# Massless limit of transport theory for massive Fermions*

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**Abstract:** We studied the $m = 0$ limit of different components of Wigner functions for massive fermions. Comparing with the chiral kinetic theory, we separated the vanishing and non-vanishing parts of vector and axial-vector components, up to the first order of $\hbar$. Then, we discussed the possible physical meaning of the vanishing and non-vanishing parts and their different behaviors at thermal equilibrium.

**Keywords:** spin, quantum kinetic theory, mass correction

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## 1 Introduction

It is widely believed that in relativistic heavy-ion collisions (HICs), a new phase of matter called quark-gluon plasma (QGP) is created [1, 2]. In non-central HICs, a strong magnetic field is created by fast moving nuclei [3, 4]. Furthermore, these systems have a large total angular momentum or vorticity in the fluid picture [5]. The effects of magnetic field and vorticity on QGP systems have been of great interest, both experimentally and theoretically, in recent years. These include various anomalous transport phenomena, such as the chiral magnetic effect (CME) [6-8] and chiral vortical effect (CVE) [8-10]. The vorticity also has a direct effect on the polarization of the hadrons produced during freeze-out [11-13]. The polarization of $\Lambda$ hyperons has been measured by the STAR collaboration [14], and the results indeed indicate a non-zero spin alignment in the direction of the total angular momentum.

Due to chiral symmetry restoration, in QGP, the u and d quarks have only a very small current mass and are often treated as chiral fermions. However, it is still worth checking whether the effect of finite mass is really negligible. Moreover, $\Lambda$ hyperons consist of s quarks, whose mass is larger and cannot be ignored. Therefore, the theoretic study of massive fermions is as important as that of massless ones.

Because these phenomena are closely related to the evolution of quantum systems, kinetic theory is a natural choice for studying them [15-18]. Based on the Wigner function formalism [19-22], the kinetic theories of both massive [23-25] and massless fermions [26-34] are studied. One of the advantages of the Wigner function formalism is that it includes spin degree of freedom naturally, which is essential in the study of magnetic field and vorticity related phenomena. Further, it is possible to connect the kinetic description to the hydrodynamic one [31, 35-37], as the latter is used in the quantitative simulation of HIC events.

Although for spin 1/2 particles one expects that the $m = 0$ limit of massive representation would connect to the massless one smoothly, this is not explicitly shown in the Wigner function formalism, especially when quantum correction is taken into consideration. This is also related to the translation or definition of different components in this theory. More detailed discussions can be found in [23, 24].

In this paper, we propose a way of splitting the axial vector component of the Wigner function. With this splitting, we show that the massive Wigner function can indeed go back to the massless one. We also show that our splitting has a certain physical meaning. We will first introduce the structure of Wigner function formalism for both massive and massless fermions and their solution up to the 1st order of $\hbar$. Then, we demonstrate how to separate both the vector component and the axial vector component into a vanishing part and a non-vanishing part in the $m = 0$ limit, as well as the physical meaning of the non-vanishing part. At the end, we briefly discuss some possible equilibrium distributions.

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## 2 Covariant Wigner function

The covariant Wigner function for spin 1/2 fermions in the presence of an external electromagnetic field is defined as [20]

$$W_{ab}(x,p)=\int \mathrm{d}^4 y \mathrm{e}^{\mathrm{i}\frac{py}{\hbar}}\langle\bar{\psi}_b(x-\frac{y}{2})\mathrm{e}^{\mathrm{i}\frac{q}{\hbar}\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}sA(x+sy)y}\psi_a(x+\frac{y}{2})\rangle, \tag{1}$$

where $q$ is the charge of a fermion, and $\mathrm{e}^{\mathrm{i}q\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}sA(x+sy)y}$ is the gauge link that ensures gauge invariance. As we are considering "free" fermions without fermion-fermion interactions, the Wigner function follows the kinetic equation

$$(\gamma^\mu\Pi_\mu+\gamma^\mu\frac{\mathrm{i}\hbar}{2}\mathrm{D}_\mu-m)W=0, \tag{2}$$

where

$$\Pi_\mu=p_\mu-q\hbar\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}ssF_{\mu\nu}(x-\mathrm{i}\hbar s\partial_p)\partial_p^\nu, \tag{3}$$

$$\mathrm{D}_\mu=\partial_\mu-q\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}sF_{\mu\nu}(x-\mathrm{i}\hbar s\partial_p)\partial_p^\nu. \tag{4}$$

