Transverse Ward-Takahashi identities and full vertex functions in different representations of QED_3*

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Abstract: We derive the transverse Ward-Takahashi identities (WTI) of *N*-dimensional quantum electrodynamics by means of the canonical quantization method and the path integration method, and subsequently attempt to prove that QED_3 is solvable based on the transverse and longitudinal WTI, indicating that the full vector and tensor vertices functions can be expressed in terms of the fermion propagators in QED_3. Further, we discuss the effect of different γ matrix representations on the full vertex function.

Keywords: Ward-Takahashi identities, vertex functions, QED_3

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1 Introduction

The normal (longitudinal) Ward-Takahashi identities (WTI) [1, 2] play an important role in various problems in quantum field theory, for example, they provide a consistency condition in the perturbative and non-perturbative approach of any quantum field theory and the proof of renormalizability of gauge theories [3]. In the Dyson-Schwinger equations (DSEs) approach, the fermion-boson vertex function is an important quantity to be specified. To use DSEs for actual calculations, we must artificially cut off the coupling of the *n*-point Green's and the higher-order Green's function to properly close the DSEs. If one can express the three-point vertices in terms of the two-point functions, the DSEs that consist of an infinite set of coupled integral equations will form a closed system for the two-point functions. Among the numerous vertex approximations, the most famous one is the Ball-Chiu Ansatz [4], except for the bare vertex approximation. How to properly break through the bare vertex approximation and the Ball-Chiu Ansatz is a highly challenging problem.

In what follows, we attempt to address this area to determine conditions that can make the DSEs closed. One possible approach to this problem is to use the WTI to constrain the form of the vertex function. However, the normal WTI only contains the longitudinal part of the vertex functions, leaving its transverse part undetermined. To find further constraints on the vertex function, Takahashi derived so called transverse relations [5] relating Green's functions of different orders to complement the normal WTI, which have the potential to determine the full fermion-boson vertex in terms of the renormalization functions of the fermion propagator [6]. Subsequently, He, Takahashi [7, 8], and Kondo [9], *etc*., found that the complete set of transverse WTI and longitudinal WTI for the vector, axial-vector, and tensor vertex functions can form complete solutions for these vertex functions in four-dimensions gauge theories. When this ignores the contribution of the three integral-term involving the Wilson line and chiral limit $m \to 0$, the full vector vertex functions are expressed in terms of the two-point functions. Subsequently, Pennington and R. Williams [10-12] tested the transverse WT identity for the fermion-boson vertex to the one-loop order.

Several authors also attempted to study this problem in various ways, such as via constraints [13-19] and direct numerical solution [20-24]. For instance, Qin *etc.,* [25] consider the coupling of a dressed-fermion to an Abelian gauge boson, and describe a unified treatment and solution of the longitudinal and transverse WTI. The vector vertex is discussed by using twelve independent tensor structures. What we need to emphasize here is that

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although people have made important progress in constructing the fermion-boson vertex functions, it is still not possible to represent the full vector, axial-vector, and tensor vertex functions in the four-dimensional gauge theory by two-point functions. That is to say, in four-dimensional quantum electrodynamics (QED), one cannot construct a completely closed DSEs by three-point Green's functions and two-point Green's functions.

However, when the dimension of the gauge theory is reduced, it will change significantly. For example, in the case of two-dimensional gauge theory, Kondo first pointed out in Ref. [9] that "the transverse together with the usual (longitudinal) Ward-Takahashi identity are applied to specify the fermion-boson vertex function. It is especially shown that in two dimensions, it becomes the exact and closed Schwinger-Dyson equation that can be exactly solved". Because QED_3 can be regarded as an effective theory of high temperature superconductivity or as a toy model of quantum chromodynamics (QCD), this makes the research on QED_3 particularly interesting (related details can be found in the relevant literature in Ref. [26]). Thus, a very natural question arises, can people obtain a completely closed DSEs in three-dimensional QED (QED_3)?

The main purpose of this work is to attempt to construct a closed DSEs by three-point and two-point Green's functions in the case of QED_3, based on normal and transverse WTI. In addition, given that there are two different expressions of the γ matrix in QED_3, we will also discuss the effect of different γ matrix representations on the full vertex functions.

2 Full vertices functions

2.1 Vertex functions in *N*-dimensional gauge theory

The longitudinal (normal) WT identity determines its divergence, *i.e.*, $\partial_\mu \Gamma^\mu(x;y,z)$. The transverse WT identity [5] specifies the curl of the vertex function $\partial^\mu \Gamma^\nu(x;y,z) - \partial^\nu \Gamma^\mu(x;y,z)$, where $\Gamma^\mu(x;y,z)$ is the fermion-boson (photon) vertex function. This was derived by Takahashi in 1986. The transverse Ward-Takahashi identity can be converted to

$$\partial_x^\mu \langle 0|T j(x)\psi(y)\bar{\psi}(z)|0\rangle - \partial_x^\nu \langle 0|T j(x)\psi(y)\bar{\psi}(z)|0\rangle, \quad (1)$$

where $j(x)$ is the current operators. The above relation is valid for both QED and QCD.

First, one introduces two bilinear covariant current operators,

$$V^{\rho\mu\nu\lambda}(x) = \frac{1}{4}\bar{\psi}(x)[[\gamma^\rho,\sigma^{\mu\nu}],\gamma^\lambda]\psi(x) = g^{\rho\mu} j^{\nu\lambda}(x) - g^{\rho\nu} j^{\mu\lambda}(x),$$
$$V^{\rho\mu\nu}(x) = \frac{-i}{2}\bar{\psi}[\gamma^\rho,\sigma^{\mu\nu}]\psi = g^{\rho\mu} j^\nu(x) - g^{\rho\nu} j^\mu(x). \quad (2)$$

One needs to calculate the curl of the time-ordered products of the fermion's three point functions involving the vector, axial-vector, and tensor current operators, namely $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, $j^{\mu\nu}(x) = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$, and $j_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$, respectively. Then the transverse WTI for fermion's vertex functions can be obtained by the curl of the T products of the corresponding fermion's three-point function

$$\partial_\rho^x \langle 0|T V^{\rho\mu\nu(\lambda)}(x)\psi(y)\bar{\psi}(z)|0\rangle$$
$$= \partial_x^\mu \langle 0|T j^{\nu(\lambda)}(x)\psi(y)\bar{\psi}(z)|0\rangle - \partial_x^\nu \langle 0|T j^{\mu(\lambda)}(x)\psi(y)\bar{\psi}(z)|0\rangle. \quad (3)$$

For the convenience of the discussion in N-dimensional gauge theory, we only use the relations of gamma matrices that do not depend on the space–time dimensions and doe not introduce the γ_5 matrix, which can be expressed as follows

$$\{\gamma^\mu,\gamma^\nu\}_N = 2g^{\mu\nu}, \quad \frac{i}{2}[\gamma^\mu,\gamma^\nu]_N = \sigma^{\mu\nu},$$
$$\frac{1}{2}[\gamma^\rho,\sigma^{\mu\nu}]_N = i[g^{\rho\mu}\gamma^\nu - g^{\rho\nu}\gamma^\mu]. \quad (4)$$

There are two ways to compute the curl of the time-ordered products of the above three-point functions; one is the canonical quantization method, and the other is the path integration method. The derivation is provided in the Appendix. Through the canonical quantization and path integration method, we arrive at the transverse WT relations for the fermion's vertex functions in N-dimensional gauge theory in the configuration space,

$$\partial^\mu \langle 0|T j^\nu(x)\psi(y)\bar{\psi}(z)|0\rangle - \partial^\nu \langle 0|T j^\mu(x)\psi(y)\bar{\psi}(z)|0\rangle$$
$$= \lim_{x'\to x}(\partial_\rho^{x'} - \partial_\rho^x)\langle 0|T\bar{\psi}(x')\frac{i}{2}\{\gamma^\rho,\sigma^{\mu\nu}\}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle$$
$$+ i\sigma^{\mu\nu}\delta^4(x-y)\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle$$
$$+ i\langle 0|T\psi(y)\bar{\psi}(x)0\rangle\sigma^{\mu\nu}\delta^4(x-z)$$
$$+ 2m\langle 0|T\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)\psi(y)\bar{\psi}(z)|0\rangle \quad (5)$$

and

$$\partial^\mu \langle 0|T j^{\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle - \partial^\nu \langle 0|T j^{\mu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle$$
$$= -\frac{1}{2}\{\sigma^{\mu\nu},\gamma^\lambda\}\delta^4(x-y)\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle$$
$$+ \langle 0|T\psi(y)\bar{\psi}(x)|0\rangle\frac{1}{2}\{\sigma^{\mu\nu},\gamma^\lambda\}\delta^4(x-z)$$
$$- (\partial_\rho^{x'} - \partial_\rho^x)\langle 0|T\bar{\psi}(x')\frac{1}{4}\left[\gamma^\rho,\left\{\sigma^{\mu\nu},\gamma^\lambda\right\}\right]U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle$$
$$- (\partial^{\lambda(x')} + \partial^{\lambda(x)})\langle 0|T\bar{\psi}(x')\sigma^{\mu\nu}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle. \quad (6)$$

2.2 Anomaly

The symmetry of classical theory may be destroyed by quantum anomaly, and there is a corresponding anomalous WT identity [27, 28]; this must be considered in advance when studying the full vertex functions. In four-dimensional gauge theories, by using the perturbative method and Pauli-Villars regularization and dimensional regularization, Sun, *et al.*, [29] found that there is no transverse anomaly term for both the axial-vector and vector current. The absence of transverse anomalies for both the axial-vector current in QED_2 theory and vector (tensor) current in QED_3 theory are also verified [30]. Thus, in the case of transverse WT identity, one does not need to discuss the problem of transverse quantum anomalies. However, the quantum anomaly of longitudinal WTI draws our attention.

