\in\Delta_4}x_j+a_i\prod_{v_j\in\Delta_4}x_j^{\langle v,v_i^*\rangle+1})\\&=\sum_{k=0}^{n}b_k\prod_{v_j\in\Delta_4}x_j^{k\langle v,v_i^*\rangle+n},\end{aligned} \tag{6}$$

whose irreducible components lie in a single parameter family of divisor $\mathcal{D}_s\equiv\phi a_0\prod_{v_j\in\Delta_4}x_j+a_i\prod_{v_j\in\Delta_4}x_j^{\langle v,v_i^*\rangle+1}$. The parameters b_k encode the open-string deformations of the n parallel branes. In other words, the brane-component-deformation parameter ϕ_m describes the position of the mth individual D-brane component. The parallel D-brane geometry corresponds to the Coulomb phase of the gauge theory, which gives rise to the $U(1)\times U(1)\times...\times U(1)$ group, the product of n $U(1)$'s, and each $U(1)$ group describes electromagnetism including the Coulomb field. Similar to the construction in Ref. [13], the combinatoric data of the parallel D-branes phase can be recovered in a one-dimensional higher polyhedron $\tilde{\nabla}_5$ defining the non-compact 4-fold $\tilde{W}_4$. Then the relevant vertices shaping the $\tilde{\nabla}_5$ are:

$$\tilde{v}_j^*=\begin{cases}(v_j^*,0) & j=0,...,p-1,\\(mv_i^*,1) & j=p+m,0\leqslant m\leqslant n.\end{cases} \tag{7}$$

When parallel D-branes coincide, the $U(1)\times U(1)\times...\times U(1)$ gauge group obtains an enhancement, becoming a $SU(n)$ group. This corresponds to a phase transition from the Coulomb branch to the Higgs branch of the gauge theory. Geometrically, the non-Abelian gauge group relates to the singularities on the Calabi-Yau manifolds. In toric language the singular curve corresponds to a one-dimensional edge of the dual polyhedron with integral lattice points on it. The resolution process is standard in terms of toric geometry [17]. It adds all the interior points on the one-dimensional edges into the points configuration, and each of these vertices corresponds to an exceptional divisor in the blow-up of the Calabi-Yau manifold. Inversely, if we remove the n–1 interior points, we can recover the singular 4-fold which gives rise to the enhanced gauge symmetry $U(n)$, matching the expectation in the coincident D-brane system [18].

In addition to compactifying the non-compact fourfold $\tilde{W}_4$, we add one more point into the points configuration beneath the hyperplane $Y=\{v\in R^5|v_5=0\}$. Together with the points for D-brane geometry, all the points define the enhanced polyhedron ∇_5 corresponding to the compact Calabi-Yau 4-fold W_4.

3 Generalized GKZ system, local solutions and relative periods

The relative periods satisfy a set of differential equations named the Picard-Fuchs equations. The differential operators of the Picard-Fuchs equations can be derived from the generalized GKZ system as follows:

$$\mathcal{L}(l^a)=\prod_{k=1}^{l_0^a}(\vartheta_0-k)\prod_{l_j^a>0}\prod_{k=0}^{l_j^a-1}(\vartheta_j-k)$$
$$-(-1)^{l_0^a}z_a\prod_{k=1}^{-l_0^a}(\vartheta_0-k)\prod_{l_j^a<0}\prod_{k=0}^{-l_j^a-1}(\vartheta_j-k), \qquad (8)$$

where $\vartheta_j=a_j\frac{\partial}{\partial a_j}$ are logarithmic derivatives with respect to the parameters a_j, and the l^a are the generators of the Mori cone [19–22] of $P_{\Sigma(\nabla_5)}$. They are also known as the charge vectors of the gauged linear sigma model (GLSM)[23] for $a=1,...,k=h^{1,1}(W_4)$.

On one hand, the charge vectors l^a which correspond to the maximal triangulation of ∇_5 lead to a set of local coordinates of the pitch around the large complex structure limit point in the complex structure moduli space of M_4:

$$z_a=(-1)^{l_0^a}\prod_j a_j^{l_j^a}. \qquad (9)$$

They are torus invariant algebraic coordinates on the large complex structure phase, where the non-perturbative instanton corrections are suppressed exponentially. On the other hand, the duality between the Mori cone and the Kähler cone gives rise to the basis J_a of $H^{1,1}(W_4)$ dual to the l^a and naturally gives the local coordinates k_a around the large radius limit point mirror to the large complex structure point. The coordinates k_a are also known as the flat coordinates.

From the point of view of F-theory, the GKZ system is derived from the combinatoric data of the 5-dimensional polyhedra corresponding to the 4-folds. So the solutions relate to the periods of the fourfolds relying on the complex structure moduli of the 4-fold. However, from the viewpoint of Type-II theory, since the enhanced polyhedra correspond to the D-brane geometry, these solutions encode the relative periods giving rise to the open-closed mirror maps and the D-brane superpotentials which rely on the open and closed string moduli.

