Near horizon OTT black hole asymptotic symmetries and soft hair *

B. Cvetković D. Simić[1)]

Institute of Physics, University of Belgrade, P. O. Box 57, 11001 Belgrade, Serbia

Abstract: We study the near horizon geometry of both static and stationary extremal Oliva Tempo Troncoso (OTT) black holes. For each of these cases, a set of consistent asymptotic conditions is introduced. The canonical generator for the static configuration is shown to be regular. For the rotating OTT black hole, the asymptotic symmetry is described by the time reparametrization, the chiral Virasoro and centrally extended $u(1)$ Kac-Moody algebras.

Keywords: near horizon geometry, OTT black hole, 3D gravity

PACS: 04.50.Kd, 04.60.Kz, 04.70.Bw **DOI:** 10.1088/1674-1137/43/1/013109

1 Introduction

The long-standing problem of the origin of black hole entropy is one of the most important open questions in contemporary physics. There are many proposals for interpreting the black hole entropy and the corresponding micro-states, such as: entanglement entropy [1], fuzz-ball [2] or soft hair on the horizon [3, 4]. The issue has also been the starting point of many ingenious discoveries, the most impressive of which is the holographic nature of gravity [5].

Holographic duality [6] states that the gravitational theory in an asymptotically anti de Sitter (AdS) space-time is dual to a non-gravitational theory defined on the conformal boundary of space-time. Although it is still a conjecture, there is a large number of results supporting this statement. Let us mention, for the purpose of this paper, that holographic duality offers many insights into the black hole physics, including the black hole information paradox and the origin of the black hole micro-states. In fact, holography provides a a way to derive the black hole entropy from the near horizon micro-states via the Cardy formula [7], whose applicability crucially relies on the existence of $2D$ conformal symmetry as a subgroup of the asymptotic symmetry group. In spite of this, the present understanding of the holographic duality is not sufficient for the most general purpose, and we need further generalizations. A notable progress represents the derivation of the Cardy-like formula in the Warped Conformal Field Theory (WCFT), see ref. [8].

A particularly interesting generalization is given in [9], where the authors propose that the extremal Kerr black hole is dual to the chiral $2D$ CFT. There are indications that this chiral CFT should arise as Discrete Light Cone Quantized (DLCQ) [10]. More precisely, the extremal black hole, non necessarily Kerr-like, possesses an intriguing feature that its near horizon geometry is an exact solution of the theory. This allows to study the physics on the horizon by investigating the properties of the near horizon geometry. For a review of the subject see [11].

In this article, we analyse the near horizon limit of a black hole with soft hair known as the Oliva-Tempo-Troncoso (OTT) black hole [12], which is the solution of the BHT gravity [13], as well as of the Poincaré gauge theory of gravity [14] for the special choice of action parameters. The leading idea of this analysis is a study of the influence of the hair parameter on the micro-states of the extremal black hole. The obtained near horizon geometries exist, without any reference to the extremal black hole, as independent solutions and are important on their own.

We first analyse the static OTT black hole, which becomes extremal for the specific value of the hair parameter, and obtain the corresponding near horizon geometry. We then study the asymptotic structure of the near horizon geometry and obtain the asymptotic symmetry group.

We continue with a study of the rotating OTT black hole which can be made extremal in two different ways: either by tuning the hair parameter or the angular momentum. The solution obtained by tuning the hair parameter, surprisingly, leads to the same near horizon geometry as in the non-rotating case. The extremal OTT

Received 25 April 2018, Revised 5 November 2018, Published online 7 December 2018

∗ Supported by the Serbian Science Foundation (171031)

1) E-mail: cbranislav@ipb.ac.rs, dsimic@ipb.ac.rs

black hole with maximal angular momentum leads to a geometry with a richer structure. We conclude that the asymptotic symmetry is a direct sum of the time reparametrization, the Virasoro algebra and the centrally extended $u(1)$ Kac-Moody algebra. The entropy of the extremal rotating OTT black hole can be expressed in terms of the central extension of the Kac-Moody algebra and the on-shell value of the zero mode Virasoro generator

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\text{on-shell}}\kappa}. \tag{1}$$

Our conventions are the same as in Ref. [14]: the Latin indices $(i,j,k,...)$ refer to the local Lorentz frame, the Greek indices $(\mu,\nu,\rho,...)$ refer to the coordinate frame, e^i is the orthonormal triad (coframe 1-form), ω^{ij} is the Lorentz connection (1-form), the respective field strengths are the torsion $T^i=de^i+\omega^i{}_m\wedge e^m$ and the curvature $R^{ij}=d\omega^{ij}+\omega^i{}_k\wedge\omega^{kj}$ (2-forms), the frame h_i dual to e^j is defined by $h_i\rfloor e^j=\delta_i{}^j$, the signature of the metric is $(+,-,-)$, totally antisymmetric symbol ε^{ijk} is normalized to $\varepsilon^{012}=1$, the Lie dual of an antisymmetric form X^{ij} is $X_i:=-\varepsilon_{ijk}X^{jk}/2$, the Hodge dual of a form α is ${}^\star\alpha$, and the exterior product of forms is implicit.