2.3 Representation and full vertices functions

In the above, we established the relationships of transverse WTI (5, 6) using only matrix relations (4), which is suitable for $N-$dimensional time-space. As we will see shortly, the representation of symmetrized part $\{\gamma^\rho,\sigma^{\mu\nu}\}$ depends on the space–time dimensions. In $3+1$ dimensions time-space, substituting $\{\gamma^\rho,\sigma^{\mu\nu}\}=-2\epsilon^{\rho\mu\nu\lambda}\gamma_\lambda\gamma_5$ into transverse WTI (5, 6), it is easy to find that our results are exactly the same as those given in Ref. [7]. This can be seen as a self examination of the transverse WTI (5, 6).

Here, we turn to consider the $2+1$ dimensional case, and we choose the following gamma matrices

$$\gamma^0=\sigma^3,\quad \gamma^1=i\sigma^1,\\ \gamma^2=i\sigma^2,\quad \frac{1}{2}\{\gamma^\rho,\sigma^{\mu\nu}\}=\epsilon^{\rho\mu\nu},\tag{7}$$

where σ^i denotes the Pauli matrix. In this case, we do not have the freedom to construct additional gamma matrices that anti-commute with all γ^μ in the 2×2 representation. This means that the flavor symmetry of fermions is the same whether, they have mass or not.

Substituting relations (7) into Eqs. (5), (6), the transverse Ward-Takahashi identity for the vector and the tensor vertex can be written in momentum space by introducing the standard definition for the three-point function,

$$\begin{aligned}&q^\mu\Gamma_V^\nu(p_1,p_2)-q^\nu\Gamma_V^\mu(p_1,p_2)\\ =&-iS_F^{-1}(p_1)\sigma^{\mu\nu}-i\sigma^{\mu\nu}S_F^{-1}(p_2)\\ &+i\epsilon^{\rho\mu\nu}(p_{1\rho}+p_{2\rho})\Gamma_S(p_1,p_2)\\ &-2im\Gamma_T^{\mu\nu}(p_1,p_2)-i\int\frac{d^3k}{(2\pi)^3}2k_\rho\epsilon^{\rho\mu\nu}\Gamma_S(p_1,p_2,k)\end{aligned}\tag{8}$$

and

$$\begin{aligned}&q^\mu\Gamma_T^{\nu\lambda}(p_1,p_2)-q^\nu\Gamma_T^{\mu\lambda}(p_1,p_2)+q^\lambda\Gamma_T^{\mu\nu}(p_1,p_2)\\ =&\epsilon^{\mu\nu\lambda}S_F^{-1}(p_1)-\epsilon^{\mu\nu\lambda}S_F^{-1}(p_2),\end{aligned}\tag{9}$$

where $\Gamma_S,\Gamma_V^\mu,\Gamma_T^{\mu\nu}$ are the scalar, vector, and tensor vertex functions, respectively, and $q=(p_1-k)-(p_2-k)$. The last term in Eq. (8) is called the integral-term, involving the vertex function $\Gamma_S(p_1,p_2;k)$ with the internal momentum k of the gauge boson appearing in the Wilson line, which is defined by the Fourier transformation

$$\begin{aligned}&\int d^3x d^3x' d^3x_1 d^3x_2\langle 0|T\bar\psi(x)\psi(x)\bar\psi(x_1)\psi(x_2)U(x',x)|0\rangle\cdot\\ &e^{i(p_1x-p_2x_2-(p_1-k)x'+(p_2-k)x)}\\ =&(2\pi)^3\delta^3(p_1-p_2-q)iS_F(p_1)\Gamma_S(p_1,p_2,k)iS_F(p_2).\end{aligned}$$

The integral-term to one-loop order in four dimensional gauge theory has been calculated in Ref. [11].

Noting that if one chooses the basic fermion field to be a four component spinor, the three 4×4 γ matrices can be assumed as

$$\gamma^0=\begin{bmatrix}\sigma_3&0\\0&-\sigma_3\end{bmatrix},\gamma^1=i\begin{bmatrix}\sigma_1&0\\0&-\sigma_1\end{bmatrix},\gamma^2=i\begin{bmatrix}\sigma_2&0\\0&-\sigma_2\end{bmatrix},\tag{10}$$

where we can define a 4×4 matrices γ^5 that anti-commute with all γ^μ

$$\gamma^3=i\begin{bmatrix}0&I\\I&0\end{bmatrix},\gamma^5=i\begin{bmatrix}0&I\\-I&0\end{bmatrix}.\tag{11}$$

This is different from the 2×2 representation, where there is no γ_5 matrix and dynamical chiral symmetry breaking. Because such differences in symmetry are expected to equally manifest in the vertices as well as the propagators, it can be expected that the vertices cannot be equal in these different representations.

At this point, we have

$$\frac{1}{2}\{\gamma^\rho,\sigma^{\mu\nu}\}={}^{\rho\mu\nu}\gamma_M,\quad \gamma_M=\begin{bmatrix}I&0\\0&-I\end{bmatrix},\quad [\gamma_M,\gamma^\rho]=0.\tag{12}$$

Thus in this case, through similar derivation steps, the above relation Eqs. (8), (9) will be modified as follows:

$$\begin{aligned}&q^\mu\Gamma_V^\nu(p_1,p_2)-q^\nu\Gamma_V^\mu(p_1,p_2)\\ =&i\epsilon^{\rho\mu\nu}(p_{1\rho}+p_{2\rho})\Gamma_M(p_1,p_2)-i\int\frac{d^3k}{(2\pi)^3}2k_\rho\epsilon^{\rho\mu\nu}\Gamma_M(p_1,p_2,k)\\ &-iS_F^{-1}(p_1)\sigma^{\mu\nu}-i\sigma^{\mu\nu}S_F^{-1}(p_2)-2im\Gamma_T^{\mu\nu}(p_1,p_2)\end{aligned}\tag{13}$$

and

$$\begin{aligned}&q^\mu\Gamma_T^{\nu\lambda}(p_1,p_2)-q^\nu\Gamma_T^{\mu\lambda}(p_1,p_2)+q^\lambda\Gamma_T^{\mu\nu}(p_1,p_2)\\ =&\epsilon^{\mu\nu\lambda}S_F^{-1}(p_1)\gamma_M-\epsilon^{\mu\nu\lambda}\gamma_M S_F^{-1}(p_2),\end{aligned}\tag{14}$$

where Γ_M denotes the vertex function $\langle 0|T\bar\psi(x)\gamma_M\psi(x)\psi(y)\bar\psi(z)|0\rangle$ in momentum space. Comparing Eqs. (13), (14), with Eqs. (8), (9), it is found that the full vertex function does depend on the different γ matrix representation we use.