$W$ satisfies the relationship $\gamma_0W^\dagger\gamma_0=W$, so it can be decomposed using the Dirac matrices [38]

$$W=\frac{1}{4}\Big[F(x,p)+\mathrm{i}\gamma_5P(x,p)+\gamma_\mu V^\mu(x,p)+\gamma_\mu\gamma_5A^\mu(x,p)+\frac{1}{2}\sigma_{\mu\nu}S^{\mu\nu}(x,p)\Big]. \tag{5}$$

Inserting Eq. (5) into Eq. (2), we arrive at the equations for all components:

$$\Pi^\mu V_\mu=mF, \tag{6}$$

$$\frac{\hbar}{2}\mathrm{D}^\mu A_\mu=mP, \tag{7}$$

$$\Pi_\mu F-\frac{1}{2}\hbar\mathrm{D}^\nu S_{\nu\mu}=mV_\mu, \tag{8}$$

$$-\hbar\mathrm{D}_\mu P+\epsilon_{\mu\nu\sigma\rho}\Pi^\nu S^{\sigma\rho}=2mA_\mu, \tag{9}$$

$$\frac{1}{2}\hbar(\mathrm{D}_\mu V_\nu-\mathrm{D}_\nu V_\mu)+\epsilon_{\mu\nu\sigma\rho}\Pi^\sigma A^\rho=mS_{\mu\nu}, \tag{10}$$

and

$$\hbar\mathrm{D}^\mu V_\mu=0, \tag{11}$$

$$\Pi^\mu A_\mu=0, \tag{12}$$

$$\frac{1}{2}\hbar D_\mu F+\Pi^\nu S_{\nu\mu}=0, \tag{13}$$

$$\Pi_\mu P+\frac{\hbar}{4}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^\nu S^{\sigma\rho}=0, \tag{14}$$

$$\Pi_\mu V_\nu-\Pi_\nu V_\mu-\frac{\hbar}{2}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^\sigma A^\rho=0. \tag{15}$$

These equations can be solved by expanding all operators and functions as series of $\hbar$ and finding solutions order by order. For each order, not all components are independent, as they must follow the constraints given by the equations above. Thus, we can use a certain number of components to express all others. For these "free" components, their corresponding kinetic equations are given by the equations that are one order higher, as the derivatives always contain $\hbar$.

The 0th order solution is [23]:

$$P^{(0)}=0, \tag{16}$$

$$F^{(0)}=mf_V^{(0)}\delta(p^2-m^2), \tag{17}$$

$$V_\mu^{(0)}=p_\mu f_V^{(0)}\delta(p^2-m^2), \tag{18}$$

$$S_{\mu\nu}^{(0)}=\frac{1}{m}\epsilon_{\mu\nu\sigma\rho}p^\sigma A^{(0)\rho}. \tag{19}$$

Here, we choose $f_V$, or $F$, together with $A_\mu$ to be the independent components. There is also one constraining equation for $A_\mu^{(0)}$:

$$p^\mu A_\mu^{(0)}=0, \tag{20}$$

Thus, there are all-together 4 degrees of freedom. It is also possible to use $S_{\mu\nu}$ instead of $A_\mu$ as a free component. The total degree of freedom is the same after taking into consideration all the constraint equations.