2 Conformally flat Riemannian solutions in PGT

In the sector with a unique AdS ground state, the BHT gravity possesses an interesting black hole solution, the OTT black hole [12]. One of the key features of this solution is its conformal flatness, such that it is also a Riemannian solution of PGT in vacuum [14], for the special choice of the Lagrangian parameters.

The general parity preserving Lagrangian 3-form of PGT, which is mostly quadratic in field strengths is given by:

$$\begin{aligned} L_G &= -a_0\varepsilon_{ijk}e^iR^{jk}-\frac{1}{3}\Lambda_0\varepsilon_{ijk}e^ie^je^k+L_{T^2}+L_{R^2}, \\ L_{T^2} &= T^{i\star}\left(a_1{}^{(1)}T_i+a_2{}^{(2)}T_i+a_3{}^{(3)}T_i\right), \\ L_{R^2} &= \frac{1}{2}R^{ij\star}\left(b_4{}^{(1)}R_{ij}+b_5{}^{(5)}R_{ij}+b_6{}^{(6)}R_{ij}\right). \end{aligned} \tag{2}$$

where ${}^{(a)}T_i$ and ${}^{(a)}R_{ij}$ are irreducible components of the torsion and the RC curvature, see [15], $a_0=1/16\pi G$, Λ_0 is a cosmological constant, and (a_1,a_2,a_3) and (b_1,b_2,b_3) are the coupling constants in the torsion and the curvature sector, respectively. In [14], it was shown that any conformally flat solution of the BHT gravity (in particular the OTT black hole) is also a Riemannian solution of PGT, provided that

$$b_4+2b_6=0. \tag{3}$$

The conformal properties of 3D spacetime, where the Weyl curvature vanishes identically, are characterized by the Cotton 2-form C^i [16], defined by $C^i:=\nabla L^i=dL^i+\omega^i{}_mL^m$ where $L^m:=Ric^m-\frac{1}{4}Re^m$ is the Schouten 1-form. The conformal flatness of space-time is expressed by the condition $C^i=0$.

By using the BHT condition that ensures the existence of the unique maximally symmetric background [14], the identification (3) can be expressed in the following way:

$$\Lambda_0=-a_0/2\ell^2, \qquad b_4=2a_0\ell^2. \tag{4}$$

3 Canonical generator and conserved charges

The usual construction of the canonical generator of the Poincaré gauge transformations, including diffeomorphisms and Lorentz rotations [17], makes use of the canonical structure of the theory. The construction can be substantially simplified by using the first order formulation of the theory, in which the Lagrangian (3-form) reads:

$$L_G=T^i\tau_i+\frac{1}{2}R^{ij}\rho_{ij}-V(e,\tau,\rho),$$

see [15]. In this formulation, τ^m and ρ_{ij} are independent dynamical variables, the covariant field momenta conjugate to e^i and ω^{ij}. The presence of the potential V ensures the validity of the on-shell relations $\tau_i=H_i$, $\rho_{ij}=H_{ij}$. These relations can be used to transform L_G into its standard quadratic form (2).

The construction of the canonical generator G in the first order formulation can be found in [15]. The action of G on the basic dynamical variables is defined via the Poisson bracket operation, so that G has to be a differentiable phase space functional. The examination of the differentiability of G starts from its variation

$$\begin{aligned} \delta G &= -\int_\Sigma d^2x(\delta G_1+\delta G_2), \\ \delta G_1 &= \varepsilon^{t\alpha\beta}\xi^\mu\big(e^i{}_\mu\partial_\alpha\delta\tau_{i\beta}+\omega^i{}_\mu\partial_\alpha\delta\rho_{i\beta}+\tau^i{}_\mu\partial_\alpha\delta e_{i\beta} \\ &\qquad +\rho^i{}_\mu\partial_\alpha\delta\omega_{i\beta}\big)+\mathcal{R}, \qquad\qquad (5a) \\ \delta G_2 &= \varepsilon^{t\alpha\beta}\theta^i\partial_\alpha\delta\rho_{i\beta}+\mathcal{R}. \qquad\qquad (5b) \end{aligned}$$

Here, Σ is the spatial section of spacetime, the variation is performed in the set of adopted asymptotic states, $\mathcal{R}$ stands for regular (differentiable) terms, and we use ρ^i and ω^i, the Lie duals of ρ_{mn} and ω_{mn}, to simplify the formulas. Diffeomorphisms are parametrized by ξ^μ, and the parameters of local Lorentz rotations are θ^i.