The above Eqs. (9), (8) show that the transverse part of the vertex function is related to the inverse of the fermion propagator and other vertex functions, namely the full vertex functions are coupled to each other and form a set of coupled equations. For instance, Eq. (8) shows that the transverse part of the vector vertex function is related to the fermion propagator, the tensor and scalar vertex functions. Noting that the transverse WT relation for the tensor vertex functions in four-dimensions has a psudoscalar vertex functions term [7], which is different from the case of three-dimensions (see Eqs. (9) and (14)). The reason why the result in three-dimensions is quite different from that in four-dimensions is due to following facts:

(a) γ matrices: The γ matrices representation in QED_3 is different from that in QED_4, and the commutative relations of γ matrices are also different (see the Eq. (A11), where $\left[\gamma^\rho,\left\{\sigma^{\mu\nu},\gamma^\lambda\right\}\right]=0$ in QED_3, but not in QED_4), which leads to the transverse Ward-Takahashi identities in QED_3 for vertex functions to be simpler;

(b) Integral-term involving the vertex function: In QED_4, the transverse Ward-Takahashi identities for the vector vertex function contain the integral-term involving the axial-vector vertex function, but the axial-vector vertex function cannot be expressed by the two-point Green's function. However, in QED_3, the vector vertex function contains the integral-term involving the scalar vertex function $q_\nu\int\frac{\mathrm{d}^3k}{(2\pi)^3}2k_\rho\epsilon^{\rho\mu\nu}\Gamma_S(p_1,p_2,k)$, while the scalar vertex function Γ_S can be expressed by two-point Green's function (refering to Eq. (18)), due to the antisymmetry of $\epsilon^{\rho\mu\nu}$ and $\sigma^{\mu\nu}$.

For the above reasons, we do not need to make any approximation in the current study to obtain completely closed DSEs in QED_3, which is the largest difference between our present and past studies. Now, we begin to derive the complete expression of the vertex function.

The well-known normal Ward-Takahashi identities

$$\begin{aligned}q_\mu\Gamma^\mu_V(p_1,p_2)=&S_F^{-1}(p_1)-S_F^{-1}(p_2)\\ iq_\mu\Gamma^{\nu\mu}_T(p_1,p_2)=&S_F^{-1}(p_1)\gamma^\nu+\gamma^\nu S_F^{-1}(p_2)\\ &+2m\Gamma^\nu_V(p_1,p_2)+(p_1^\nu+p_2^\nu)\Gamma_S(p_1,p_2),\end{aligned}\tag{15}$$

denote the longitudinal part of the three-point vertex function, which along with the transverse WT relation form a complete set of WT-type constraint relations for the fermion's three-point vertex functions in QED_3 theories. Then, by this complete set of constraint relations, one can obtain complete solutions for these vector and tensor vertex functions.

Evidently, in four-dimensions space-time, it is extremely difficult to consider the full contributions of the above three Wilson integral-terms in Eqs. (5), (6). To obtain a set of closed DSEs, He [7, 8] first ignored the contribution of the integral-term involving the vertex functions and observed what follows. However, in three-dimensions space-time, we find that the full vertex function has a very simple expression (no need to ignore the integral term), which can be expressed in terms of the fermion propagators. Using Eqs. (9), (8) and normal Ward-Takahashi identities Eq. (15), the complete expression in 2×2 representation for the vector vertex can be obtained as follows

$$\begin{aligned}\Gamma^\mu_V(p_1,p_2)=&\frac{1}{(q^2-4m^2)}\Big\{q^\mu\left[S_F^{-1}(p_1)-S_F^{-1}(p_2)\right]\\ &+iq_\nu\left[S_F^{-1}(p_1)\sigma^{\mu\nu}+\sigma^{\mu\nu}S_F^{-1}(p_2)\right]\\ &+2m\left[S_F^{-1}(p_1)\gamma^\mu+\gamma^\mu S_F^{-1}(p_2)\right]\\ &+\left[2m(p_1^\mu+p_2^\mu)-i\epsilon^{\rho\mu\nu}q_\nu(p_{1\rho}+p_{2\rho})\right]\Gamma_S(p_1,p_2)\\ &+i\int\frac{\mathrm{d}^3k}{(2\pi)^3}2k_\rho q_\nu\epsilon^{\rho\mu\nu}\Gamma_S(p_1,p_2,k)\Big\}.\end{aligned}\tag{16}$$

Herein, the tensor vertex function is

$$\begin{aligned}q^2\Gamma^{\mu\nu}_T(p_1,p_2)=&i\Big\{S_F^{-1}(p_1)(q^\mu\gamma^\nu-q^\nu\gamma^\mu-iq_\lambda\epsilon^{\mu\nu\lambda})\\ &+(q^\mu\gamma^\nu-q^\nu\gamma^\mu+iq_\lambda\epsilon^{\mu\nu\lambda})S_F^{-1}(p_2)\\ &+2m[q^\mu\Gamma^\nu_V(p_1,p_2)-q^\nu\Gamma^\mu_V(p_1,p_2)]\\ &+[q^\mu(p_1^\nu+p_2^\nu)-q^\nu(p_1^\mu+p_2^\mu)]\Gamma_S(p_1,p_2)\Big\},\end{aligned}\tag{17}$$

where

$$\begin{aligned}q_\mu(p_1^\mu+p_2^\mu)\Gamma_S(p_1,p_2)=&-2m\left[S_F^{-1}(p_1)-S_F^{-1}(p_2)\right]\\ &-\left[S_F^{-1}(p_1)\gamma^\mu q_\mu+\gamma^\mu q_\mu S_F^{-1}(p_2)\right].\end{aligned}\tag{18}$$

It is highly interesting here to examine the possible kinematic singularities that the dressed vertex function Eqs. (16)–(19) may have. In the case of the chiral limit, we have

$$\Gamma_S(p_1,p_2)=\frac{-1}{(p_1^\mu+p_2^\mu)}\times\left[S_F^{-1}(p_1)\gamma^\mu+\gamma^\mu S_F^{-1}(p_2)\right],\tag{19}$$

there is no singularity for $\Gamma_S(p_1,p_2)$ [in the limit $q_\mu\to0$(which requires both $q^2\to0$ and $p_1^2\to p_2^2$ in Minkowski metric, it is also a limit in Euclidean space) and the limit $q^2\to0$ for $p_1^2\neq p_2^2$ in Minkowski metric]. In the case of the non-Chiral limit, there is also no singularity for $\Gamma_S(p_1,p_2)$ in the limit $q^2\to0$ for $p_1^2\neq p_2^2$ in Minkowski metric. Thus, the vector vertex function Γ^μ_V does not suffer from singularities in the limit $q^2\to0$ for $p_1^2\neq p_2^2$, due to $q_\mu\neq0$. However, the tensor vertex function $\Gamma^{\mu\nu}_T(p_1,p_2)$ always has singularity. This singularity is worthy of careful consideration.

The full vertex functions depend on the different γ matrix representation we use. If we replace Γ_S with Γ_M, in the above equation (16) we obtain the vector vertex function in 4×4 representation. Similarly, by replacing $\epsilon^{\mu\nu\lambda}$ with $\epsilon^{\mu\nu\lambda}\gamma_M$ in the equation (17), subsequently we obtain the tensor vertex function in 4×4 representation. The full vector and tensor vertex functions in three-dimensional space-time can be expressed in terms of fermion propagators only, which is different from vertex functions in the four-dimensional space-time (in four-dimensions gauge theory, only in the chiral limit, Γ^μ_V and Γ^μ_A at tree level are expressed in terms of the fermion propagators). On the basis of the above, at this, point the closedness of the DSE can be established.

2.4 Two-point Green's function in QED_3

Let us now discuss two important two-point Green's functions in QED_3, namely, the photon propagator and the fermion propagator. The photon propagator can be written as

$$iD^{-1}_{\mu\nu}(q) = -q^2\left[g_{\mu\nu} + \left(\frac{1}{\lambda} - 1\right)\frac{q_\mu q_\nu}{q^2}\right] + \Pi^{\mu\nu}(q), \tag{20}$$

where $q=p_1-p_2$ and $\Pi^{\mu\nu}(q)$ is the photon polarization vector

$$\Pi^{\mu\nu}(q) = \frac{iN_f e^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 Tr_D[\gamma^\mu S_F(p_1)\Gamma^\nu(p_1,p_2)S_F(p_2)], \tag{21}$$

where the full fermion propagator $S^{-1}(p) = \gamma\cdot pA(p^2)+B(p^2)$. As mentioned above, the vector vertex function Γ^μ_V does not suffer from singularities in the limit $q^2\to 0$ for $p_1^2 \neq p_2^2$ and $q_\mu \neq 0$ in Minkowski metric. In this case, substituting the vector vertex function Γ^μ_V (16) into the photon polarization vector (21), then the photon polarization vector $\Pi^{\mu\nu}(q)$ is obtained, as shown in Eq. (B1) of the Appendix B.

