Asymptotic structure of Einstein-Gauss-Bonnet theory in lower dimensions*

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Abstract: Recently, an action principle for the $D\to 4$ limit of Einstein-Gauss-Bonnet gravity has been proposed. It is a special scalar-tensor theory that belongs to the family of Horndeski gravity. It also has well defined $D\to 3$ and $D\to 2$ limits. In this work, we examine this theory in three and four dimensions in the Bondi-Sachs framework. In both three and four dimensions, we find that there is no news function associated with the scalar field, which means that there is no scalar propagating degree of freedom in the theory. In four dimensions, the mass-loss formula is not affected by the Gauss-Bonnet term. This is consistent with the fact that there is no scalar radiation. However, the effects of the Gauss-Bonnet term are quite significant in the sense that they arise just one order after the integration constants and also arise in the quadrupole of the gravitational source.

Keywords: asymptotic structure, Einstein-Gauss-Bonnet theory, Horndeski gravity

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I. INTRODUCTION

Einstein-Gauss-Bonnet (EGB) gravity is the simplest case of Lovelock's extension of Einstein gravity [1]. The theory exists naturally in higher dimensions and becomes important with the development of string theory. Its black hole solutions [2-5] play an important role in studying anti-de Sitter/conformal field theory (AdS/CFT) correspondence. In four dimensions, the Gauss-Bonnet combination is a topological invariant and does not affect the classical equations of motion. Einstein's general relativity is widely believed to be the unique Lagrangian theory yielding second order equations of motion for the metric in four dimensions. The Lovelock type of construction requires additional scalar or vector fields, giving rise to Hordenski gravities [6] or generalized Galilean gravities [7-9].

However, this has been recently challenged by a novel four dimensional EGB solution [10], which is encoded in the dimensional regularization. After a rescaling of the coupling constant $\alpha\to\frac{\alpha}{D-4}$, the $D\to 4$ limit can be taken smoothly at the solution level, yielding a nontrivial new black hole. This created a great deal of interest [11-40], as well as controversy [41], as one would expect that higher-derivative theories of finite order that are ghost free in four-dimensions cannot be pure metric theories but are of the Hordenski type. In fact, the resolution of the divergence at the action level is far less clear, and the action principle for the $D=4$ solution is not given in [10]. One consistent approach is to consider a compactification of D-dimensional EGB gravity on a maximally symmetric space of $(D-p)$ dimensions, where $p\leqslant 4$, keeping only the breathing mode characterizing the size of the internal space such that the theory is minimum. The $D\to p$ limit can then be smoothly applied [42], leading to an action principle admitting the four dimensional EGB solution [10, 43, 44] (see also [45, 46]). In fact, the analogous $D\to 2$ limit of Einstein gravity was proposed many years ago [47] (see also the recent work in [48, 49]). It turns out that the resulting theory is indeed a special Horndeski theory. The action contains a Horndeski scalar that coupled to the Gauss-Bonnet term, as well as the metric field. The lower dimensional action is given by [42][1)]

$$S_p=\int d^px\sqrt{-g}\Big[R+\alpha\phi\mathcal{G}+\alpha(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-4(\nabla\phi)^2\nabla^2\phi+2(\nabla\phi)^2(\nabla\phi)^2)\Big],\tag{1}$$

where $G^{\mu\nu}$ is the Einstein tensor, and

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1) We have chosen that the curvature tensor of the internal maximally symmetric $D-p$ space vanishes.

$$\mathcal{G} \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \tag{2}$$

is the Gauss-Bonnet term.

There are several interesting features in the new theory (1.1). First, there is no scalar kinematic term; thus, a scalar propagator should be absent. Second, the classical solution of the Minkowski vacuum admits two independent scalar solutions, namely, $\phi = 0$, which we refer to as the ordinary vacuum, and $\phi = \log\frac{r}{r_0}$, which we refer to as the logarithm vacuum.[1)] Last but not the least, the α correction is inherited from the higher-dimensional counterparts. Hence, it includes not only the four dimensional Gauss-Bonnet term coupled with a scalar field but also scalar terms that are non-minimally coupled to gravity. The latter seems to be more significant than the former in the corrections to the classical solution of Einstein gravity.