The explicit form of the generator of Lorentz rotations, see [15], implies that there is only one possible nonregular term on the rhs of the variation of the Lorentz rotations generator G_2, which is of the form (5b).

In general $\delta G\neq\mathcal{R}$, so that G is not differentiable. This problem can be, in principle, easily solved by going over to the improved generator $\tilde{G}:=G+\Gamma$, where the boundary term Γ is constructed so that $\delta\tilde{G}=\mathcal{R}$. By making a partial integration in δG, one finds that Γ is defined by the following variational equation

$$\begin{aligned}\delta\Gamma &= \delta\Gamma_1+\delta\Gamma_2,\\ \delta\Gamma_1 &= \int_{\partial\Sigma}\xi^\mu\left(e^i{}_\mu\delta\tau_i+\omega^i{}_\mu\delta\rho_i+\tau^i{}_\mu\delta e_i+\rho^i{}_\mu\delta\omega_i\right), \quad (6a)\\ \delta\Gamma_2 &= \int_{\partial\Sigma}\theta^i\delta\rho_i. \quad (6b)\end{aligned}$$

In many cases the asymptotic conditions ensure the regularity of the Lorentz rotations generator and $\Gamma_2=0$. However, it is worth noting that in the particular problem we are solving the contribution of the surface term of the Lorentz rotations generator is non-trivial, as we shall see in section 5.2.

4 Static OTT black hole orbifold

Extremal static OTT black hole. The metric of the static OTT black hole is given by:

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-N^{-2}\mathrm{d}r^2-r^2\mathrm{d}\varphi^2, \quad (7)$$

where $N^2=-\mu+br+\frac{r^2}{\ell^2}$. Black hole horizons are located at:

$$r_\pm=\frac{1}{2}\left(-b\ell^2\pm\ell\sqrt{b^2\ell^2+4\mu}\right).$$

The black hole is extremal if the horizons coincide, $r_+=r_-$. This condition is satisfied when $b^2\ell^2+4\mu=0$. Let us note that the existence of the extremal black hole horizon implies $b<0$.

Orbifold. Let us now consider the following coordinate transformation:

$$t\to\frac{t}{\varepsilon},\qquad r\to r_++\varepsilon\rho. \quad (8)$$

The metric now becomes:

$$\mathrm{d}s^2=\frac{\rho^2}{\ell^2}\mathrm{d}t^2-\frac{\ell^2}{\rho^2}\mathrm{d}\rho^2-(r_++\varepsilon\rho)^2\mathrm{d}\varphi^2.$$

In the limit $\varepsilon\to 0$, the metric (with the prescription $\rho\to r$) reads:

$$\mathrm{d}s^2=\frac{r^2}{\ell^2}\mathrm{d}t^2-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_+^2\mathrm{d}\varphi^2. \quad (9)$$

It represents a perfectly regular solution, an orbifold.

We choose the triad fields in the simple diagonal form:

$$e^0=\frac{r}{\ell}\mathrm{d}t,\qquad e^1=\frac{\ell}{r}\mathrm{d}r,\qquad e^2=r_+\mathrm{d}\varphi. \quad (10a)$$

The Levi-Civita connection that corresponds to the triad field reads

$$\omega^0=0,\qquad \omega^1=0,\qquad \omega^2=-\frac{e^0}{\ell}. \quad (10b)$$

The curvature 2-form has only one non-vanishing component:

$$R^0=0,\qquad R^1=0,\qquad R^2=\frac{1}{\ell^2}e^0e^1, \quad (11a)$$

the scalar curvature is constant, $R=\frac{2}{\ell^2}$, and the Ricci and Shoutten 1-forms are given by:

$$\begin{aligned}&Ric^0=\frac{e^0}{\ell^2},\qquad Ric^1=\frac{e^1}{\ell^2},\qquad Ric^2=0,\\ &L^0=\frac{e^0}{2\ell^2},\qquad L^1=\frac{e^1}{2\ell^2},\qquad L^2=-\frac{e^2}{2\ell^2}. \quad (11b)\end{aligned}$$

The solution is conformally flat (as the OTT black hole), i.e. the Cotton 2-form $C^i=\nabla L^i$ vanishes and solves the equations of motion of both BHT gravity and PGT in the sector $b_4+2b_6=0$.