The DSE for the fermion propagator of QED_3 in momentum space,

$$S_F^{-1}(p_2) = \not{p}_2 - m - \frac{ie^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 \gamma^\mu S(p_1) \times\Gamma^\nu(p_1,p_2;(p_1-p_2))D_{\nu\mu}(p_1-p_2). \tag{22}$$

Finally, substituting Eqs. (20)–(21) into Eq. (22), we obtain the closed DSE for the fermion propagator in QED_3,

$$\begin{aligned}
2B(p_2^2) =& TrS_F^{-1}(p_2) = Tr(\not{p}_2 - m) - \frac{ie^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 Tr[\gamma^\mu S(p_1)\Gamma^\nu(p_1,p_2;(p_1-p_2))]D_{\nu\mu}(p_1-p_2).\\
=& -2m - \frac{ie^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 \frac{D_{\nu\mu}(p_1-p_2)}{(q^2-4m^2)[p_1^2A^2(p_1^2)-B^2(p_1^2)]}\Big\{-q^\nu[-2i\epsilon^{\mu\rho\sigma}p_{1\rho}p_{2\sigma}A(p_1^2)A(p_2^2)+2p_1^\mu A(p_1^2)B(p_2^2)\\
&-2p_2^\mu B(p_1^2)A(p_2^2)] + iq_\lambda\Big(2\epsilon^{\mu\nu\lambda}[p_1^2A^2(p_1^2)-B^2(p_1^2)] + [ip_{1\rho}p_{2\sigma}M^{\mu\rho\nu\lambda\sigma}(p_1,p_2)A(p_1^2)A(p_2^2)\\
&+4ip_{1\rho}(-g^{\mu\nu}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\nu})A(p_1^2)B(p_2^2) - i4p_{2\rho}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\lambda}g^{\nu\rho})B(p_1^2)A(p_2^2) - 2\epsilon^{\mu\nu\lambda}B(p_1^2)B(p_2^2)]\Big)\\
&+2m\Big(2g^{\mu\nu}[p_1^2A^2(p_1^2)-B^2(p_1^2)] + [4p_{1\rho}p_{2\sigma}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\nu})A(p_1^2)A(p_2^2) - 2i\epsilon^{\mu\lambda\nu}p_{1\lambda}A(p_1^2)B(p_2^2)\\
&+2i\epsilon^{\mu\nu\lambda}p_{2\lambda}B(p_1^2)A(p_2^2) - 2g^{\mu\nu}B(p_1^2)B(p_2^2)]\Big) + \frac{[2m(p_1^\nu+p_2^\nu)-i\epsilon^{\nu\lambda\rho}q_\lambda(p_{1\rho}+p_{2\rho})]}{-q_\mu(p_1^\mu+p_2^\mu)}\Big[q_\tau\Big(2g^{\mu\tau}[p_1^2A^2(p_1^2)-B^2(p_1^2)]\\
&+[4p_{1\rho}p_{2\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})A(p_1^2)A(p_2^2) - 2i\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)B(p_2^2) + 2i\epsilon^{\mu\tau\rho}p_{2\rho}B(p_1^2)A(p_2^2) - 2g^{\mu\tau}B(p_1^2)B(p_2^2)]\Big)\\
&-2m[-2i\epsilon^{\mu\rho\sigma}p_{1\rho}p_{2\sigma}A(p_1^2)A(p_2^2)+2p_1^\mu A(p_1^2)B(p_2^2)-2p_2^\mu B(p_1^2)A(p_2^2)]\Big]\\
&+i\int\frac{\mathrm{d}^3k}{(2\pi)^3}2k_\rho q_\lambda\epsilon^{\rho\nu\lambda}\frac{1}{-q_\mu(p_1^\mu+p_2^\mu-2k^\mu)}\Big(q_\tau[4p_{1\rho}p_{3\sigma}(g^{\mu\rho}g^{\sigma\tau}-g^{\mu\sigma}g^{\rho\tau}+g^{\mu\tau}g^{\rho\sigma})A(p_1^2)A(p_3^2)\\
&+4p_{1\rho}p_{4\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})A(p_1^2)A(p_4^2) - i2\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)B(p_3^2) + i2\epsilon^{\mu\rho\tau}p_{3\rho}B(p_1^2)A(p_3^2) - 2g^{\mu\tau}B(p_1^2)B(p_3^2)\\
&-2i\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)B(p_4^2) + 2i\epsilon^{\mu\tau\rho}p_{4\rho}B(p_1^2)A(p_4^2) - 2g^{\mu\tau}B(p_1^2)B(p_4^2)] + 2m[-i2\epsilon^{\mu\rho\sigma}p_{1\rho}p_{3\sigma}A(p_1^2)A(p_3^2) + 2p_1^\mu A(p_1^2)B(p_3^2)\\
&-2p_3^\mu B(p_1^2)A(p_3^2) + 2i\epsilon^{\mu\rho\sigma}p_{1\rho}p_{4\sigma}A(p_1^2)A(p_4^2) - 2p_1^\mu A(p_1^2)B(p_4^2) + 2p_4^\mu B(p_1^2)A(p_4^2)]\Big)\Big\},
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
2p_2^\pi A(p_2^2) =& Tr\gamma^\pi S_F^{-1}(p_2) = Tr(\gamma^\pi \not{p}_2 A(p_2^2) + \gamma^\pi B(p_2^2)) = 2p_2^\pi - \frac{ie^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 \gamma^\pi\gamma^\mu S(p_1)\Gamma^\nu(p_1,p_2;(p_1-p_2))D_{\nu\mu}(p_1-p_2)\\
=& 2p_2^\pi - \frac{ie^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 \frac{D_{\nu\mu}(p_1-p_2)}{(q^2-4m^2)[p_1^2A^2(p_1^2)-B^2(p_1^2)]}\Big\{q^\nu\Big[2g^{\pi\mu}[p_1^2A^2(p_1^2)-B^2(p_1^2)] - [4(g^{\pi\mu}p_1p_2 - p_1^\pi p_2^\mu + p_2^\pi p_1^\mu)A(p_1^2)A(p_2^2)
\end{aligned}$$