To test the above interesting features, we will study the asymptotic structure of the lower dimensional EGB theory (1.1) in the Bondi-Sachs framework [50, 51] in the present work. In 1960s, Bondi *et al.* established an elegant framework of asymptotic expansions to understand the gravitational radiation in axisymmetric isolated systems in the Einstein theory [50]. The metric fields are expanded in inverse powers of a radius coordinate in a suitable coordinate system, and the equations of motion are solved order by order with respect to proper boundary conditions. In this framework [50], the radiation is characterized by a single function from the expansions of the metric fields, which is called the news function. Meanwhile, the mass of the system always decreases whenever there is a news function. Sachs then extended this framework to asymptotically flat spacetime [51]. This is a good starting point to study the asymptotic structure of the theory (1.1) in three dimensions. We obtain the asymptotic form of the solution space. There is no news function in three dimensions. This is a direct demonstration that there is no scalar propagating degree of freedom. Next, we turn to the four dimensional case. Two scalar solutions of the vacuum lead to two different boundary conditions for the scalar fields. The solution spaces are obtained in series expansions with respect to different boundary conditions. For both cases, there is no news function in the expansion of the scalar field, which means that a scalar propagating degree of freedom does not exist in four dimensions. In addition, the α corrections are transparent in the solution space. They arise just one order after the integration constants and also arise in the quadrupole, i.e., the first radiating source in the multipole expansion. In the logarithm vacuum, the α corrections even live at the linearized level. We show the precise formula of the α corrections in the quadrupole. Hence, the two different vacua are indeed experimentally distinguishable.

The organization of this paper is quite simple. In the next section, we study the asymptotic structure in three dimensions. We perform the same analysis in four dimensions in Section II, with special emphasis on α corrections in the gravitational solutions and the classical radiating source. After a brief conclusion and a discussion on some future directions, we complete the article with an appendix, where some useful relations are listed.

II. ASYMPTOTIC STRUCTURE OF EINSTEIN-GAUSS-BONNET THEORY IN THREE DIMENSIONS

As a toy model, it is worthwhile to examine the EGB theory (1.1) in three dimensions to determine if the Bondi-Sachs framework is applicable to this theory. In three dimensions, the Gauss-Bonnet term is identically zero. Applying the relations in Appendix A, the variation of the action is obtained as

$$\begin{aligned}\delta S_3 = \int \mathrm{d}^3x\sqrt{-g}\Big\{&-\frac{1}{2}g_{\tau\gamma}\delta g^{\tau\gamma}\Big[R+\alpha(4G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-4(\nabla\phi)^2\nabla^2\phi+2(\nabla\phi)^2(\nabla\phi)^2)\Big]+R_{\mu\nu}\delta g^{\mu\nu}+\nabla_\mu\left(g_{\alpha\beta}\nabla^\mu\delta g^{\alpha\beta}-\nabla_\nu\delta g^{\mu\nu}\right)\\
&+\alpha\Big[2\left(g_{\rho\sigma}\nabla^\mu\nabla^\nu\delta g^{\rho\sigma}-\nabla_\rho\nabla^\mu\delta g^{\rho\nu}-\nabla_\rho\nabla^\nu\delta g^{\rho\mu}+\nabla^2\delta g^{\mu\nu}\right)\nabla_\mu\phi\nabla_\nu\phi+4R^\mu_\rho\nabla_\mu\phi\nabla_\nu\phi\delta g^{\nu\rho}+4R^\nu_\rho\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\rho}\\
&-2\Big(R\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu}+(\nabla\phi)^2R_{\rho\sigma}\delta g^{\rho\sigma}+g_{\rho\sigma}(\nabla\phi)^2\nabla^2\delta g^{\rho\sigma}-(\nabla\phi)^2\nabla_\rho\nabla_\sigma\delta g^{\rho\sigma}\Big)-4\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\nabla^2\phi\\
&-4(\nabla\phi)^2\nabla_\rho\nabla_\sigma\phi\delta g^{\rho\sigma}+4\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi(\nabla\phi)^2+2(\nabla\phi)^2g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\delta g^{\mu\nu}-4(\nabla\phi)^2\nabla_\rho\phi\nabla_\mu\delta g^{\rho\mu}\Big]\\
&+\alpha\Big[8G^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi-8g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi\nabla^2\phi+8g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi(\nabla\phi)^2-4(\nabla\phi)^2\nabla^2\delta\phi\Big]\Big\}.\end{aligned} \tag{3}$$