In terms of the periods of the fourfolds, according to Ref. [16], the local solutions to the GKZ system can be derived from the fundamental period,

$$w_0(z;\rho)=\sum_{m_1,...,m_a\geqslant 0}\frac{\Gamma(-\sum_a(m_a+\rho_a)l_0^a+1)}{\prod_{1\leqslant i\leqslant p}\Gamma(\sum_a(m_a+\rho_a)l_i^a+1)}z^{m+\rho}, \qquad (10)$$

using the Frobenius method. The whole periods vector reads:

$$\vec{\Pi}(z)=\begin{pmatrix}\Pi_0= & w_0(z;\rho)|_{\rho=0}\\ \Pi_{1,a}= & \partial_{\rho_a}w_0(z;\rho)|_{\rho=0}\\ \Pi_{2,n}= & \sum_{a_1,a_2}K_{a_1,a_2;n}\partial_{\rho_{a_1}}\partial_{\rho_{a_2}}w_0(z;\rho)|_{\rho=0}\\ & \vdots\end{pmatrix}, \qquad (11)$$

where $n\in\{1,...,h\}$ and h stands for the dimension of $H^4(M_4)$. The $K_{a_1,a_2;n}$ are the combinatoric coefficients of the second derivative of w_0. Guided by the mirror hypothesis, the period vector $\vec{\Pi}(z)$ has a dual description on the A-side with the general form:

$$\vec{\Pi}^*(k)=\begin{pmatrix}\Pi_0^*= & 1\\ \Pi_{1,a}^*= & k_a\\ \Pi_{2,n}^*= & \sum_{a_1,a_2}K^*_{a_1,a_2;n}k_{a_1}k_{a_2}+b_n+F_n^{\text{inst}}\\ & \vdots\end{pmatrix}, \qquad (12)$$

where $k_a=\Pi_{1,a}/\Pi_0$ are the flat coordinates and $K^*_{a_1,a_2;n}=K_{a_1,a_2;n}$. The coefficients $K^*_{a_1,a_2;n}$ of leading terms relating to the classical sector in the periods can be easily determined once the 4-cycle $\pi_4^*\in H_4(W_4,Z)$ relating to the 4-form flux G_4 has been selected. The constants b_n are not relevant in our discussion and the F_n^{inst} stand for the instanton correction sector of the solutions.

In terms of the relative periods of D-brane geometry, after taking the decoupling limit, these solutions give rise to the relative periods corresponding to the mirror map, bulk potential and superpotential. The relative period vector is of the form:

$$\lim_{g_s\to 0}\vec{\Pi}^*(k)=(1,t,\hat{t},F_t(t),\mathcal{W}(t,\hat{t})...)), \qquad (13)$$

where t and $\hat{t}$ are the flat coordinates on the moduli space after splitting the open and closed string moduli. The solution $F_t(t)\equiv\partial_tF(t)$ is determined only by the closed moduli, where $F(t)$ is the $N=2$ prepotential. Its corresponding part in the open sector is the superpotential $\mathcal{W}(t,\hat{t})$. The flat coordinates can be found in the large radius regime of the A-model (k_a) and the large complex structure regime of the B-model (z_a). The relation between k_a and z_a defines the mirror map as follows,

$$k_a(z)=\frac{\Pi_{1,a}(z)}{\Pi_0}, \tag{14}$$

while t and $\hat{t}$ equal the special integer linear combination of k_a.

The instanton corrections are encoded as a power series expansion of $q_i=\exp(2\pi i t_i)$ and $\hat{q}_i=\exp(2\pi i\hat{t}_i)$:

$$F^{inst}(t,\hat{t})=\sum_{\vec{r},\vec{s}}G_{\vec{r},\vec{s}}q^{\vec{r}}\hat{q}^{\vec{s}}=\sum_n\sum_{\vec{r},\vec{s}}\frac{N_{\vec{r},\vec{s}}}{n^2}q^{n\vec{r}}\hat{q}^{n\vec{s}}. \tag{15}$$

In Eq. (15), $\{G_{\vec{r},\vec{s}}\}$ are open Gromov-Witten invariants labeled by a relative homology class, where $\vec{s}$ represent the elements of $H_1(L)$ and $\vec{r}$ represent the elements of $H_2(W_3)$, and $\{N_{\vec{r},\vec{s}}\}$ are Ooguri-Vafa invariants.

4 Models

In this section we select the same compactifying point $\tilde{v}^*_c=(0,0,0,0,-1)$ in terms of combinatoric data in all cases[1)].

4.1 D-branes on the sextic

The sextic is defined as a hypersurface,

$$P=a_1x_1^6+a_2x_2^6+a_3x_3^6+a_4x_4^6+a_5x_5^3+a_0x_1x_2x_3x_4x_5. \tag{16}$$

In the ambient toric variety $P_{\Sigma(\Delta_4)}$, it is determined by the vertices of the polyhedron Δ_4:

$$v_1=(2,-1,-1,-1),v_2=(-1,5,-1,-1),v_3=(-1,-1,5,-1),$$
$$v_4=(-1,-1,-1,5),v_5=(-1,-1,-1,-1). \tag{17}$$

4.1.1 Parallel D-branes phase

We consider parallel branes which are described by the reducible divisor $\mathcal{D}=\mathcal{D}_1+\mathcal{D}_2$ realized by the degree-12 homogeneous equation

$$Q=b_0(x_1x_2x_3x_4x_5)^2+b_1x_1x_2x_3x_4x_5^4+b_2x_5^6 \tag{18}$$

$$\sim\prod_{i=1}^{2}(\phi_ia_0x_1x_2x_3x_4x_5+a_1x_5^3). \tag{19}$$

According to the construction in subsection 2, the open-closed system is encoded in the enhanced polyhedron $\tilde{\nabla}_5$ whose vertices are

$$\tilde{v}^*_0=(0,0,0,0,0),\tilde{v}^*_1=(1,0,0,0,0),\tilde{v}^*_2=(0,1,0,0,0),$$
$$\tilde{v}^*_3=(0,0,1,0,0),\tilde{v}^*_4=(0,0,0,1,0),\tilde{v}^*_5=(-2,-1,-1,-1,0),$$
$$\tilde{v}^*_6=(0,0,0,0,1),\tilde{v}^*_7=(1,0,0,0,1),\tilde{v}^*_8=(2,0,0,0,1). \tag{20}$$

The geometry for the F-theory compactification, the compact 4-fold W_4, is defined by the polyhedron ∇_5 with vertices given in Section (20) and $\tilde{v}^*_c$.