$$-2i\epsilon^{\pi\mu\rho}p_{1\rho}A(p_1^2)B(p_2^2)+2i\epsilon^{\pi\mu\rho}p_{2\rho}B(p_1^2)A(p_2^2)-2g^{\pi\mu}B(p_1^2)B(p_2^2)]\Big]+iq_\lambda\Big[4i(-g^{\pi\nu}g^{\mu\lambda}+g^{\pi\lambda}g^{\mu\nu})[p_1^2A^2(p_1^2)-B^2(p_1^2)]$$
$$+[i4p_{1\rho}p_{2\sigma}E^{\pi\mu\rho\nu\lambda\sigma}A(p_1^2)A(p_2^2)+ip_{1\rho}Q^{\pi\mu\rho\nu\lambda}A(p_1^2)B(p_2^2)-ip_{2\rho}M^{\pi\mu\nu\lambda\rho}B(p_1^2)A(p_2^2)-4i(-g^{\pi\nu}g^{\mu\lambda}+g^{\pi\lambda}g^{\mu\nu})B(p_1^2)B(p_2^2)]\Big]$$
$$+2m\Big[-2i\epsilon^{\pi\mu\nu}[p_1^2A^2(p_1^2)-B^2(p_1^2)]+[p_{1\rho}p_{2\sigma}M^{\pi\mu\rho\nu\sigma}A(p_1^2)A(p_2^2)+4p_{1\rho}(g^{\pi\mu}g^{\rho\nu}-g^{\pi\rho}g^{\mu\nu}+g^{\pi\nu}g^{\mu\rho})A(p_1^2)B(p_2^2)$$
$$-4p_{2\rho}(g^{\pi\mu}g^{\nu\rho}-g^{\pi\nu}g^{\mu\rho}+g^{\pi\rho}g^{\mu\nu})B(p_1^2)A(p_2^2)+2i\epsilon^{\pi\mu\nu}B(p_1^2)B(p_2^2)]\Big]$$
$$+\frac{[2m(p_1^\nu+p_2^\nu)-i\epsilon^{\nu\lambda\rho}q_\lambda(p_{1\rho}+p_{2\rho})]}{-q_\nu(p_1^\nu+p_2^\nu)}\Big[q_\nu\Big(-2i\epsilon^{\pi\mu\nu}[p_1^2A^2(p_1^2)-B^2(p_1^2)]+[p_{1\rho}p_{2\sigma}M^{\pi\mu\rho\nu\sigma}A(p_1^2)A(p_2^2)$$
$$+4p_{1\rho}(g^{\pi\mu}g^{\rho\nu}-g^{\pi\rho}g^{\mu\nu}+g^{\pi\nu}g^{\mu\rho})A(p_1^2)B(p_2^2)-4p_{2\rho}(g^{\pi\mu}g^{\nu\rho}-g^{\pi\nu}g^{\mu\rho}+g^{\pi\rho}g^{\mu\nu})B(p_1^2)A(p_2^2)+2i\epsilon^{\pi\mu\nu}B(p_1^2)B(p_2^2)]\Big)$$
$$+2m\Big(2g^{\pi\mu}[p_1^2A^2(p_1^2)-B^2(p_1^2)]-[4(g^{\pi\mu}p_1p_2-p_1^\pi p_2^\mu+p_2^\pi p_1^\mu)A(p_1^2)A(p_2^2)-2i\epsilon^{\pi\mu\rho}p_{1\rho}A(p_1^2)B(p_2^2)$$
$$+2i\epsilon^{\pi\mu\rho}p_{2\rho}B(p_1^2)A(p_2^2)-2g^{\pi\mu}B(p_1^2)B(p_2^2)]\Big)\Big]+i\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{2k_\alpha q_\beta\epsilon^{\alpha\nu\beta}}{-q_\nu(p_1^\nu+p_2^\nu-2k^\nu)}\Big[q_\nu\Big([p_{1\rho}p_{3\sigma}M^{\pi\mu\rho\sigma\nu}A(p_1^2)A(p_3^2)$$
$$+4p_{1\rho}(g^{\pi\mu}g^{\rho\nu}-g^{\pi\rho}g^{\mu\nu}+g^{\pi\nu}g^{\mu\rho})A(p_1^2)B(p_3^2)-4p_{3\rho}(g^{\pi\mu}g^{\rho\nu}-g^{\pi\rho}g^{\mu\nu}+g^{\pi\nu}g^{\mu\rho})B(p_1^2)A(p_3^2)+i2\epsilon^{\pi\mu\nu}B(p_1^2)B(p_3^2)]$$
$$+[p_{1\rho}p_{4\sigma}M^{\pi\mu\rho\nu\sigma}A(p_1^2)A(p_4^2)+4p_{1\rho}(g^{\pi\mu}g^{\rho\nu}-g^{\pi\rho}g^{\mu\nu}+g^{\pi\nu}g^{\mu\rho})A(p_1^2)B(p_4^2)-4p_{2\rho}(g^{\pi\mu}g^{\nu\rho}-g^{\pi\nu}g^{\mu\rho}+g^{\pi\rho}g^{\mu\nu})B(p_1^2)A(p_4^2)$$
$$+2i\epsilon\pi\mu\nu B(p_1^2)B(p_4^2)]\Big)+2m\Big([4(g^{\pi\mu}p_1p_3-p_1^\pi p_3^\mu+p_3^\pi p_1^\mu)A(p_1^2)A(p_3^2)-2i\epsilon^{\pi\mu\rho}p_{1\rho}A(p_1^2)B(p_3^2)+2i\epsilon^{\pi\mu\rho}p_{3\rho}B(p_1^2)A(p_3^2)$$
$$-2g^{\pi\mu}B(p_1^2)B(p_3^2)]-[4(g^{\pi\mu}p_1p_4-p_1^\pi p_4^\mu+p_4^\pi p_1^\mu)A(p_1^2)A(p_4^2)-2i\epsilon^{\pi\mu\rho}p_{1\rho}A(p_1^2)B(p_4^2)$$
$$+2i\epsilon^{\pi\mu\rho}p_{4\rho}B(p_1^2)A(p_4^2)-2g^{\pi\mu}B(p_1^2)B(p_4^2)]\Big)\Big]\Big\},\tag{24}$$

where

$$E^{\pi\mu\rho\nu\lambda\sigma}=\frac{1}{8}[Tr(\gamma^\pi\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\sigma)-Tr(\gamma^\pi\gamma^\mu\gamma^\rho\gamma^\lambda\gamma^\nu\gamma^\sigma)]=g^{\pi\mu}(g^{\rho\nu}g^{\lambda\sigma}-g^{\rho\lambda}g^{\nu\sigma})-g^{\pi\rho}(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\lambda}g^{\nu\sigma})$$
$$+g^{\pi\nu}(g^{\mu\rho}g^{\lambda\sigma}-g^{\mu\lambda}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\lambda})-g^{\pi\lambda}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\nu})$$
$$+g^{\pi\sigma}(-g^{\mu\nu}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\nu}),$$
$$Q^{\pi\mu\rho\nu\lambda}=\frac{1}{2}[Tr(\gamma^\pi\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\lambda)-Tr(\gamma^\pi\gamma^\mu\gamma^\rho\gamma^\lambda\gamma^\nu)]=\frac{1}{2}(M^{\pi\mu\rho\nu\lambda}-M^{\pi\mu\rho\lambda\nu})=-2ig^{\pi\mu}\epsilon^{\rho\nu\lambda}+2ig^{\pi\rho}\epsilon^{\mu\nu\lambda}-ig^{\pi\nu}\epsilon^{\mu\rho\lambda}$$
$$+i2\epsilon^{\pi\lambda m}(g^{\mu\rho}g^\nu_m-g^{\mu\nu}g^\rho_m+g^\mu_m g^{\rho\nu})+ig^{\pi\lambda}\epsilon^{\mu\rho\nu}-i2\epsilon^{\pi\nu m}(g^{\mu\rho}g^\lambda_m-g^{\mu\lambda}g^\rho_m+g^\mu_m g^{\rho\lambda}),$$
$$\frac{1}{2}(M^{\pi\mu\rho\nu\lambda}-M^{\pi\mu\nu\rho\lambda})=M^{\pi\mu\rho\nu\lambda}.\tag{25}$$

Based on the coupled equations satisfied by $A(p^2)$ and $B(p^2)$ above, in principle we can strictly solve the complete fermion propagator by the numerical iterative method, after which the chiral symmetry spontaneous breaking and confinement characteristics of QED_3 can be analytically analyzed. However, importantly, the coupled integral equations Eqs. (23), (24) are extremely complex, which is a significant challenge for the rigorous numerical solutions. We will address this problem in future work.

3 Summary and conclusion

To summarize, we first derive the transverse WTI of N-dimensional gauge theory by means of the canonical quantization method and the path integration method, and subsequently, using the characteristics of the γ matrix representation in three-dimensional gauge theory, it is shown that the normal (longitudinal) WTI together with the transverse WTI form a complete set of Ward-Takahashi type constraint relations for the fermion-boson vertex functions in QED_3 theory. By solving this complete set, the full scalar, vector, and tensor vertex functions $(\Gamma_S,\Gamma_V^\mu,\Gamma_T^{\mu\nu})$ can be expressed in terms of the fermion's two-point functions, which is completely different from the situation in four-dimensions gauge theory (where only in the chiral limit, Γ_V^μ and Γ_A^μ at tree level are expressed in terms of the fermion propagators). It is found that the full tensor vertex function in 4×4 representation is different from that in 2×2 representation. This means that when we study the dynamic behavior of three-dimensional gauge theory related to the tensor vertex function, we must specify the γ matrix representation in advance. Furthermore, we examine the possible kinematic singularities that the dressed vertex function may have, and find that the vector vertex function Γ_V^μ does not suffer from singularities in the limit $q^2 \to 0$ for $p_1^2 \neq p_2^2, q_\mu \neq 0$ in the Minkowski metric. Then, substituting the vector vertex function into the DSEs for the fermion and photon propagators, finally we obtain the closed DSE in QED_3. Based on this set of closed coupled nonlinear integral equations, in principle we can numerically solve the two-point Green functions and three-point Green functions by a numerical iteration method to analyze the mechanism of the chiral symmetry spontaneous breaking and confinement in QED_3.

Finally, we need to emphasize that low-dimensional gauge theory has a very wide range of applications in condensed matter physics. In particular, QED_3 was suggested to be the effective low-energy field theory for the anomalous normal state of high-T_c cuprate superconductors [31–33]. It also provides a promising field-theoretic description for such exotic quantum many-body state as $U(1)$ quantum spin liquid [34]. When massless Dirac fermions are coupled to $U(1)$ gauge boson, they acquire a finite anomalous dimension due to the strong gauge interaction [31–33, 35]. This may lead to intriguing Luttinger-like behaviors, which has been used to understand the absence of well-defined quasiparticle peaks in the normal state of high-T_c cuprate superconductors [31-33, 35]. To reveal the nature of these Luttinger-like behaviors, one needs to compute certain types of Green's function very carefully. The gauge invariance must be preserved during the analytical calculations [36–40]. In principle, these Green's functions can be self-consistently obtained by solving a close set of DSEs. We expect that the generic WTI obtained in this work would be utilized to calculate the gauge invariant Green's functions by means of DSEs.