The same procedure can be used for massless fermions. However, in the massless case, the vector components $V_\mu$ and $A_\mu$ decouple from the others, and at the 0th order, they can be expressed as [28]

$$V_\mu^{(0)}=p_\mu f_V^{(0)}\delta(p^2), \tag{21}$$

$$A_\mu^{(0)}=p_\mu f_A^{(0)}\delta(p^2). \tag{22}$$

## 3 Massless limit

At first glimpse, the $m\to 0$ limit for $V_\mu$ is quite clear, but for $A_\mu$ it is not. Actually, in the massive case, there is no very simple and unique expression for $A_\mu$, with only one constraining equation. In order to make a comparison with the massless case, we propose the following separation:

$$A_\mu^{(0)}=(p_\mu f_A^{(0)}-\theta_\mu^{(0)})\delta(p^2-m^2). \tag{23}$$

From Eq. (20), we can determine the relation between $f^{(0)}$ and $\theta_\mu^{(0)}$:

$$(p^2f_A^{(0)}-p\cdot\theta^{(0)})\delta(p^2-m^2)=0. \tag{24}$$

However, there is still one redundant degree of freedom. For a given $A_\mu$, we can change $\theta_\mu^{(0)}$ by an arbitrary vector that is parallel to $p_\mu$ and modify $f_A^{(0)}$ according to Eq. (24). The new set will also give the same $A_\mu^{(0)}$. To remove this arbitrariness, we must fix $f_A^{(0)}$. This can be achieved by introducing an auxiliary time-like vector $n_\mu$ and requiring

$$\theta^{(0)}\cdot n = 0. \tag{25}$$

These will lead to the relation

$$f_A^{(0)} = \frac{A^{(0)}\cdot n}{p\cdot n}. \tag{26}$$

Different choices of $n$ correspond to different values of $f_A^{(0)}$ and $\theta_\mu^{(0)}$. Therefore, it is important to choose an $n_\mu$ with proper physical meaning. One of the natural choices is relating $n_\mu$ to the local "average velocity," which corresponds to fluid velocity in ideal hydrodynamics [39]. Then, this can be related to the physical fact that when observed in different reference frames, the helicity of massive fermions can be different. For comparison, although we can write the same expresion for $f_V^{(0)}$ and $V_\mu^{(0)}$, $f_V^{(0)}$ actually does not depend on $n$ because $V_\nu^{(0)}$ is proportional to $p_\mu$.

Regarding the masslese case, from Eq. (21), we can see that relation in Eq. (26) holds naturally. Now, we only have to prove that $\theta_\mu^{(0)}$ vanishes when $m$ goes to zero. This can also be demonstrated using $n_\mu$. If we assume that the Wigner function will not diverge when $m$ goes to 0, then Eq. (19) requires that $\theta_\mu^{(0)}$ vanishes. We can rewrite it in another form to see this more clearly:

$$\theta_\mu^{(0)}\delta(p^2-m^2) = -\frac{m}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu S^{\sigma\rho}. \tag{27}$$

As $p\cdot n$ is generally not zero, $\theta_\mu^{(0)}$ must vanish when $m$ goes to 0.

Therefore, we can see that the $m\to 0$ limits for both, $V_\mu^{(0)}$ and $A_\mu^{(0)}$ are the same as those for the massless solutions.

Next, we proceed to the 1st order components. Similar to the 0th order case, we can also use $f_V$ and $A_\mu$ as free components; then, we will have [25]:

$$P^{(1)} = \frac{1}{2m}\mathrm{D}^\mu A_\mu^{(0)}, \tag{28}$$

$$F^{(1)} = mf^{(1)}\delta(p^2-m^2) - \frac{1}{2m(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}p^\mu \mathrm{D}^\nu p^\sigma A^{(0)\rho}, \tag{29}$$

$$\begin{aligned} V_\mu^{(1)} =& p_\mu f_V^{(1)}\delta(p^2-m^2) + \frac{qp_\mu}{2m^2(p^2-m^2)}\epsilon_{\alpha\beta\sigma\rho}F^{\sigma\rho}p^\alpha A^{(0)\beta} \\ &+ \frac{1}{2m^2}\epsilon_{\mu\nu\sigma\rho}\mathrm{D}^\nu p^\sigma A^{(0)\rho}, \end{aligned} \tag{30}$$

$$A_\mu^{(1)} = (p_\mu f_A^{(1)} - \theta_\mu^{(1)})\delta(p^2-m^2) - \frac{1}{2(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}p^\nu \mathrm{D}^\sigma V^{(0)\rho}, \tag{31}$$

$$S_{\mu\nu}^{(1)} = \frac{1}{2m}(\mathrm{D}_\mu V_\nu^{(0)} - \mathrm{D}_\nu V_\mu^{(0)}) + \frac{1}{m}\epsilon_{\mu\nu\sigma\rho}p^\sigma A^{(1)\rho}. \tag{32}$$