4.1 Asymptotic conditions

Let us consider the following asymptotic conditions for the metric in the region $r\to\infty$:

$$g_{\mu\nu}\sim\begin{pmatrix}\mathcal{O}_{-2} & \mathcal{O}_2 & \mathcal{O}_1\\ \mathcal{O}_2 & -\frac{\ell^2}{r^2}+\mathcal{O}_3 & \mathcal{O}_1\\ \mathcal{O}_1 & \mathcal{O}_1 & \mathcal{O}_0\end{pmatrix}, \quad (12)$$

where $\mathcal{O}_n$ denotes a term with an asymptotic behaviour r^{-n} or faster. In accordance with (12), the the triad fields behave as:

$$e^i{}_\mu\sim\begin{pmatrix}\mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_2\\ \mathcal{O}_1 & \frac{\ell}{r}+\mathcal{O}_2 & \mathcal{O}_0\\ \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0\end{pmatrix} \quad (13)$$

The condition $T^i=0$, together with (13), gives the following asymptotic behaviour of the spin connection

$$\omega^i{}_\mu\sim\begin{pmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1\\ \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_1\\ \mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_2\end{pmatrix} \quad (14)$$

The diffeomorphisms that leave the metric (12) invariant are given by:

$$\begin{aligned}\xi^t &= T(t)+\mathcal{O}_3,\\ \xi^r &= rU(\varphi)+\mathcal{O}_0,\\ \xi^\varphi &= S(\varphi)+\mathcal{O}_1. \quad (15)\end{aligned}$$

Lorentz transformations that leave the asymptotic conditions invariant are

$$\theta^0=\mathcal{O}_2,\qquad \theta^1=\mathcal{O}_2,\qquad \theta^2=\mathcal{O}_2. \quad (16)$$

In terms of the Fourier modes $\ell_n:=\delta_0(S=\mathrm{e}^{in\varphi})$ and $j_n:=\delta_0(U=\mathrm{e}^{im\varphi})$, the algebra of the residual gauge transformations takes the form of a semi-direct sum of the

Virasoro and the Kac-Moody algebras:

$$[\ell_m,\ell_n] = -i(m-n)\ell_{m+n},$$
$$[\ell_m,j_n] = inj_{m+n},$$
$$[j_n,j_m] = 0. \qquad (17)$$

4.2 Algebra of charges

The gauge generator is not a priori well-defined because, for given asymptotic conditions, its functional derivatives may be ill-defined, as already mentioned in section 3. This problem can be solved by constructing an improved generator, which includes suitable surface terms [18]. Since our solution is Riemannian, $\tau_i=0$, relation (6b) reduces to:

$$\delta G=\int_{\partial\Sigma}\xi^\mu(\omega^i{}_\mu\delta\rho_i+\rho^i{}_\mu\delta\omega_i) \qquad (18)$$

For the particular asymptotic conditions adopted in this paper, we conclude that the gauge generator is differentiable, so that there is no need for adding any surface term,

$$\Gamma=0. \qquad (19)$$

As a consequence, both the central charge of the Virasoro algebra and the level of the $u(1)$ Kac-Moody algebra both vanish.

5 Near-horizon geometry of rotating OTT

Rotating OTT black hole. The rotating OTT black hole is defined by the metric

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-F^{-2}\mathrm{d}r^2-r^2(\mathrm{d}\varphi+N_\varphi\mathrm{d}t)^2, \qquad (20a)$$

where

$$F=\frac{H}{r}\sqrt{\frac{H^2}{\ell^2}+\frac{b}{2}H(1+\eta)+\frac{b^2\ell^2}{16}(1-\eta)^2-\mu\eta},$$
$$N=AF, \qquad A=1+\frac{b\ell^2}{4H}(1-\eta),$$
$$N_\varphi=\frac{\ell}{2r^2}\sqrt{1-\eta^2}(\mu-bH),$$
$$H=\sqrt{r^2-\frac{\mu\ell^2}{2}(1-\eta)-\frac{b^2\ell^4}{16}(1-\eta)^2}. \qquad (20b)$$

The roots of $N=0$ are

$$r_\pm=\ell\sqrt{\frac{1+\eta}{2}}\left(-\frac{b\ell}{2}\sqrt{\eta}\pm\sqrt{\mu+\frac{b^2\ell^2}{4}}\right).$$

The metric (20) depends on three free parameters, μ, b and η. For $\eta=1$, it represents the static OTT black hole, and for $b=0$, it reduces to the rotating BTZ black hole with parameters (m,j), such that $4Gm:=\mu$ and $4Gj:=\mu\ell\sqrt{1-\eta^2}$.

The conserved charges of the rotating black hole take the following form:

$$E = \frac{1}{4G}\left(\mu+\frac{1}{4}b^2\ell^2\right), \qquad (21a)$$
$$J = \ell\sqrt{1-\eta^2}E. \qquad (21b)$$

The rotating OTT black hole is a three-parameter solution, so that the extremal limit can be achieved in two different ways. The first is the same as in the non-rotating case, by requiring $4\mu+b^2\ell^2=0$. As a simple consequence, the resulting geometry is the same as if the black hole were non-rotating. This is not a surprising result if we note that both energy and angular momentum vanish in this case.

The second way to obtain an extremal black hole is to take $\eta=0$, which means that angular momentum takes the maximal possible value. This corresponds to the usual procedure for the Kerr black hole.