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Appendix A

First, we introduce two bilinear covariant current operators,

$$V^{\rho\mu\nu\lambda}(x)=\frac{1}{4}\bar{\psi}(x)\Big[[\gamma^\rho,\sigma^{\mu\nu}],\gamma^\lambda\Big]\psi(x)=g^{\rho\mu}j^{\nu\lambda}(x)-g^{\rho\nu}j^{\mu\lambda}(x),$$
$$V^{\rho\mu\nu}(x)=\frac{-i}{2}\bar{\psi}[\gamma^\rho,\sigma^{\mu\nu}]\psi=g^{\rho\mu}j^\nu(x)-g^{\rho\nu}j^\mu(x). \tag{A1}$$

In the canonical quantization method, we note here the general identity [3]

$$\begin{aligned}&\partial_\lambda^x\langle 0|TV^{\lambda\mu\nu(\alpha)}(x)\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)|0\rangle\\ =&\sum_{i=1}^n\delta_{\lambda 0}\langle 0|T\Big\{V^{\lambda\mu\nu(\alpha)}(x),\psi(x_i)]\delta(x^0-x_i^0)\bar{\psi}(y_i)\\ &+\psi(x_i)[V^{\lambda\mu\nu(\alpha)}(x),\bar{\psi}(y_i)]\delta(x^0-y_i^0)\Big\}\\ &\times\psi(x_1)\bar{\psi}(y_1)\ldots\ulcorner\psi(x_i)\bar{\psi}(y_i)\lrcorner\ldots\psi(x_n)\bar{\psi}(y_n)|0\rangle\\ &+\langle 0|T\partial_\lambda^x V^{\lambda\mu\nu(\alpha)}(x)\psi(x_1)\bar{\psi}(y_1)\ldots\psi(x_n)\bar{\psi}(y_n)|0\rangle,\end{aligned} \tag{A2}$$

where the delimiter $\ulcorner\ \lrcorner$ term above denotes its omission. The last term in above equation leads to a similar situation of $\langle 0|T\bar{\psi}(x)N(\overrightarrow{\partial}_\lambda^x+\overleftarrow{\partial}_\lambda^x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle$, normally, where N is matrix with an anti-communication relation. This means that the transverse WT identity exhibits a different appearance, depending on the dimensionality of space-time, because the anti-communication relation depends on the space-time dimension.

Substituting the relations (A1) into Eqs. (A2), there are

$$\begin{aligned}&\partial_\rho\langle 0|TV^{\rho\mu\nu}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\partial^\mu\langle 0|Tj^\nu(x)\psi(y)\bar{\psi}(z)|0\rangle-\partial^\nu\langle 0|Tj^\mu(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&-\delta^4(x-y)\gamma^0\frac{i}{2}[\sigma^{\mu\nu},\gamma^0]\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &+\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle\frac{i}{2}[\sigma^{\mu\nu},\gamma^0]\gamma^0\delta^3(x-z)\\ &+\langle 0|T\partial_\rho V^{\rho\mu\nu}(x)\psi(y)\bar{\psi}(z)|0\rangle\end{aligned} \tag{A3}$$

and

$$\begin{aligned}&\partial_\rho\langle 0|TV^{\rho\mu\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\partial^\mu\langle 0|Tj^{\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle-\partial^\nu\langle 0|Tj^{\mu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&-\gamma^0\frac{1}{4}\Big[[\gamma^0,\sigma^{\mu\nu}],\gamma^\lambda\Big]\delta^4(x-y)\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &+\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle\frac{1}{4}\Big[[\gamma^0,\sigma^{\mu\nu}],\gamma^\lambda\Big]\gamma^0\delta^4(x-z)\\ &+\langle 0|T\partial_\rho V^{\rho\mu\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle.\end{aligned} \tag{A4}$$

To relate the last term in the above equation to a definite Green's function and to make the equations above more concise, here one needs to consider two conditions. First, the equation of motion for fermions with mass $\bar{\psi}(i\overleftarrow{\not{D}}+m)=0,(i\overrightarrow{\not{D}}-m)\psi=0$s is introduced to make the last term more concise. Thus the term $\gamma^\mu\partial_\mu\psi(x)$ and $\partial_\mu\bar{\psi}(x)\gamma^\mu$ need to be shown in the equations as

$$\begin{aligned}&\langle 0|T\partial_\rho V^{\rho\mu\nu}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\langle 0|i\bar{\psi}(x)\sigma^{\mu\nu}\gamma^\rho\partial_\rho\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &-\langle 0|i\partial_\rho\bar{\psi}(x)\gamma^\rho\sigma^{\mu\nu}\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &+\langle 0|\bar{\psi}(x)\frac{i}{2}\{\sigma^{\mu\nu},\gamma^\rho\}(\overleftarrow{\partial}_\rho-\overrightarrow{\partial}_\rho)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\end{aligned} \tag{A5}$$

and

$$\begin{aligned}&\langle 0|T\partial_\rho V^{\rho\mu\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\frac{1}{4}\langle 0|\partial_\rho[\bar{\psi}(x)(\gamma^\rho\sigma^{\mu\nu}\gamma^\lambda+\sigma^{\mu\nu}\gamma^\lambda\gamma^\rho)\psi(x)]\psi(y)\psi(z)|0\rangle\\ &+\frac{1}{4}\langle 0|\partial_\rho[\bar{\psi}(x)(\gamma^\rho\gamma^\lambda\sigma^{\mu\nu}+\gamma^\lambda\sigma^{\mu\nu}\gamma^\rho)\psi(x)]\psi(y)\bar{\psi}(z)|0\rangle\\ &-\langle 0|\partial_\rho\Big\{\bar{\psi}(x)g^{\rho\lambda}\sigma^{\mu\nu}\psi(x)\Big\}\psi(y)\psi(z)|0\rangle.\end{aligned} \tag{A6}$$

To further simplify the calculations, here we need to use the following relations to the first item of Eq. (A6).

$$\begin{aligned}&\frac{1}{4}\langle 0|\partial_\rho[\bar{\psi}(x)(\gamma^\rho\sigma^{\mu\nu}\gamma^\lambda+\sigma^{\mu\nu}\gamma^\lambda\gamma^\rho)\psi(x)]\psi(y)\psi(z)|0\rangle\\ =&\frac{1}{2}\langle 0|\partial_\rho\bar{\psi}(x)\gamma^\rho\sigma^{\mu\nu}\gamma^\lambda\psi(x)\psi(y)\psi(z)|0\rangle\\ &+\frac{1}{2}\langle 0|\bar{\psi}(x)\sigma^{\mu\nu}\gamma^\lambda\gamma^\rho\partial_\rho\psi(x)\psi(y)\psi(z)|0\rangle\\ &-\langle 0|\bar{\psi}(x)A^{\rho\mu\nu\lambda}(\overleftarrow{\partial}^x_\rho-\overrightarrow{\partial}^x_\rho)\psi(x)\psi(y)\psi(z)|0\rangle,\end{aligned}\tag{A7}$$

where we have defined $\frac{1}{4}[\gamma^\rho,\sigma^{\mu\nu}\gamma^\lambda]=A^{\rho\mu\nu\lambda}$. In a similar procedure, we derive the second item of Eq. (A6), and define $\frac{1}{4}[\gamma^\rho,\gamma^\lambda\sigma^{\mu\nu}]=B^{\rho\lambda\mu\nu}$.

Second, we must move the derivative operators out of the T-product. To this end, we can write the form $\langle 0|T\bar{\psi}(x)N\psi(x)\psi(y)\bar{\psi}(z)|0\rangle$ as $\langle 0|T\bar{\psi}(x')N\psi(x)\psi(y)\bar{\psi}(z)|0\rangle$ and then take $x'\to x$. The above new expression including the nonlocal current is not gauge invariant. It needs to introduce a Wilson line $U(x,x')=P\exp[-ig\int_x^{x'}\mathrm{d}y^\rho A_\rho(y)]$, joining the two space-time points (x,x') to ensure that the current operators are locally gauge invariant. The comprehensive use of the Wilson line, the Eq. (A2) and the equation of motion for fermions, there eventually are two relations

$$\begin{aligned}&(\partial^{x'}_\rho+\partial^x_\rho)\langle 0|T\bar{\psi}(x')M^{\rho\mu\nu\lambda}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\langle 0|T\bar{\psi}(x)M^{\rho\mu\nu\lambda}(\overleftarrow{\partial}^x_\rho+\overrightarrow{\partial}^x_\rho)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &-\delta^4(x-y)\gamma^0M^{0\mu\nu\lambda}\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &+\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle M^{0\mu\nu\lambda}\gamma^0\delta^4(x-z)\end{aligned}\tag{A8}$$

and

$$\begin{aligned}&(\partial^{x'}_\rho-\partial^x_\rho)\langle 0|T\bar{\psi}(x')M^{\rho\mu\nu\lambda}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\langle 0|T\bar{\psi}(x)M^{\rho\mu\nu\lambda}(\overleftarrow{\partial}^x_\rho-\overrightarrow{\partial}^x_\rho)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &-\delta^4(x-y)\gamma^0M^{0\mu\nu\lambda}\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &-\langle 0|T\psi(y)\bar{\psi}(x)|\rangle M^{0\mu\nu\lambda}\gamma^0\delta^4(x-z)0\\ &-2igA_\rho\langle 0|T\bar{\psi}(x)M^{\rho\mu\nu\lambda}\psi(x)\psi(y)\bar{\psi}(z)|0\rangle,\end{aligned}\tag{A9}$$

where $M^{\rho\mu\nu\lambda}$ denotes a matrix.