The generators of a Mori cone of the toric variety determined by ∇_5 are given by:

$$\begin{array}{rcl} & & \quad 0 \quad 1 \quad 2\,3\,4\,5 \quad 6 \quad 7 \quad 8 \quad c \\ l^1 &=& (\;-4\;\; 0\;\; 1\,1\,1\,1\; -2\;\; 2\;\;\; 0\;\; 0\;) \\ l^2 &=& (\;\;\,0\;\;\; 0\;\; 0\,0\,0\,0\;\;\; 1\;\; -2\;\; 1\;\; 0\;) \\ l^3 &=& (\;-1\;\; 1\;\; 0\,0\,0\,0\;\;\; 0\;\;\;\; 1\;\; -1\;\; 0\;) \\ l^4 &=& (\;\;\,0\;\; -2\;\, 0\,0\,0\,0\;\;\; 0\;\;\;\; 0\;\;\; 1\;\; 1\;). \end{array} \tag{21}$$

The Kähler form is $J=\sum_a k_aJ_a$, where J_a denotes the basis of $H^{1,1}(W_4)$ dual to the Mori cone generated by (21), and k_a are flat coordinates on the Kähler moduli space of the mirror four-fold W_4. Then we choose the basis elements of $H_4(W_4)$ which are defined by intersections of the toric divisors D_i corresponding to the $\tilde{v}^*_i$, namely

$$\gamma_1=D_1\cap D_9,\ \gamma_2=D_2\cap D_6,\ \gamma_3=D_2\cap D_7. \tag{22}$$

After changing the variables as follows to visualize the closed and open moduli

$$t=k_1+2(k_2+k_3),\ \ \hat{t}_1=k_2+k_3,\ \ \hat{t}_2=k_3, \tag{23}$$

the leading terms of the periods are

$$\tilde{\Pi}^*_{2,1}=3t^2,\ \ \tilde{\Pi}^*_{2,2}=(t-2\hat{t}_1)^2,\ \ \tilde{\Pi}^*_{2,3}=(t-2\hat{t}_2)^2. \tag{24}$$

corresponding to γ_1, γ_2, and $\gamma_1+\gamma_3$. $\tilde{\Pi}^*_{2,1}$ only depends on the closed moduli t and is supposed to be the leading term of the bulk potential function $F_t(t)$, while $\tilde{\Pi}^*_{2,2}$, $\tilde{\Pi}^*_{2,3}$ are supposed to lead the D-brane superpotential $\mathcal{W}(t,\hat{t})$, which depends on both open ($\hat{t}$) and closed (t) parameters.

We identify the bulk potential and the superpotentials with the exact solutions to the GKZ system which are led by $\tilde{\Pi}^*_{2,1}$,$\tilde{\Pi}^*_{2,2}$ and $\tilde{\Pi}^*_{2,3}$ respectively. Following Eq. (10), in terms of algebraic coordinates (9),

$$z_1=\frac{a_2a_3a_4a_5b_1^4}{a_0^4b_0^4},\quad z_2=\frac{b_0b_2}{b_1^2},\quad z_3=\frac{a_1b_1}{a_0b_2}, \tag{25}$$

the fundamental period and the logarithmic periods,

$$\Pi_0(z)=w_0(z;0),\qquad \Pi_{1,i}(z)=\partial_{\rho_i}w_0(z;\rho)|_{\rho_i=0},$$
$$\Pi_{2,n}(z)=\sum_{i,j}K_{i,j;n}\partial_{\rho_i}\partial_{\rho_j}w_0(z;\rho)|_{\rho=0}, \tag{26}$$

solve the generalized GKZ system governed by the charge vectors (21). The flat coordinates are given by

$$k_i=\frac{\Pi_{1,i}(z)}{\Pi_0(z)}=\frac{1}{2\pi i}\log\ z_i+...\ . \tag{27}$$

1) The compactifying point is different from the choice in the [4], which picked other integral points on the hyperplane $Y=\{v\in R^5|v_5=-1\}$. However, the detail of the P^1 compactification dominates the subleading terms in g_s and is irrelevant in the decoupling limit.

Then the mixed inverse mirror maps in terms of $q_i=\exp(2\pi i k_i)$ for $\{i=1,2,3\}$ are

$$\begin{aligned} z_1=&q_1-24q_1^2+2q_1q_2-24q_1^2q_2+q_1q_2^2-4q_1q_2q_3+336q_1^2q_2q_3\\ &-4q_1q_2^2q_3+288q_1^2q_2^2q_3+4q_1q_2^2q_3^2-2928q_1^2q_2^2q_3^2+...\\ z_2=&q_2+12q_1q_2+414q_1^2q_2-2q_2^2-48q_1q_2^2-1944q_1^2q_2^2-q_2q_3\\ &-12q_1q_2q_3-414q_1^2q_2q_3+5q_2^2q_3-24q_1q_2^2q_3\\ &-1764q_1^2q_2^2q_3-3q_2^2q_3^2+72q_1q_2^2q_3^2+3708q_1^2q_2^2q_3^2+...\\ z_3=&q_3+q_2q_3+12q_1q_2q_3+414q_1^2q_2q_3+12q_1q_2^2q_3+q_3^2+q_2^2q_3^2\\ &-240q_1q_2^2q_3^2-5652q_1^2q_2^2q_3^2+... \; . \end{aligned} \tag{28}$$