The horizon is located at

$$r_0=\frac{\ell\sqrt{b^2\ell^2+4\mu}}{2\sqrt{2}}. \qquad (22)$$

The coordinate change is given as

$$r\to r_0+\epsilon r$$
$$t\to\frac{t}{\epsilon^2} \qquad (23)$$
$$\varphi\to\varphi-\frac{t}{\ell\epsilon^2}.$$

An interesting departure from the usual redefinition of the coordinates in literature is that, in order to obtain a non-singular metric, we have to scale the time coordinate with the same parameter as used in the rescaling of the radial coordinate but to the power of minus two, instead of the standard minus one.

After changing the coordinates and taking the limit $\epsilon\to0$, we obtain the near-horizon metric

$$\mathrm{d}s^2=\frac{32(b^2\ell^2+4\mu)}{b^4\ell^4}\frac{r^4}{\ell^4}\mathrm{d}t^2-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\left(\mathrm{d}\varphi-\frac{16r^2}{b^2\ell^5}\mathrm{d}t\right)^2, \qquad (24)$$

or

$$\mathrm{d}s^2=2r_0^2\frac{16r^2}{b^2\ell^5}\mathrm{d}t\mathrm{d}\varphi-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\mathrm{d}\varphi^2. \qquad (25)$$

It is convenient to further rescale the time coordinate and obtain a more convenient form of the metric

$$\mathrm{d}s^2=\frac{2r^2r_0}{\ell^2}\mathrm{d}t\mathrm{d}\varphi-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\mathrm{d}\varphi^2. \qquad (26)$$

We again choose the triad fields in the diagonal form

$$e^0=\frac{r^2}{\ell^2}\mathrm{d}t, \quad e^1=\frac{\ell}{r}\mathrm{d}r, \quad e^2=\frac{r^2}{\ell^2}\mathrm{d}t-r_0\mathrm{d}\varphi. \qquad (27)$$

The Levi-Civita connection is given by:

$$\omega^{01}=-\frac{2e^0}{\ell}+\frac{e^2}{\ell}, \quad \omega^{02}=\frac{e^1}{\ell}, \quad \omega^{12}=\frac{e^0}{\ell}. \qquad (28)$$

The solution is maximally symmetric and therefore we have:

$$R^{ij}=\frac{1}{\ell^2}e^ie^j,\quad Ric^i=\frac{2e^i}{\ell^2},\quad L^i=\frac{e^i}{2\ell^2},\quad C^i=0. \quad (29)$$

The rotating OTT black hole for $b=0$ reduces to the rotating BTZ black hole. What can be said about the corresponding near-horizon geometries? If we introduce $\rho=r^2$, we obtain a near-horizon BTZ black hole geometry with two times smaller ℓ, and a different r_0 [11]. The only trace of the hair parameter is hidden in r_0, and it will lead to different values of the central charges. Thus, we are able to recover the results for the near-horizon BTZ black hole geometry from those of the OTT black hole, but not by simply taking $b=0$.

5.1 Asymptotic conditions

We consider the following asymptotic form of the metric

$$g_{\mu\nu}\sim\begin{pmatrix}\mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_{-2}\\ \mathcal{O}_3 & -\frac{\ell^2}{r^2}+\mathcal{O}_4 & \mathcal{O}_1\\ \mathcal{O}_{-2} & \mathcal{O}_1 & \mathcal{O}_0\end{pmatrix}. \quad (30)$$

The asymptotic form of the triad fields is chosen in accordance with the asymptotic behaviour of the metric (30)

$$e^i{}_\mu\sim\begin{pmatrix}\frac{r^2}{\ell^2}+\mathcal{O}_1 & \mathcal{O}_5 & \mathcal{O}_0\\ \mathcal{O}_1 & \frac{\ell}{r}+\mathcal{O}_3 & \mathcal{O}_0\\ \frac{r^2}{\ell^2}+\mathcal{O}_1 & \mathcal{O}_5 & \mathcal{O}_0\end{pmatrix} \quad (31)$$

The asymptotic form of the spin connection reads

$$\omega^i{}_\mu\sim\begin{pmatrix}-\frac{r^2}{\ell^3}+\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0\\ \mathcal{O}_0 & -\frac{1}{r}+\mathcal{O}_2 & \mathcal{O}_0\\ -\frac{r^2}{\ell^3}+\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0\end{pmatrix} \quad (32)$$