After dropping many boundary terms, one obtains the Einstein equation

$$G_{\mu\nu} - \alpha T_{\mu\nu} = 0, \tag{4}$$

where

1) We set a constant radial scale r_0 to compensate the length dimension in ϕ.

$$
\begin{aligned}
T_{\mu\nu} =& g_{\mu\nu}[4R^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi+2\nabla_\sigma\nabla_\rho\phi\nabla^\rho\nabla^\sigma\phi-2(\nabla^2\phi)^2+(\nabla\phi)^2(\nabla\phi)^2+4\nabla_\rho\nabla_\sigma\phi\nabla^\rho\phi\nabla^\sigma\phi] \\
&+4\nabla_\mu\nabla_\nu\phi\nabla^2\phi-4\nabla_\rho\nabla_\mu\phi\nabla^\rho\nabla_\nu\phi+4\nabla_\mu\phi\nabla_\nu\phi\nabla^2\phi-4\nabla_\rho\nabla_\mu\phi\nabla_\nu\phi\nabla^\rho\phi-4\nabla_\rho\nabla_\nu\phi\nabla_\mu\phi\nabla^\rho\phi \\
&-4\nabla_\mu\phi\nabla_\nu\phi(\nabla\phi)^2-4R^\rho_\nu\nabla_\mu\phi\nabla_\rho\phi-4R^\rho_\mu\nabla_\nu\phi\nabla_\rho\phi+2R\nabla_\mu\phi\nabla_\nu\phi+2G_{\mu\nu}(\nabla\phi)^2-4R_{\mu\rho\nu\sigma}\nabla^\rho\phi\nabla^\sigma\phi, \qquad (5)
\end{aligned}
$$

and the scalar equation

$$
G^{\mu\nu}\nabla_\mu\nabla_\nu\phi+R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+\nabla^2\phi(\nabla\phi)^2-(\nabla^2\phi)^2+2\nabla_\rho\nabla_\sigma\phi\nabla^\sigma\phi\nabla^\rho\phi+\nabla_\rho\nabla_\sigma\phi\nabla^\sigma\nabla^\rho\phi=0. \qquad (6)
$$

A. Bondi gauge

In order to study three dimensional Einstein theory at future null infinity, the Bondi gauge was adapted to three dimensions with the gauge fixing ansatz [52, 53]

$$
\mathrm{d}s^2=\frac{V}{r}\mathrm{e}^{2\beta}\mathrm{d}u^2-2\mathrm{e}^{2\beta}\mathrm{d}u\mathrm{d}r+r^2(\mathrm{d}\phi-U\mathrm{d}u)^2, \qquad (7)
$$

in (u,r,φ) coordinates, and β,U,V are functions of (u,r,φ). Suitable fall-off conditions that preserve asymptotic flatness are

$$
U=O(r^{-2}),\quad V=O(r),\quad \beta=O(r^{-1}),\quad \phi=O(r^{-1}). \qquad (8)
$$