According to the leading terms (24), we find the relative periods which correspond to the closed-string period and D-brane superpotentials in the A-model as follows:

$$F_t(t)\equiv\Pi_{2,1}=3t^2+\frac{1}{4\pi^2}(15768q+24117750q^2+...)$$

$$\begin{aligned} \mathcal{W}_1(t,\hat{t}_1)\equiv\Pi_{2,2}=(t-\hat{t}_1)^2+\frac{1}{4\pi^2}\Big(&13248q+19396368q^2\\ &+\frac{9}{2}\hat{q}_1^4+8\hat{q}_1^3+18\hat{q}_1^2-6552q\hat{q}_1^2+72\hat{q}_1\\ &+11856q\hat{q}_1-4896q\hat{q}_1^{-1}-6942384q^2\hat{q}_1^{-1}\\ &+480q\hat{q}_1^{-2}+1728504q^2\hat{q}_1^{-2}+...\Big)\\ \mathcal{W}_2(t,\hat{t}_2)\equiv\Pi_{2,3}=(t-\hat{t}_2)^2+\frac{1}{4\pi^2}\Big(&13248q+19396368q^2\\ &+\frac{9}{2}\hat{q}_2^4+8\hat{q}_2^3+18\hat{q}_2^2-6552q\hat{q}_2^2+72\hat{q}_2\\ &+11856q\hat{q}_2-4896q\hat{q}_2^{-1}-6942384q^2\hat{q}_2^{-1}\\ &+480q\hat{q}_2^{-2}+1728504q^2\hat{q}_2^{-2}+...\Big), \end{aligned} \tag{29}$$

where $q=\exp(2\pi i t)$, $\hat{q}_1=\exp(2\pi i\hat{t}_1)$ and $\hat{q}_2=\exp(2\pi i\hat{t}_2)$. Absolutely, the bulk potential $F_t(t)$ only depends on the closed modulus t and the D-brane superpotentials $\mathcal{W}_1(t,\hat{t}_2),\mathcal{W}_2(t,\hat{t}_1)$ have kept the Z_2 symmetry with respect to $\hat{t}_1$ and $\hat{t}_2$ after adding the instanton correction, meeting our expectations.

The first several orders of $U(1)$ Ooguri-Vafa invariants for the parallel phase are listed in Table 1.

4.1.2 Coincident D-branes phase

According to the enhanced polyhedron ∇_5, the defining polynomial of the dual 4-fold M_4 on the B-model side is

$$\begin{aligned} \tilde{P}=&a_1x_1^6x_6^4+a_2x_2^6x_7^4+a_3x_3^6x_8^4+a_4x_4^6x_9^4+a_5x_5^3x_6x_7x_8x_9x_{10}^3\\ &+a_6x_1^2x_2^2x_3^2x_4^2x_5^2a_7x_5^4x_1x_2x_3x_4x_{10}^2+a_8x_5^6x_{10}^4\\ &+a_9x_6^2x_7^2x_8^2x_9^2x_{10}^2+a_0x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}. \end{aligned} \tag{30}$$

To simplify the notation, we denote the coefficients of the monomials in the polynomial as a_i's, and the relation between the notations in the D-brane geometry and the 4-fold is as follows:

$$a_i=\begin{cases} a_i & 0\leqslant i\leqslant 5,\\ b_{i-6} & 6\leqslant i\leqslant 8,\\ c & i=9. \end{cases} \tag{31}$$

When $b_1^2=4b_0b_2$, the defining equation for the parallel D-branes becomes:

$$Q\sim(\phi a_0x_2x_3x_4x_5+a_5x_5^3)^2, \tag{32}$$

which means the two individual D-branes coincide as $\phi_1=\phi_2=\phi$. Correspondingly, the equivalent description

$$a_7^2=4a_6a_8 \tag{33}$$

gives rise to the perfect square $(x_1x_2x_3x_4x_5\pm x_5^3x_{10}^2)^2$ in $\tilde{P}$. Obviously M_4 becomes singular and the condition (33) constricts the complex moduli space of the 4-fold to a submanifold which is the coincident D-branes phase. The dual description of the coincident on the A-model side is the blow-down of the exceptional divisor developing the curve singularity on the W_4.

Table 1. $U(1)$ Ooguri-Vafa invariants N_{n_1,n_2,n_3} for the superpotential $\mathcal{W}_1(t,\hat{t}_1)$ for one of the two parallel branes on the sextic.

n_1 \ $n_2=n_3$	0	1	2	3	4
0	0	72	0	0	0
1	480	-4896	13248	11856	-6552
2	19680	-290448	1729728	-6942384	19393056
3	1633440	-32754240	285912576	-1530550656	5859206496
4	179674560	-4597315416	52996510080	-375868552680	1882007237376

The points $\tilde{v}_6^*,\tilde{v}_7^*,\tilde{v}_8^*$ lie on the one-dimensional edge and we ignore the interior point $\tilde{v}_7^*$ to recover the singularity corresponding to the coincidence of the branes. The interior points on the edge span the Dynkin diagram of the A_1 and the dual Calabi-Yau 4-fold develops the A_1 singularity when the two parallel D-branes coincide.

Then we obtain the charge vectors (34) for the coincident D-branes phase corresponding to the maximal triangulation of the point configuration without $\tilde{v}_7^*$:

$$\begin{array}{cccccccccccc} & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & c & \\ l^1 & =(& -4 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 0 &) \\ l^2 & =(& -2 & 2 & 0 & 0 & 0 & 0 & 1 & -1 & 0 &) \\ l^3 & =(& 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 &) \end{array} \tag{34}$$

The Kähler form is $J=\sum_a k_a J_a$, where J_a denotes the basis of $H^{1,1}(W_4)$ dual to the Mori cone generated by Eq. (34), and k_a are flat coordinates on the Kähler moduli space of the mirror four-fold W_4. Then we choose basis elements (35) of $H_4(W_4)$ which are defined by intersections of the toric divisors D_i corresponding to the $\tilde{v}_i^*$,

$$\gamma_1=D_2\cap D_6,\ \gamma_2=D_2\cap D_7 \tag{35}$$

After transforming the variables as follows,

$$t=k_1+k_2,\ \ \hat{t}=k_2, \tag{36}$$

the leading terms of the period integrals are

$$\tilde{\Pi}^*_{2,1}=\frac{3}{2}t^2,\ \ \tilde{\Pi}^*_{2,2}=(t-\hat{t})^2. \tag{37}$$

corresponding to $\gamma_1+\gamma_2$ and γ_1. In Eq. (37), the $\tilde{\Pi}^*_{2,1}$ depends on the closed modulus t purely leading the bulk potential, and the $\tilde{\Pi}^*_{2,2}$ rely on closed (t) and open ($\hat{t}$) modulus both leading the superpotential. There is only one open modulus in the coincident branes phase, since the coincidence condition reduces the degree of freedom of the open-closed parameter space. The open deformation $\hat{t}$ can be interpreted as the position parameter of the two coincident D-branes.