The condition of vanishing torsion $T^i=0$, together with (31) and (32), leads to the following constraints

$$\omega_r^2+\omega_r^0=\mathcal{O}_5, \quad (33a)$$

$$\omega_\varphi^1-\frac{e_\varphi^1}{\ell}=\mathcal{O}_2, \quad (33b)$$

$$\frac{e^0{}_\varphi}{\ell}-\frac{e^2{}_\varphi}{\ell}+\omega_\varphi^2-\omega_\varphi^0=\mathcal{O}_2, \quad (33c)$$

$$\omega_\varphi^2-\frac{e_\varphi^2}{\ell}=\mathcal{O}_1, \quad (33d)$$

$$\omega_\varphi^0-\frac{e_\varphi^0}{\ell}=\mathcal{O}_1. \quad (33e)$$

The diffeomorphisms that leave the metric (30) invariant are given by

$$\begin{aligned}\xi^t &= T(t)+\mathcal{O}_3,\\ \xi^r &= rU(\varphi)+\mathcal{O}_1,\\ \xi^\varphi &= S(\varphi)+\mathcal{O}_4.\end{aligned} \quad (34)$$

Lorentz transformations that leave the asymptotic form of the triads and the spin connection invariant are

$$\begin{aligned}\theta^0 &= \partial_r\xi^t\frac{e^2{}_t}{e^1{}_r}+\mathcal{O}_2,\\ \theta^1 &= -\frac{2\xi^r}{r}+\partial_t\xi^t+\mathcal{O}_4,\\ \theta^2 &= \frac{e^0{}_t}{e^1{}_r}\partial_r\xi^t+\mathcal{O}_2.\end{aligned} \quad (35)$$

5.2 Algebra of charges

The improved generator is given by

$$\tilde{G}=G+\Gamma. \quad (36)$$

A direct calculation yields the surface term

$$\begin{aligned}\Gamma = -4a_0\int_0^{2\pi}\mathrm{d}\varphi\Big[T(t)\frac{r^2}{\ell^2}\left(\omega^0{}_\varphi-\frac{e^0{}_\varphi}{\ell}-\omega^2{}_\varphi+\frac{e^2{}_\varphi}{\ell}\right)\\ +S(\varphi)\omega^i{}_\varphi e_{i\varphi}+(2U(\varphi)+\partial_tT(t))e^1{}_\varphi\Big]\end{aligned} \quad (37)$$

The charge is finite due to the conditions that follow from the constraint $T^i=0$. By using the composition law for the local Poincaré transformations

$$\begin{aligned}\xi''^\mu &= \xi^\alpha\partial_\alpha\xi'^\mu-\xi'^\alpha\partial_\alpha\xi^\mu,\\ \theta''^i &= \epsilon^i{}_{jk}\theta^j\theta'^k+\xi^\alpha\partial_\alpha\theta'^i-\xi'^\alpha\partial_\alpha\theta^i\end{aligned} \quad (38)$$

we derive the Poisson bracket algebra for the improved canonical generators (which are also well-defined [19]). The Virasoro algebra is not centrally extended

$$\{L_m,L_n\} = -i(m-n)L_{m+n}, \quad (39)$$

$$\{L_m,J_n\} = inJ_{m+n}, \quad (40)$$

whereas the Kac-Moody algebra does have a central charge κ

$$\{J_m,J_n\}=-i16\pi a_0 m\delta_{m+n,0}, \quad (41)$$

whose value is

$$\kappa=16\pi a_0=\frac{\ell}{G}. \quad (42)$$

For related studies, see [20, 21].

The entropy of the extremal OTT black hole $S=\frac{\pi r_0^2}{G}$ can be reproduced in terms of purely algebraic quantities via a peculiar formula

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\mathrm{on-shell}}\kappa}, \quad (43)$$

where $L_0^{\text{on-shell}}$ is the value of the Virasoro generator L_0 on the shell

$$L_0^{\text{on-shell}}=\frac{r_0^2}{2\ell G}. \tag{44}$$

The entropy formula has a striking resemblance to the entropy formula of [8]. In our case $J_0^{\text{on-shell}}=0$, so that our formula is a consequence of the general expression for entropy in WCFT if

$$L_0^{\text{vac}}-\frac{(J_0^{\text{vac}})^2}{2\kappa}=-\frac{\kappa}{8}. \tag{45}$$

One might intuitively expect that the formula for the black hole entropy in WCFT should correctly reproduce the entropy of an extremal OTT black hole. The expectation relies on the resemblance of the algebra (39), (40) with the Euclidean WCFT algebra, and it is anticipated that the same derivation as in [8] holds in our case.

6 Sugawara-Sommerfeld construction

It is well-known that the Virasoro algebra can be constructed as a bilinear combination of the elements of the Kac-Moody algebra. We apply this procedure, known as the Sugawara-Sommerfeld construction [22], to the algebra obtained in the previous section.