Taking into account the above equations, substituting relations (A7, A8, A9) into relations (A3, A4, A5, A6), we arrive at the transverse WT relations for the fermion's vertex functions in gauge theories in configuration space

$$\begin{aligned}&\partial^\mu\langle 0|Tj^\nu(x)\psi(y)\bar{\psi}(z)|0\rangle-\partial^\nu\langle 0|Tj^\mu(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\lim_{x'\to x}(\partial^{x'}_\rho-\partial^x_\rho)\langle 0|T\bar{\psi}(x')\frac{i}{2}\{\gamma^\rho,\sigma^{\mu\nu}\}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &+i\sigma^{\mu\nu}\delta^4(x-y)\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &+i\langle 0|T\psi(y)\bar{\psi}(x)0\rangle\sigma^{\mu\nu}\delta^4(x-z)\\ &+2m\langle 0|T\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\end{aligned}\tag{A10}$$

and

$$\begin{aligned}&\partial^\mu\langle 0|Tj^{\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle-\partial^\nu\langle 0|Tj^{\mu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&-\frac{1}{2}\{\sigma^{\mu\nu},\gamma^\lambda\}\delta^4(x-y)\langle 0|T\psi(x)\bar{\psi}(z)|0\rangle\\ &+\langle 0|T\psi(y)\bar{\psi}(x)|0\rangle\frac{1}{2}\{\sigma^{\mu\nu},\gamma^\lambda\}\delta^4(x-z)\\ &-(\partial^{x'}_\rho-\partial^x_\rho)\langle 0|T\bar{\psi}(x')\frac{1}{4}\left[\gamma^\rho,\left\{\sigma^{\mu\nu},\gamma^\lambda\right\}\right]U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle\\ &-(\partial^{\lambda(x')}+\partial^{\lambda(x)})\langle 0|T\bar{\psi}(x')\sigma^{\mu\nu}U(x',x)\psi(x)\psi(y)\bar{\psi}(z)|0\rangle.\end{aligned}\tag{A11}$$

In the path integration method [9], in the Abelian case, there is the identity

$$\begin{aligned}&\langle i\gamma^\mu[\partial_\mu-ieA_\mu(x)]\psi(x)-m\psi(x)+\eta(x)\rangle_J=0\\ &\langle\bar{\psi}(x)i\gamma^\mu[\overleftarrow{\partial}_\mu+ieA_\mu(x)]+m\bar{\psi}(x)-\bar{\eta}(x)\rangle_J=0.\end{aligned}\tag{A12}$$

Hence, one needs only to pay attention to the fermionic part

$$\mathcal{L}_F=\bar{\psi}i\gamma^\mu(\partial_\mu-ieA_\mu)\psi-\bar{\psi}m\psi+\bar{\eta}\psi+\bar{\psi}\eta.\tag{26}$$

If one identifies A_μ in $\mathcal{L}_F$ as $A_\mu=A^\alpha_\mu T^\alpha$ with the generator T^α of the gauge group G, the following relations also hold for the non-Abelian case, irrespective of the gauge part. Then, one can multiply Eq. (A12) by the matrix S from the left (right), where S may be a matrix of spinor, flavors, and colors spaces. Operating the differential operator $\frac{\delta}{\delta\eta(y)}(\frac{\delta}{\delta\bar{\eta}(y)})$ to the resulting equation, an then adding or subtracting, subsequently taking derivatives of both side with respect to $\frac{\delta}{\delta\bar{\eta}(y)}$ and $\frac{\delta}{\delta\eta(z)}$ and setting all the source terms to zero, we obtain the transverse WT identity

$$\begin{aligned}&\partial_\rho\langle\bar{\psi}(x)\frac{i}{2}\{S,\gamma^\rho\}\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ =&-\langle\bar{\psi}(x)\frac{i}{2}[S,\gamma^\rho](\overrightarrow{\partial}_\rho-\overleftarrow{\partial}_\rho)\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &-e\langle\bar{\psi}(x)[S,\gamma^\rho A_\rho]\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &+\langle\bar{\psi}(x)[S,m]\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &+\langle\psi(y)\bar{\psi}(x)\rangle_cS\delta^d(x-z)+S\langle\psi(x)\bar{\psi}(z)\rangle_c\delta^d(x-y)\end{aligned}\tag{A14}$$

and

$$\begin{aligned}&\partial_\rho\langle\bar{\psi}(x)\frac{i}{2}[S,\gamma^\rho]\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ =&-\langle\bar{\psi}(x)\frac{i}{2}\{S,\gamma^\rho\}(\overrightarrow{\partial}_\rho-\overleftarrow{\partial}_\rho)\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &-e\langle\bar{\psi}(x)\{S,\gamma^\rho A_\rho\}\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &+\langle\bar{\psi}(x)\{S,m\}\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &-\langle\psi(y)\bar{\psi}(x)\rangle_cS\delta^d(x-z)-S\langle\psi(x)\bar{\psi}(z)\rangle_c\delta^d(x-y).\end{aligned}\tag{A15}$$

Let $S=S_s\otimes S_f\otimes S_c$ be a direct product of operators within the space of spinor, flavor and color. If choose $S=I_s\otimes I_f\otimes I_c$ is chosen, we obtain the normal WT identity. Fnally, the transverse WT identity for vector current is obtained from Eq. (A14) by choosing $S=\sigma_{\mu\nu}\otimes I_f\otimes I_c$.

From the derivation of the above formula, the transverse WT identity exhibits different appearance depending on the dimensionality of space–time. However, it is not easy to calculate the transverse WT identity for tensor current. To use the above relations (A14, A15), we need to slightly modify the bilinear covariant current operators (A1):

$$\begin{aligned}V^{\rho\mu\nu\lambda}(x)=&\frac{1}{4}\bar{\psi}(x)\left[[\gamma^\rho,\sigma^{\mu\nu}],\gamma^\lambda\right]\psi(x)=g^{\rho\mu}j^{\nu\lambda}(x)-g^{\rho\nu}j^{\mu\lambda}(x)\\ =&\bar{\psi}(x)\frac{1}{4}\left\{\gamma^\rho,\{\sigma^{\mu\nu},\gamma^\lambda\}\right\}\psi(x)-\bar{\psi}(x)g^{\rho\lambda}\sigma^{\mu\nu}\psi(x).\end{aligned}\tag{A16}$$

Through the above relations (A14, A16), the transverse WTI for fermion's vertex functions can be obtained by

$$\begin{aligned}&\partial^x_\rho\langle 0|TV^{\rho\mu\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle\\ =&\partial_\rho\langle\bar{\psi}(x)\frac{i}{2}\{S,\gamma^\rho\}\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ &-\partial^\lambda\langle\bar{\psi}(x)\sigma^{\mu\nu}\psi(x);\psi(y)\bar{\psi}(z)\rangle_c\\ =&\partial^\mu_x\langle 0|Tj^{\nu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle-\partial^\nu_x\langle 0|Tj^{\mu\lambda}(x)\psi(y)\bar{\psi}(z)|0\rangle,\end{aligned}\tag{A17}$$

where $S=\frac{-i}{2}\{\sigma^{\mu\nu},\gamma^\lambda\}$. Then it can be verified that the the transverse WT identity (A10, A11) are obtained by the path integration method (A14, A15).

As shown above, the transverse and longitudinal WT identities in the four-dimensional gauge theory do not specify the vertex function with a two-point Green's function, thus forming a closed DSEs. However, in the case of low-dimension gauge theory, such as QED_3, the basic situation changed significantly. In QED_3 theory, we find that a set of transverse WT relations (for the vector and tensor vertex function) are coupled to each other, and the transverse relations together with the longitudinal WT identities would lead to a complete set of WT-type constraint relations for the three-point functions. Then, the complete expressions for three vertex functions can be deduced by solving this complete set of WT relations.