Following Eq. (10), in terms of algebraic coordinates (9),

$$z_1=\frac{a_2a_3a_4a_5a_7}{a_0^4a_6},\quad z_2=\frac{a_1^2a_6}{a_0^2a_7}, \tag{38}$$

the fundamental period and the logarithmic periods,

$$\Pi_0(z)=w_0(z;0),\quad \Pi_{1,i}(z)=\partial_{\rho_i}w_0(z;\rho)|_{\rho_i=0},$$
$$\Pi_{2,n}(z)=\sum_{i,j}K_{i,j;n}\partial_{\rho_i}\partial_{\rho_j}w_0(z;\rho)|_{\rho=0}, \tag{39}$$

solve the generalized GKZ system governed by charge vectors (34). The flat coordinates are given by

$$k_i=\frac{\Pi_{1,i}(z)}{\Pi_0(z)}=\frac{1}{2\pi i}\log\ z_i+\dots\ . \tag{40}$$

Then the mixed inverse mirror maps in terms of $q_i=\exp(2\pi i k_i)$ for $\{i=1,2\}$ are

$$z_1=q_1+24q_1^2-q_1q_2-2112q_1^2q_2+1452q_1^2q_2^2+\dots$$
$$z_2=q_2-24q_1q_2+972q_1^2q_2+q_2^2-708q_1q_2^2-38232q_1^2q_2^2+\dots\ . \tag{41}$$

We identify the bulk potential and the superpotentials (42) with the exact solutions to the GKZ system which lead by $\tilde{\Pi}^*_{2,1}$ and $\tilde{\Pi}^*_{2,2}$ respectively, where $q=\exp(2\pi i t)$, as:

$$F_t(t)=\frac{3}{2}t^2+\frac{1}{4\pi^2}\left(7884q+12058875q^2+35701252536q^3+\frac{553612044149883}{4}q^4+\frac{15574379291641267884}{25}q^5+308278315993934991468 6q^6+\dots\right),$$
$$\mathcal{W}_c(t,\hat{t})=(t-\hat{t})^2+\frac{1}{4\pi^2}\left(-160q_1+6600q_1^2-\frac{4900480q_1^3}{9}+14q_2+4832q_1q_2-583608q_1^2q_2+94807552q_1^3q_2+\frac{25q_2^2}{6}+7768q_1q_2^2+6267312q_1^2q_2^2-1416535392q_1^3q_2^2+\frac{94q_2^3}{45}-\frac{43632}{5}q_1q_2^3+\frac{74621484}{5}q_1^2q_2^3+\frac{166008561536}{9}q_1^3q_2^3+\dots\right). \tag{42}$$

In Table 2, the first several orders of $U(2)$ Ooguri-Vafa invariants for the coincident phase are listed. Due to the numeric normalization of the open- and closed-string flat coordinates, t and $\hat{t}$, the invariants in Table 2 are rational. The same situation also appears in the paper of Hans Jockers and Masoud Soroush [7]. As they mentioned, the invariants listed in their Table 1 on page 33 and Table 2 on page 41 are fractional because of the normalization.

4.2 D-branes on the octic

The octic is defined as a hypersurface

$$P=a_1x_1^8+a_2x_2^8+a_3x_3^8+a_4x_4^8+a_5x_5^2+a_0x_1x_2x_3x_4x_5. \tag{43}$$

In the ambient toric variety $P_{\Sigma(\Delta_4)}$, it is determined by the vertices of the polyhedron Δ_4,

$$v_1=(1,-1,-1,-1),v_2=(-1,7,-1,-1),v_3=(-1,-1,7,-1),$$
$$v_4=(-1,-1,-1,7),v_5=(-1,-1,-1,-1). \tag{44}$$

Table 2. $U(2)$ Ooguri-Vafa invariants N_{n_1,n_2} for the off-shell superpotential $\mathcal{W}_c(t,\hat{t})$ of coincident branes on the sextic.

n_1 \ n_2	0	1	2	3	4
0	0	14	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{26}{105}$
1	-160	4832	7768	$-\frac{43632}{5}$	$\frac{125088}{35}$
2	6640	-583608	6266104	$-\frac{74621484}{5}$	$-\frac{1353668264}{35}$
3	-544480	94807552	-1416535392	$\frac{55336185568}{3}$	$-\frac{1681292771616}{35}$
4	59891520	-17376236956	404955896632	$-\frac{26201731369652}{5}$	71041365850176

4.2.1 Parallel D-branes phase

We consider the parallel branes which are described by the reducible divisor $\mathcal{D}=\mathcal{D}_1+\mathcal{D}_2$ realized by the degree-16 homogeneous equation

$$Q=b_0(x_1x_2x_3x_4x_5)^2+b_1x_1x_2x_3x_4x_5^3+b_2x_5^4 \tag{45}$$

$$\sim\prod_{i=1}^{2}(\phi_i a_0x_1x_2x_3x_4x_5+a_5x_5^2). \tag{46}$$

According to the construction in Section 2, the open-closed system is encoded in the enhanced polyhedron $\tilde{\nabla}_5$ whose vertices are

$$\tilde{v}_0^*=(0,0,0,0,0),\tilde{v}_1^*=(1,0,0,0,0),\tilde{v}_2^*=(0,1,0,0,0),$$
$$\tilde{v}_3^*=(0,0,1,0,0),\tilde{v}_4^*=(0,0,0,1,0),\tilde{v}_5^*=(-4,-1,-1,-1,0),$$
$$\tilde{v}_6^*=(0,0,0,0,1),\tilde{v}_7^*=(1,0,0,0,1),\tilde{v}_8^*=(2,0,0,0,1). \tag{47}$$

The geometry for the F-theory compactification, the compact 4-fold W_4, is defined by the polyhedron ∇_5 with vertices given in Eq. (47) and $\tilde{v}_c^*$.