$V_\mu^{(1)}$ can be rewritten as

$$\begin{aligned} V_\mu^{(1)} =& p_\mu f_V^{(1)}\delta(p^2-m^2) \\ &- \frac{p_\mu}{2p^2 p\cdot n}\epsilon_{\alpha\beta\sigma\rho}p^\alpha n^\beta(\mathrm{D}^\sigma\theta^{(0)\rho})\delta(p^2-m^2) \\ &+ \frac{q}{2p^2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}p^\nu f_A^{(0)}\delta(p^2-m^2) \\ &+ \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu(\mathrm{D}^\rho\theta^{(0)\sigma})\delta(p^2-m^2) \\ &- \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu p^\sigma(\mathrm{D}^\rho f_A^{(0)})\delta(p^2-m^2). \end{aligned} \tag{33}$$

In this derivation, we used the Schouten identity $p_\mu\epsilon_{\nu\sigma\rho\lambda} + p_\nu\epsilon_{\sigma\rho\lambda\mu} + p_\sigma\epsilon_{\rho\lambda\mu\nu} + p_\rho\epsilon_{\lambda\mu\nu\sigma} + p_\lambda\epsilon_{\mu\nu\sigma\rho} = 0$ and also $\mathrm{D}_\mu\delta(p^2-m^2) = \frac{2F_{\mu\nu}p^\nu}{p^2-m^2}\delta(p^2-m^2)$. If we take the massless limit, $\theta_\mu^{(0)}$ vanishes and one can easily check that Eq. (33) goes back to

$$\begin{aligned} V_\mu^{(1)} =& p_\mu f_V^{(1)}\delta(p^2-m^2) + \frac{q}{2p^2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}p^\nu f_A^{(0)}\delta(p^2-m^2) \\ &- \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu p^\sigma(\mathrm{D}^\rho f_A^{(0)})\delta(p^2-m^2), \end{aligned} \tag{34}$$

which is the same as the result for chiral fermions [28]. For $n_\mu$, we can use the same definition as in the 0th order case.

For $A_\mu^{(1)}$, we rewrite it as

$$\begin{aligned} A_\mu^{(1)} =& (p_\mu f_A^{(1)} - \tilde{\theta}_\mu^{(1)})\delta(p^2-m^2) - \frac{1}{2p^2}\epsilon_{\mu\nu\sigma\rho}p^\nu \mathrm{D}^\sigma V^{(0)\rho} \\ &- \frac{1}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu p^\sigma(\mathrm{D}^\rho f_V^{(0)})\delta(p^2-m^2), \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{\theta}_\mu^{(1)}\delta(p^2-m^2) =& -\frac{m}{2p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\nu S^{(1)\sigma\rho} \\ &- \frac{m^2 q}{2p\cdot n(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}n^\nu F^{\sigma\rho}f_V^{(0)}\delta(p^2-m^2), \end{aligned} \tag{36}$$

$$\begin{aligned} f_A^{(1)}\delta(p^2-m^2) =& \frac{A^{(1)}\cdot n}{p\cdot n} \\ &- \frac{q}{2p\cdot np^2}\epsilon_{\mu\nu\sigma\rho}n^\mu p^\nu F^{\sigma\rho}f_V^{(0)}\delta(p^2-m^2). \end{aligned} \tag{37}$$