First we introduce the auxiliary operators

$$K_n=\frac{1}{2\kappa}\sum_i J_i J_{n-i}, \tag{46}$$

which obey the following commutation relations

$$i\{K_m,J_n\} = -nJ_{m+n}, \tag{47}$$

$$i\{K_m,K_n\} = (m-n)K_{m+n}, \tag{48}$$

$$i\{K_m,L_n\} = (m-n)K_{m+n}. \tag{49}$$

Then, we define generators of the first Virasoro algebra as

$$L_n^{\mathrm{R}}=L_n-K_n, \tag{50}$$

which satisfy the commutation relations

$$i\{J_m,J_n\} = \kappa m\delta_{m+n,0}, \tag{51}$$

$$i\{J_m,L_n^{\mathrm{R}}\} = 0, \tag{52}$$

$$i\{L_m^{\mathrm{R}},L_n^{\mathrm{R}}\} = (m-n)L_{m+n}^{\mathrm{R}}. \tag{53}$$

The generators of the second Virasoro algebra are defined as

$$L_n^{\mathrm{L}}=-K_{-n}-in\alpha J_{-n}+\frac{c^{\mathrm{L}}}{24}\delta_{n,0}. \tag{54}$$

The generators L_n^{L} and L_n^{R} define the two commuting Virasoro algebras

$$i\{L_m^{\mathrm{L}},L_n^{\mathrm{L}}\} = (m-n)L_{m+n}^{\mathrm{L}}+\frac{c^{\mathrm{L}}}{12}m(m^2-1)\delta_{m+n,0}, \tag{55}$$

$$i\{L_m^{\mathrm{L}},L_n^{\mathrm{R}}\} = 0, \tag{56}$$

$$i\{L_m^{\mathrm{R}},L_n^{\mathrm{R}}\} = (m-n)L_{m+n}^{\mathrm{R}}, \tag{57}$$

with central charges

$$c^{\mathrm{L}}=12\kappa\alpha^2, \qquad c^{\mathrm{R}}=0. \tag{58}$$

In theories with conformal symmetry, it is well-known that entropy can be reproduced by the Cardy formula. Sugawara-Sommerfeld construction includes an arbitrary parameter α, whose value is fixed by requiring that the Virasoro algebra satisfies certain canonical relations. We shall fix it by requiring that the Cardy formula

$$S=2\pi\sqrt{\frac{L_0^{\mathrm{L}}c^{\mathrm{L}}}{6}}+2\pi\sqrt{\frac{L_0^{\mathrm{R}}c^{\mathrm{R}}}{6}}, \tag{59}$$

reproduces entropy correctly. For the orbifold, the values of the Virasoro zero modes are

$$L_0^{\mathrm{L}}=\frac{c^{\mathrm{L}}}{24}, \qquad L_0^{\mathrm{R}}=\frac{r_0^2}{2\ell G}, \tag{60}$$

which implies that the Cardy formula, in combination with (58), gives the entropy

$$S=\frac{\pi c^{\mathrm{L}}}{6}=2\pi\kappa\alpha^2. \tag{61}$$

Consequently, we get

$$\alpha^2=\frac{r_0}{2\ell}. \tag{62}$$

7 Thermodynamics at extremality

There is an equivalent Cardy formula in which, instead of using the background values of the Virasoro zero modes, one uses the temperature. Thus, the required additional piece of information is the temperature of the dual CFT, which may be derived from the black hole thermodynamics.

We start from the first law of black hole thermodynamics

$$\delta E=T_{\mathrm{H}}\delta S+\Omega\delta J+\Phi_i\delta q^i, \tag{63}$$

where J is the angular momentum, Ω is the angular velocity, q^i are additional conserved charges, and Φ_i are potentials conjugate to q^i. In the case of the extremal black hole (for more details on extremal black holes and the first law of thermodynamics, see [23]), for which the Hawking temperature is zero, $T_{\mathrm{H}}=0$, the first law implies that energy is a function of the conserved charges

$$E_{\mathrm{Ext}}=E_{\mathrm{Ext}}(J_{\mathrm{Ext}},q^i_{\mathrm{Ext}}). \tag{64}$$

The corresponding generalized temperatures are defined by

$$T_{\mathrm{L}}=\frac{\partial S_{\mathrm{Ext}}}{\partial J_{\mathrm{Ext}}}, \qquad T_i=\frac{\partial S_{\mathrm{Ext}}}{\partial q^i_{\mathrm{Ext}}}, \tag{65}$$

where T_{L} is the left moving temperature.