The generators of Mori cone of the toric variety determined by ∇_5 are given by:

$$\begin{array}{rcccccccccccc} & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & c & \\ l^1 &=(& -4 & 0 & 1 & 1 & 1 & 1 & -4 & 4 & 0 & 0 &) \\ l^2 &=(& 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 &) \\ l^3 &=(& -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 &) \\ l^4 &=(& 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 &). \end{array} \tag{48}$$

The Kähler form is $J=\sum_a k_aJ_a$, where J_a denotes the basis of $H^{1,1}(W_4)$ dual to the Mori cone generated by Eq. (48), and k_a are flat coordinates on the Kähler moduli space of the mirror four-fold W_4. Then we choose the basis elements of $H_4(W_4)$ which are defined by intersections of the toric divisors D_i corresponding to the $\tilde{v}_i^*$, namely

$$\gamma_1=D_2\cap D_9,\ \gamma_2=D_7\cap D_8,\ \gamma_3=D_6\cap D_7. \tag{49}$$

After changing the variables as follows to visualize the closed and open moduli

$$t=k_1+4(k_2+k_3),\ \ \hat{t}_1=k_2+k_3,\ \ \hat{t}_2=k_3, \tag{50}$$

the leading terms of the periods are

$$\tilde{\Pi}_{2,1}^*=t^2,\ \ \tilde{\Pi}_{2,2}^*=2(t-4\hat{t}_1)^2,\ \ \tilde{\Pi}_{2,3}^*=2(t-4\hat{t}_2)^2, \tag{51}$$

corresponding to γ_1, γ_2 and γ_3. $\tilde{\Pi}_{2,1}^*$ only depends on the closed moduli t and is supposed to be the leading term of the bulk potential function $F_t(t)$, while $\tilde{\Pi}_{2,2}^*$, $\tilde{\Pi}_{2,3}^*$ are supposed to lead the D-brane superpotential $\mathcal{W}(t,\hat{t})$, which depends on both open ($\hat{t}$) and closed (t) parameters.

We identify the bulk potential and the superpotentials with the exact solutions to the GKZ system which lead by $\tilde{\Pi}_{2,1}^*$,$\tilde{\Pi}_{2,2}^*$ and $\tilde{\Pi}_{2,3}^*$ respectively. Following Eq. (10), in terms of algebraic coordinates (9),

$$z_1=\frac{a_2a_3a_4a_5b_1^4}{a_0^4b_0^4},\quad z_2=\frac{b_0b_2}{b_1^2},\quad z_3=\frac{a_1b_1}{a_0b_2}, \tag{52}$$

the fundamental period and the logarithmic periods,

$$\Pi_0(z)=w_0(z;0),\quad \Pi_{1,i}(z)=\partial_{\rho_i}w_0(z;\rho)|_{\rho_i=0},$$
$$\Pi_{2,n}(z)=\sum_{i,j}K_{i,j;n}\partial_{\rho_i}\partial_{\rho_j}w_0(z;\rho)|_{\rho=0}, \tag{53}$$

solve the generalized GKZ system governed by charge vectors (48). The flat coordinates are given by

$$k_i=\frac{\Pi_{1,i}(z)}{\Pi_0(z)}=\frac{1}{2\pi i}\log\ z_i+...\ . \tag{54}$$

Then the mixed inverse mirror maps in terms of $q_i=\exp(2\pi ik_i)$ for $\{i=1,2,3\}$ are:

$$z_1=q_1-24q_1^2+4q_1q_2-72q_1^2q_2+6q_1q_2^2-72q_1^2q_2^2-8q_1q_2q_3$$
$$+400q_1^2q_2q_3-24q_1q_2^2q_3+1056q_1^2q_2^2q_3+24q_1q_2^2q_3^2$$
$$-3168q_1^2q_2^2q_3^2+...$$
$$z_2=q_2+6q_1q_2+189q_1^2q_2-2q_2^2-24q_1q_2^2-828q_1^2q_2^2-q_2q_3$$
$$-6q_1q_2q_3-189q_1^2q_2q_3+5q_2^2q_3-4q_1q_2^2q_3-1386q_1^2q_2^2q_3$$
$$-3q_2^2q_3^2+28q_1q_2^2q_3^2+2214q_1^2q_2^2q_3^2+...$$
$$z_3=q_3+q_2q_3+6q_1q_2q_3+189q_1^2q_2q_3+q_3^2+q_2^2q_3^2-52q_1q_2^2q_3^2$$
$$-3042q_1^2q_2^2q_3^2+...\ . \tag{55}$$

According to the leading terms (51), we find the relative periods which correspond to the closed-string period

Table 3. $U(1)$ Ooguri-Vafa invariants N_{n_1,n_2,n_3} for the off-shell superpotential $\mathcal{W}_1(t,\hat{t})$ of parallel branes on the octic.