Again, $\tilde{\theta}_\mu$ should vanish in the massless limit, and the remaining part is the same as the massless expression [28]. Thus, we have shown explicitly that up to the first order of $\hbar$, the massive Wigner function can connect to the massless one continuously. This is consistent with the results found in Refs. [23, 24]. Actually, at the lowest order, the dipole-momentum tensor $\Sigma_{\mu\nu}$ used in [23] can be expressed as

$$\Sigma_{\mu\nu} = \frac{1}{m^2}\epsilon_{\mu\nu\sigma\rho}p^\sigma\theta^\rho. \tag{38}$$

Using the same procedure as that for deriving Eq. (33) and Eq. (35), this can be shown to be equal to $\frac{1}{p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\sigma A^\rho$, which is the spin tensor used in [24]. When taking $m\to 0$, it will go to $-\frac{1}{p\cdot n}\epsilon_{\mu\nu\sigma\rho}p^\sigma n^\rho$, as required by [23]. Therefore the different formulations are connected, and this connection is closely related to the massless limit itself.

We should also check the meaning of $f_V$ and $f_A$. In our case

$$f_V^{(0)} = \frac{V^{(0)}\cdot n}{p\cdot n}, \tag{39}$$

$$f_A^{(0)} = \frac{A^{(0)}\cdot n}{p\cdot n}, \tag{40}$$

$$f_V^{(1)} = \frac{V^{(1)}\cdot n}{p\cdot n} - \frac{q}{2p^2 p\cdot n}\epsilon^{\mu\nu\sigma\rho}n^\mu p^\nu F^{\sigma\rho} f_A^{(0)}\delta(p^2-m^2), \tag{41}$$

$$f_A^{(1)} = \frac{A^{(1)}\cdot n}{p\cdot n} - \frac{q}{2p^2 p\cdot n}\epsilon_{\mu\nu\sigma\rho}n^\mu p^\nu F^{\sigma\rho} f_V^{(0)}\delta(p^2-m^2). \tag{42}$$

These are consistent with the massless case. It is already known that $f_V$ is the particle number density. In the massless case, $f_A$ is the difference between left and right handed particles, or the chiral imbalance. Now that we have a continuous expression, we might call $f_A$ chiral imbalance as well, but in the massive case it is not a conserved charge. However, we observe that when choosing the same $n_\mu$, the expression of $f_A^{(0)}$ becomes different in some literature, such as [24], while the massless limit is the same. $\theta_\mu$ might be viewed as the real spin degree of freedom. In the massless case, especially in chiral kinetic theory, this is not seen because the spins of chiral fermions are bound to their momentum. Of course, the degrees of freedom are not lost. We could discuss massless Dirac fermions, and they will have two degrees of freedom in the $S_{\mu\nu}$ component. It is only because $V_\mu$ and $A_\mu$ decouple from $S_{\mu\nu}$, $F$ and $P$ that the latter are not included in the usual chiral kinetic theories. In contrast, in the massive case, the chiral states are not energy eigenstates, and the spin degree of freedom is coupled to other ones, just as $S_{\mu\nu}$ is coupled to $A_\mu$. In fact, in the massive case, it is possible to use $S_{\mu\nu}$ instead of $A_\mu$ as free components to construct the entire kinetic theory [23].

## 4 Kinetic equations and solutions

The discussion above naturally leads to the question of what is the equilibrium distribution of all components. We will only consider 0th order components in this section. The transport equations for the 0th order components are:

$$p\cdot D^\mu f_V^{(0)} = 0, \tag{43}$$

$$(p\cdot D\theta_\mu^{(0)} - qF_{\mu\nu}\theta^{(0)\nu}) - p_\mu p\cdot D f_A^{(0)} = 0. \tag{44}$$

Without the collision term, one cannot determine the true equilibrium state only from the kinetic equation. However, we can still perform some general discussions. The equation for $f_V^{(0)}$ is just a Boltzmann-type equation without a collision term. It would be natural to assume that in equilibrium, $f_V^{(0)}$ takes the form of the usual Fermi-Dirac distribution. In the other equation, there is also a Boltzmann-type equation for $f_A^{(0)}$, coupled to a BMT equation [40] involving $\theta_\mu^{(0)}$. A straightforward guess is that $f_A^{(0)}$ also takes the Fermi-Dirac distribution, leaving the equation for $\theta_\mu^{(0)}$ as

$$p\cdot D\theta_\mu^{(0)} - qF_{\mu\nu}\theta^{(0)\nu} = 0. \tag{45}$$

However, if $f_A^{(0)}$ really is the chiral imbalance, as supposed, with finite mass it should be dispersed over time and, at least at a classical level, reach zero at equilibrium.