Appendix B

As mentioned above, substituting the vector vertex function Γ_V^μ (16) into the photon polarization vector (21), the photon polarization vector is obtained as follows

$$\begin{aligned}
\Pi^{\mu\nu}(q) =& \frac{iN_f e^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 Tr_D[\gamma^\mu S_F(p_1)\Gamma^\nu(p_1,p_2)S_F(p_2)]. \\
=& \frac{iN_f e^2}{(2\pi)^3}\int \mathrm{d}^3 p_1 \frac{1}{(q^2-4m^2)} Tr\Big\{ q^\nu\Big[\frac{2p_2^\mu A(p_2^2)}{p_2^2A^2(p_2^2)-B^2(p_2^2)} - \frac{2p_1^\mu A(p_1^2)}{p_1^2A^2(p_1^2)-B^2(p_1^2)}\Big] \\
&+ iq_\lambda\Big(\frac{i4p_{2\rho}[g^{\mu\nu}g^{\lambda\rho}-g^{\mu\lambda}g^{\nu\rho}]A(p_2^2)-2\epsilon^{\mu\nu\lambda}B(p_2^2)}{p_2^2A^2(p_2^2)-B^2(p_2^2)} + \frac{i4p_{1\rho}[-g^{\mu\nu}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\nu}]A(p_1^2)-2\epsilon^{\mu\nu\lambda}B(p_1^2)}{p_1^2A^2(p_1^2)-B^2(p_1^2)}\Big) \\
&+ 2m\Big[\frac{-i2p_{2\rho}\epsilon^{\mu\nu\rho}A(p_2^2)-2g^{\mu\nu}B(p_2^2)}{p_2^2A^2(p_2^2)-B^2(p_2^2)} + \frac{-i2\epsilon^{\mu\rho\nu}p_{1\rho}A(p_1^2)-2g^{\mu\nu}B(p_1^2)}{p_1^2A^2(p_1^2)-B^2(p_1^2)}\Big] \\
&+ \frac{[2m(p_1^\nu+p_2^\nu)-i\epsilon^{\rho\nu\lambda}q_\lambda(p_{1\rho}+p_{2\rho})]}{-q_\tau(p_1^\tau+p_2^\tau)}\Big(q_\tau\Big[\frac{-i2p_{2\rho}\epsilon^{\mu\tau\rho}A(p_2^2)-2g^{\mu\tau}B(p_2^2)}{p_2^2A^2(p_2^2)-B^2(p_2^2)} + \frac{-i2\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)-2g^{\mu\tau}B(p_1^2)}{p_1^2A^2(p_1^2)-B^2(p_1^2)}\Big] \\
&+ 2m\Big[\frac{2p_2^\mu A(p_2^2)}{p_2^2A^2(p_2^2)-B^2(p_2^2)} - \frac{2p_1^\mu A(p_1^2)}{p_1^2A^2(p_1^2)-B^2(p_1^2)}\Big]\Big) \\
&+ i\int\frac{\mathrm{d}^3k}{(2\pi)^3}2k_\alpha q_\beta\epsilon^{\alpha\nu\beta}\frac{q_\tau[C^{\mu\tau}(p_3,p_1,p_2)+O^{\mu\tau}(p_4,p_1,p_2)]+2m[F^\mu(p_3,p_1,p_2)-F^\mu(p_4,p_1,p_2)]}{-q_\sigma(p_3^\sigma+p_4^\sigma)}\Big\},
\end{aligned} \tag{B1}$$

where we used this relationship $q=p_1-p_2, p_3=p_1-k, p_4=p_2-k$, and the relations of $C^{\mu\tau}(p_3), O^{\mu\tau}(p_4), F^\mu(p_3), M^{\mu\nu\rho\tau\lambda}$ are defined as follows:

$$\begin{aligned}
C^{\mu\tau}(p_3,p_1,p_2) =& Tr\gamma^\mu S_F(p_1)S_F^{-1}(p_1-k)\gamma^\tau S_F(p_2) = \frac{1}{[p_1^2A^2(p_1^2)-B^2(p_1^2)][p_2^2A^2(p_2^2)-B^2(p_2^2)]}\times[p_{1\rho}p_{3\sigma}p_{2\lambda}M^{\mu\rho\sigma\tau\lambda}A(p_1^2)A(p_2^2)A(p_3^2) \\
&+4p_{1\rho}p_{2\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})A(p_1^2)A(p_2^2)B(p_3^2)-4p_{3\rho}p_{2\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})B(p_1^2)A(p_2^2)A(p_3^2) \\
&+2i\epsilon^{\mu\tau\rho}p_{2\rho}B(p_1^2)A(p_2^2)B(p_3^2)-4p_{1\rho}p_{3\sigma}(g^{\mu\rho}g^{\sigma\tau}-g^{\mu\sigma}g^{\rho\tau}+g^{\mu\tau}g^{\rho\sigma})A(p_1^2)B(p_2^2)A(p_3^2)+2i\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)B(p_2^2)B(p_3^2) \\
&-2i\epsilon^{\mu\rho\tau}p_{1\rho}B(p_1^2)B(p_2^2)A(p_3^2)+2g^{\mu\tau}B(p_1^2)B(p_2^2)B(p_3^2)],
\end{aligned} \tag{B2}$$

$$\begin{aligned}
O^{\mu\tau}(p_4,p_1,p_2) =& Tr\gamma^\mu S_F(p_1)\gamma^\tau S_F^{-1}(p_4)S_F(p_2) = \frac{1}{[p_1^2A^2(p_1^2)-B^2(p_1^2)][p_2^2A^2(p_2^2)-B^2(p_2^2)]} \\
&\times\Big[p_{1}\rho p_{4\sigma}p_{2\lambda}M^{\mu\rho\tau\sigma\lambda}A(p_1^2)A(p_2^2)A(p_4^2)+4p_{1\rho}p_{2\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})A(p_1^2)A(p_2^2)B(p_4^2) \\
&-4p_{4\rho}p_{2\sigma}(g^{\mu\tau}g^{\rho\sigma}-g^{\mu\rho}g^{\tau\sigma}+g^{\mu\sigma}g^{\tau\rho})B(p_1^2)A(p_2^2)A(p_4^2)+2i\epsilon^{\mu\tau\rho}p_{2\rho}B(p_1^2)A(p_2^2)B(p_4^2) \\
&-4p_{1\rho}p_{4\sigma}(g^{\mu\rho}g^{\tau\sigma}-g^{\mu\tau}g^{\rho\sigma}+g^{\mu\sigma}g^{\rho\tau})A(p_1^2)B(p_2^2)A(p_4^2)+2i\epsilon^{\mu\rho\tau}p_{1\rho}A(p_1^2)B(p_2^2)B(p_4^2) \\
&-2i\epsilon^{\mu\tau\rho}p_{4\rho}B(p_1^2)B(p_2^2)A(p_4^2)+2g^{\mu\tau}B(p_1^2)B(p_2^2)B(p_4^2)\Big],
\end{aligned} \tag{B3}$$

$$\begin{aligned}
F^\mu(p_3,p_1,p_2) =& Tr\gamma^\mu S_F(p_1)S_F^{-1}(p_3)S_F(p_2) = \frac{1}{[p_1^2A^2(p_1^2)-B^2(p_1^2)][p_2^2A^2(p_2^2)-B^2(p_2^2)]} \\
&\times\Big[4p_{1\rho}p_{3\sigma}p_{2\lambda}(g^{\mu\rho}g^{\sigma\lambda}-g^{\mu\sigma}g^{\rho\lambda}+g^{\mu\lambda}g^{\rho\sigma})A(p_1^2)A(p_2^2)A(p_3^2)-2i\epsilon^{\mu\rho\sigma}p_{1\rho}p_{2\sigma}A(p_1^2)A(p_2^2)B(p_3^2)+2ip_{3\rho}p_{2\sigma}\epsilon^{\mu\rho\sigma}B(p_1^2)A(p_2^2)A(p_3^2) \\
&-2p_2^\mu B(p_1^2)A(p_2^2)B(p_3^2)+2ip_{1\rho}p_{3\sigma}\epsilon^{\mu\rho\sigma}A(p_1^2)B(p_2^2)A(p_3^2)-2p_1^\mu A(p_1^2)B(p_2^2)B(p_3^2)+2p_3^\mu B(p_1^2)B(p_2^2)A(p_3^2)\Big],
\end{aligned} \tag{B4}$$

$$M^{\mu\nu\rho\tau\lambda} = Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau\gamma^\lambda) = -2ig^{\mu\nu}\epsilon^{\rho\tau\lambda}+2ig^{\mu\rho}\epsilon^{\nu\tau\lambda}-2ig^{\mu\tau}\epsilon^{\nu\rho\lambda}+i4\epsilon^{\mu\lambda m}(g^{\nu\rho}g_m^\tau-g^{\nu\tau}g_m^\rho+g_m^\nu g^{\rho\tau}). \tag{B5}$$

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