n_1 \ $n_2=n_3$	0	1	2	3	4
0	0	128	0	0	0
1	320	-3328	16256	-42752	59008
2	13120	-237952	2030336	-10705536	38614016
3	1088960	-28164352	349712128	-2756937728	15404800000
4	119783040	-4042092800	66070617600	-694387830656	5256175037440

and D-brane superpotentials in the A-model as follows:

$$\begin{aligned} F_t(t)&\equiv\Pi_{2,1}=t^2+\frac{1}{4\pi^2}(29504q+...)\\ \mathcal{W}_1(t,\hat{t}_1)&\equiv\Pi_{2,2}=2(t-4\hat{t}_1)^2+\frac{1}{4\pi^2}\Big(59008q+8\hat{q}_1^4+\frac{128}{9}\hat{q}_1^3\\ &\quad+32\hat{q}_1^2+128\hat{q}_1-42752q\hat{q}_1^{-1}+16256q\hat{q}_1^{-2}\\ &\quad-3328q\hat{q}_1^{-3}+320q\hat{q}_1^{-4}\\ &\quad+38618080q^2\hat{q}_1^{-4}+...\Big)\\ \mathcal{W}_2(t,\hat{t}_2)&\equiv\Pi_{2,3}=2(t-4\hat{t}_2)^2+\frac{1}{4\pi^2}\Big(59008q+8\hat{q}_2^4+\frac{128}{9}\hat{q}_2^3\\ &\quad+32\hat{q}_2^2+128\hat{q}_2-42752q\hat{q}_2^{-1}+16256q\hat{q}_2^{-2}\\ &\quad-3328q\hat{q}_2^{-3}+320q\hat{q}_2^{-4}\\ &\quad+38618080q^2\hat{q}_2^{-4}+...\Big), \end{aligned} \tag{56}$$

where $q=\exp(2\pi it)$, $\hat{q}_1=\exp(2\pi i\hat{t}_1)$ and $\hat{q}_2=\exp(2\pi i\hat{t}_2)$. Absolutely, the bulk potential $F_t(t)$ only depends on the closed modulus t and the D-brane superpotentials $\mathcal{W}_1(t,\hat{t}_2),\mathcal{W}_2(t,\hat{t}_1)$ have kept the Z_2 symmetry with respect to $\hat{t}_1$ and $\hat{t}_2$ after adding the instanton correction, meeting our expectations.

The first several orders of $U(1)$ Ooguri-Vafa invariants for the parallel phase are listed in Table 3.

4.2.2 Coincident D-branes phase

According to the enhanced polyhedron ∇_5, the defining polynomial of the the dual 4-fold M_4 on the B-model side is

$$\begin{aligned}\tilde{P}=&a_1x_1^8x_6^4+a_2x_2^8x_7^4+a_3x_3^8x_8^4+a_4x_4^8x_9^4+a_5x_5^2x_6x_7x_8x_9x_{10}^2\\ &+a_6x_1^2x_2^2x_3^2x_4^2x_5^2a_7x_5^3x_1x_2x_3x_4x_{10}+a_8x_5^4x_{10}^2\\ &+a_9x_6^2x_7^2x_8^2x_9^2x_{10}^2+a_0x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}.\end{aligned} \tag{57}$$

To simplify the notation, we also denote the coefficients of the monomials in this polynomial as a_i's, and the relation between the notations in the D-brane geometry and the 4-fold is as follows:

$$a_i=\begin{cases}a_i & 0\leqslant i\leqslant 5,\\ b_{i-6} & 6\leqslant i\leqslant 8,\\ c & i=9.\end{cases} \tag{58}$$

When $b_1^2=4b_0b_2$, the defining equation for the parallel D-branes becomes:

$$Q\sim(\phi a_0x_1x_2x_3x_4x_5+a_5x_5^2)^2, \tag{59}$$

which means the two individual D-branes coincide as $\phi_1=\phi_2=\phi$. Correspondingly, the equivalent description

$$a_7^2=4a_6a_8 \tag{60}$$

gives rise to the perfect square $(x_1x_2x_3x_4x_5\pm x_5^2x_{10})^2$ in $\tilde{P}$. Obviously M_4 becomes singular and the condition (60) constricts the complex structure moduli space of the 4-fold to a submanifold which is the coincident D-branes phase. The dual description of the coincident on the A-model side is the blow-down of the exceptional divisor developing the curve singularity on the W_4.

The points $\tilde{v}_6^*,\tilde{v}_7^*,\tilde{v}_8^*$ lie on the one-dimensional edge and we ignore the interior point $\tilde{v}_7^*$ to recover the singularity corresponding to the coincidence of the branes. The interior points on the edge span the Dynkin diagram of the A_1 and the dual Calabi-Yau 4-fold develops the A_1 singularity when all the 2 parallel D-branes coincide.

Then we obtain the new charge vectors (61) for the coincident D-branes phase corresponding to the maximal triangulation of the point configuration without $\tilde{v}_7^*$.

$$\begin{array}{cccccccccccc} & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & c & \\ l^1 & =(& -4 & 0 & 1 & 1 & 1 & 1 & -2 & 2 & 0 &)\\ l^2 & =(& -2 & 2 & 0 & 0 & 0 & 0 & 1 & -1 & 0 &)\\ l^3 & =(& 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 &)\end{array} \tag{61}$$

The Kähler form is $J=\sum_a k_aJ_a$, where J_a denotes the basis of $H^{1,1}(W_4)$ dual to the Mori cone generated by Eq. (61), and the k_a's are flat coordinates on the Kähler moduli space of the mirror four-fold W_4. Then we choose basis elements (62) of $H_4(W_4)$ which are defined by intersections of the toric divisors D_i corresponding to the $\tilde{v}_i^*$,

$$\gamma_1=D_2\cap D_6,\ \gamma_2=D_2\cap D_7,\ \gamma_3=D_6\cap D_7. \tag{62}$$

After transforming the variables as follows,

$$t=k_1+2k_2,\ \ \hat{t}=k_2, \tag{63}$$

the leading terms of the period integrals are

$$\tilde{\Pi}_{2,1}^*=t^2,\ \ \tilde{\Pi}_{2,2}^*=(t-2\hat{t})^2, \tag{64}$$

corresponding to $\gamma_1+\gamma_2$ and γ_3. In Eq. (64), $\tilde{\Pi}_{2,1}^*$ depends on the closed modulus t purely leading the bulk

Table 4. $U(2)$ Ooguri-Vafa invariants N_{n_1,n_2} for the off-shell superpotential $\mathcal{W}_c(t,\hat{t})$ of coincident branes on the octic.