In the massless situation, we have

$$f_V^{(0)} = f_+^{(0)} + f_-^{(0)}, \tag{46}$$

$$f_A^{(0)} = f_+^{(0)} - f_-^{(0)}, \tag{47}$$

$$f_\chi^{(0)} = \frac{1}{e^{\mathrm{sgn}(p\cdot n)\frac{p\cdot n-\mu_\chi}{T}}+1}, \tag{48}$$

where $\chi = +$ and $-$ correspond to the right-handed and left-handed components, respectively. $\mu_\chi$ is the corresponding chemical potential. If the right-handed and left-handed components are balanced, we would have $\mu_+ = \mu_- = \mu$; then

$$f_V^{(0)} = \frac{2}{e^{\mathrm{sgn}(p\cdot n)\frac{p\cdot n-\mu}{T}}+1}, \tag{49}$$

$$f_A^{(0)} = 0. \tag{50}$$

This is consistent with the massive situation we just discussed.

Of course, when $f_A^{(0)} = 0$, there is always a trivial solution $\theta_\mu^{(0)} = 0$, giving an $A_\mu^{(0)}$ value of zero. However, if $A_\mu^{(0)}$ is the average spin of the system, it should not always be zero. We can imagine starting from a special initial state, in which the spins of all particles are polarized along one direction. With the addition of a collision term, either the spin itself is conserved, or it is coupled to the orbital angular momentum; however, the total angular momentum is conserved. Either way, it is very unlikely that the average spin will evolve to exactly zero at equilibrium. Therefore, in general we should expect non-zero $A_\mu$ even at equilibrium. In the special case of only a constant thermal vorticity $\omega_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$, where $\beta_\mu = \frac{n_\mu}{T}$, there is a very interesting solution

$$\theta_\mu^{(0)} = a\epsilon_{\mu\nu\sigma\rho}n^\nu p^\sigma \omega^{\rho\lambda} p_\lambda, \tag{51}$$

$$f_A^{(0)} = 0, \tag{52}$$

where $a$ is an undetermined variable quantifying the "strength" of the polarization. We can see that now $A_\mu^{(0)}$ is non-zero, but $f_A^{(0)}$ is zero. By our previous translation of $f_A$, this solution means that there is non-zero spin polarization without any chiral imbalance, which is physical.

For another special case with a homogeneous electromagnetic field and no vorticity, there is also a similar solution

$$\theta_\mu^{(0)} = bF_{\mu\nu}p^\nu, \tag{53}$$

$$f_A^{(0)} = 0. \tag{54}$$

By this solution we require there being no electric field $F_{\mu\nu}n^\nu = 0$.

These two simplified cases show that it is at least possible to have zero chiral imbalance but non-zero average spin at equilibrium. For the more complicated case with both vorticity and magnetic field, there also are solutions, such as [41]:

$$A_\mu = -\frac{1}{2m\Gamma}\epsilon_{\mu\nu\sigma\rho}p^\nu \mathrm{D}^\sigma \beta^\rho - \frac{\hbar}{2(p^2-m^2)}\epsilon_{\mu\nu\sigma\rho}p^\nu \mathrm{D}^\sigma V^{(0)\rho}. \tag{55}$$

However, this solution is differernt from the above ones as now even at the 0th order, $A\cdot n$ is not zero, thus giving a non-zero $f_A$.

## 5 Summary

By performing a special splitting of the axial vector component of the covariant Wigner function for massive fermions, we showed that the $m=0$ limit can be taken, and that the results are consistent with the massless Wigner function. We also discussed the different meanings of the split parts and their possible equilibrium values.

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*The author became aware of related work* [42] *after the completion of this study.*

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