The entropy, energy and angular momentum of the extremal OTT black hole are given by:

$$S_{\mathrm{Ext}}=\frac{\pi r_0}{G}, \qquad E_{\mathrm{Ext}}=\frac{r_0^2}{2\ell^2 G}, \qquad J_{\mathrm{Ext}}=\frac{r_0^2}{2\ell G}. \tag{66}$$

From the variation of the entropy of extremal OTT

$$\delta S_{\rm Ext}=\frac{\delta J_{\rm Ext}}{T_{\rm L}},$$

the left moving temperature is determined as

$$T_{\rm L}=\frac{r_0}{\pi\ell}. \tag{67}$$

In the extremal case, the right moving temperature is zero

$$T_{\rm R}=0. \tag{68}$$

By requiring that the alternative form of the Cardy formula,

$$S_{\rm C}=\frac{\pi^2}{3}T_{\rm L}c^{\rm L}+\frac{\pi^2}{3}T_{\rm R}c^{\rm R},$$

reproduces the entropy of the extremal OTT black hole, we conclude that $c^{\rm R}$ is undetermined, and that the left central charge is two times bigger than the Brown-Henneaux central charge

$$c^{\rm L}=\frac{3\ell}{G}. \tag{69}$$

This can be used to fix the constant α appearing in the Sugawara-Sommerfeld construction. From equation (58) and the previous formula, we derive

$$\alpha=\frac{1}{2}. \tag{70}$$

8 Concluding remarks

We investigated the near horizon symmetry of both static and stationary OTT black holes in the quadratic PGT. In the static case, the corresponding asymptotic symmetry is trivial, whereas in the stationary case, the set of consistent asymptotic conditions leads to a symmetry described by time reparametrization and the semi-direct sum of the centrally extended $u(1)$ Kac-Moody and the chiral Virasoro algebras. The improved asymptotic conditions that follow from the vanishing of torsion (33c) can be further strengthened, thus making time reparametrization a pure gauge.

The near horizon limit corresponds to deep infrared sector of the theory, which implies that only the soft part of the charge survives. This means that the corresponding charges represent the soft hair on the black hole horizon. Formula

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\rm on-shell}\kappa},$$

shows that there is an intimate relationship between the black hole entropy and the soft hairs on the horizon, but more precise statements require further studies.

Using the Sugawara-Sommerfeld construction, we build the Virasoro algebra as a bilinear combination of the $u(1)$ Kac-Moody and chiral Virasoro algebras. The presence of conformal symmetry enables to use the Cardy formula for entropy, which correctly reproduces the black hole entropy.

References

1 L. Bombelli, R. K. Koul, J. Lee, and R. Sorkin, Phys. Rev. D, **34**: 373 (1986); C. Callan and F. Wilczek, Phys. Lett. B, **333**: 55-61 (1994); M. Srednicki, Phys. Rev. Lett., **71**: 666-669 (1993)
2 S. D. Mathur, Fortsch. Phys., **53**: 793-827 (2005)
3 S. W. Hawking, M. J. Perry, and A. Strominger, Phys. Rev. Lett., **116**: 231301 (2016)
4 H. Afshar, D. Grumiller, and M. M. Sheikh-Jabbari, Phys. Rev. D, **96**: 084032 (2017)
5 G. 't Hooft, arXiv:gr-qc/9310026
6 J. M. Maldacena, Adv. Theor. Math. Phys., **2**: 231 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. B, **636**: 99 (2002); E. Witten, Adv. Theor. Math. Phys., **2**: 253 (1998)
7 A. Strominger, JHEP, **9802**: 009 (1998)
8 S. Detournay, T. Hartman, and D. M. Hofman, Phys. Rev. D, **86**: 124018 (2012)
9 M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. D, **80**: 124008 (2009)
10 V. Balasubramanian, J. de Boer, M. M. Sheikh-Jabbari, and J. Simon, JHEP, **1002**: 017 (2010)
11 G. Compère, Living Rev. Rel., **15**: 11 (2012); Living Rev. Rel., **20**: 1 (2017)
12 J. Oliva, D. Tempo, and R. Troncoso, JHEP, **07**: 011 (2009)
13 E. A. Bergshoeff, O. Hohm, and P. K. Townsend, **102**: 201301 (2009)
14 M. Blagojević and B. Cvetković, Phys. Rev. D, **93**: 044018 (2016)
15 M. Blagojević and B. Cvetković, Phys.Rev. D, **85**: 104003 (2012)
16 A. García, F. W. Hehl, Ch. Heinicke, and A. Macías, Class. Quant. Grav., **21**: 1099-1118 (2004)
17 M. Blagojević, *Gravitation and Gauge Symmetries*, (Institute of Physics, Bristol, 2002)
18 T. Regge and C. Teitelboim, Ann. Phys. (N.Y), **88**: 286-318 (1974)
19 J. D. Brown and M. Henneaux, J. Math. Phys., **27**: 489 (1986)
20 H. Afshar, S. Detournay, D. Grumiller, and B. Oblak, JHEP, **1603**: 187 (2016)
21 H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, Phys. Rev. D, **93**: 101503 (2016)
22 H. Sugawara, Phys. Rev., **170**: 1659 (1968); C. Sommerfeld, Phys. Rev., **176**: 2019 (1968)
23 M. Johnstone, M.M. Sheikh-Jabbari, J. Simón, and H. Yavartanoo, Phys. Rev. D, **88**: 101503 (2013)