$n_1 \backslash n_2$	0	1	2	3	4
0	0	32	$\frac{4}{3}$	$\frac{16}{15}$	$\frac{52}{105}$
1	160	-5504	$\frac{124736}{3}$	$\frac{155712}{5}$	$-\frac{3271328}{105}$
2	6560	-568896	9520016	$-\frac{303915232}{5}$	$\frac{754225376}{3}$
3	544480	-91509504	$\frac{7637988352}{3}$	$-\frac{430429613824}{15}$	$\frac{899073583744}{5}$
4	59891520	-16678810208	$\frac{2110558745464}{3}$	-12541186234064	$\frac{2597111110133168}{21}$

potential, and the $\tilde{\Pi}^*_{2,2}$ rely on closed (t) and open ($\hat{t}$) modulus both leading the superpotential. There is only one open modulus in the coincident branes phase, since the coincident condition reduces the degree of freedom of the open-closed parameter space. The open deformation $\hat{t}$ can be interpreted as the position parameter of the two coincident D-branes.

The instanton corrections of the period integrals can be recovered from the solution of the generalized GKZ system corresponding to the enhanced polyhedron ∇_5. Following Eq.(10), in terms of algebraic coordinates (9),

$$z_1=\frac{a_2a_3a_4a_5a_7}{a_0^4a_6^2}, \quad z_2=\frac{a_1^2a_6}{a_0^2a_7}, \tag{65}$$

the fundamental period and the logarithmic periods,

$$\Pi_0(z)=w_0(z;0), \quad \Pi_{1,i}(z)=\partial_{\rho_i}w_0(z;\rho)|_{\rho_i=0},$$
$$\Pi_{2,n}(z)=\sum_{i,j}K_{i,j;n}\partial_{\rho_i}\partial_{\rho_j}w_0(z;\rho)|_{\rho=0}, \tag{66}$$

solve the generalized GKZ system governed by charge vectors (61). The flat coordinates are given by

$$k_i=\frac{\Pi_{1,i}(z)}{\Pi_0(z)}=\frac{1}{2\pi i}\log\ z_i+... \ . \tag{67}$$

Then the mixed inverse mirror maps in terms of $q_i=\exp(2\pi i k_i)$ for $\{i=1,2\}$ are

$$z_1=q_1-24q_1^2+4q_1q_2-72q_1^2q_2+6q_1q_2^2-72q_1^2q_2^2+...$$
$$z_2=q_2+6q_1q_2+189q_1^2q_2-2q_2^2-24q_1q_2^2-828q_1^2q_2^2+... \ . \tag{68}$$

We identify the bulk potential and the superpotentials (69) with the exact solutions to the GKZ system which lead by $\tilde{\Pi}^*_{2,1}$ and $\tilde{\Pi}^*_{2,2}$ respectively, where $q=exp(2\pi i t)$:

$$F_t(t)=t^2+\frac{1}{4\pi^2}\Big(29504q+257677200q^2 +\frac{38440454795264}{9}q^3+...\Big)$$

$$\mathcal{W}_c(t,\hat{t})=(t-\hat{t})^2+\frac{1}{4\pi^2}\Big(160q_1+6600q_1^2+32q_2-5504q_1q_2 -568896q_1^2q_2+\frac{28q_2^2}{3}+\frac{124736}{3}q_1q_2^2 +9518640q_1^2q_2^2+...\Big) \tag{69}$$

In Table 4, the first several orders of $U(2)$ Ooguri-Vafa invariants for the coincident phase are displayed. Similar to the last model, the invariants are rational up to a numeric normalization of the open- and closed-string flat coordinates, t and $\hat{t}$. Analogously, the invariants listed in Table 1 on page 33 and Table 2 on page 41 in Ref. [7] are fractional because of the normalization.

5 Summary

This paper focuses on the parallel phase and the coincident phase of two D-brane systems with compact Calabi-Yau manifold compactification which contains one closed modulus, namely D-branes on the sextic and the D-branes on the octic. The open-closed duality connects the D-brane geometry based on the Calabi-Yau 3-fold for Type-II theory with a Calabi-Yau 4-fold for F-theory, and furthermore, equalizes the open and closed moduli as the complex structure moduli of the dual 4-fold. Taking advantage of this, we conveniently compute the D-brane superpotentials for parallel and coincident phases near the large radius limit point in the open-closed deformation space on the F-theory side, and extract the disk invariants from their instanton expansion.

The parallel and coincident phases of the D-brane system are also known as the Coulomb branch and the Higgs branch respectively in terms of gauge theory. When the n parallel D-branes approach each other and finally coincide, the phase transition appears and the $U(1)\times U(1)\times...\times U(1)$ gauge theory corresponding to the parallel phase receives a symmetry enhancement, giving rise to the $U(n)$ gauge theory. The difference can be observed from the disk invariants of the two phases, in terms of there being no suitable normalization that can make $N_{n_1,n_2,n_3=n_2}=N_{n_1,n_2}$ for every (n_1,n_2). It means the two phases have different BPS state spectra.

In future work, we will further study the phase transition for multiple D-brane systems with several closed-string deformations and the physical properties of the phase transition from the viewpoint of changing of Ooguri-Vafa invariants. It is also interesting to calculate the D-brane superpotential from the A_∞-structure in the derived category of coherent sheaves of Calabi-Yau manifold and path algebras of quivers.

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