One of the advantages of the Bondi gauge is encoded in the organization of the equations of motion [50, 51, 53] (also see [54, 55] for the generalization to matter coupled theories). There are four types of equations of motion, namely the main equation, standard equation, supplementary equation, and trivial equation. The terminology characterizes their special properties. The main equations determine the r-dependence of the unknown functions β,U,V, while the standard equation controls the time evolution of the scalar field. Because of the Bianchi identities, the supplementary equations are left with only one order in the $1/r$ expansion undetermined, and the trivial equation is fulfilled automatically when the main equations and the standard equation are satisfied. In three dimensional EGB theory (1.1), the components $G_{rr}-\alpha T_{rr}=0$, $G_{r\varphi}-\alpha T_{r\varphi}=0$, and $G_{ru}-\alpha T_{ru}=0$ are the main equations. The scalar equation is the standard equation; $G_{u\varphi}-\alpha T_{u\varphi}=0$ and $G_{uu}-\alpha T_{uu}=0$ are the supplementary equations. Finally, $G_{\varphi\varphi}-\alpha T_{\varphi\varphi}=0$ is the trivial equation.

B. Solution space

Once the scalar field is given as initial data in the series expansion

$$
\phi(u,r,\varphi)=\sum_{a=1}^{\infty}\frac{\phi_a(u,\varphi)}{r^a}, \qquad (9)
$$

the unknown functions β,U,V can be solved explicitly. In asymptotic form, they are

$$
\begin{aligned}
\beta =& \frac{3\alpha\phi_1\partial_u\phi_1}{4r^3}+\frac{\alpha}{2r^4}\Big[2M\phi_1^2+4(\partial_\varphi\phi_1)^2-2\phi_1\partial_\varphi^2\phi_1 \\
&+5\phi_1^2\partial_u\phi_1+6\phi_2\partial_u\phi_1+2\phi_1\partial_u\phi_2\Big]+O(r^{-5}), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
U =& \frac{N(u,\varphi)}{r^2}-\frac{\alpha}{6r^4}\Big[20\partial_u\phi_1\partial_\varphi\phi_1+\phi_1\Big(3\partial_\varphi M\partial_u\phi_1 \\
&-\partial_uN\partial_u\phi_1-4\partial_u\partial_\varphi\phi_1\Big)\Big]+O(r^{-5}), \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
V =& -rM(u,\varphi)-\frac{1}{r}\left[N^2-2\alpha\partial_u\phi_1(2\partial_u\phi_1-\phi_1\partial_uM)\right] \\
&+\frac{\alpha}{3r^2}\Big[4(1-M)\phi_1^2\partial_uM-8\phi_2\partial_u\phi_1\partial_uM-4\phi_1\partial_u\phi_2\partial_uM \\
&+\partial_\varphi M\partial_\varphi\phi_1\partial_u\phi_1-2\partial_\varphi\phi_1\partial_uN\partial_u\phi_1-4\partial_\varphi^2\phi_1\partial_u\phi_1 \\
&+24\partial_u\phi_1\partial_u\phi_2+16\partial_\varphi\phi_1\partial_u\partial_\varphi\phi_1+\phi_1\Big(16M\partial_u\phi_1 \\
&-3\partial_\varphi M\partial_u\phi_1+8(\partial_u\phi_1)^2+6\partial_u\phi_1\partial_u\partial_\varphi N+\partial_\varphi M\partial_u\partial_\varphi\phi_1 \\
&-2\partial_uN\partial_u\partial_\varphi\phi_1-4\partial_u\partial_\varphi^2\phi_1\Big)\Big]+O(r^{-3}), \qquad (12)
\end{aligned}
$$

where $N(u,\varphi)$ and $M(u,\varphi)$ are integration constants. Compared to the pure Einstein case [53], the α corrections are at least two orders after the integration constants. The solution space is no longer in a closed form.

The time evolution of every order of the scalar field is controlled from the standard equation. This means that there is no news function from the scalar field. We list the first two orders of the standard equation

$$
2(\partial_u\phi_1)^2+\phi_1(\partial_uM+4\partial_u^2\phi_1)=0, \qquad (13)
$$

$$
\begin{aligned}
&4\partial_u(\phi_1\partial_u\phi_2)+8\partial_\varphi\phi_1\partial_u\partial_\varphi\phi_1+12\phi_2\partial_u^2\phi_1+2\phi_1^2\partial_u^2\phi_1-4\partial_\varphi^2\phi_1\partial_u\phi_1+\phi_1\left(\partial_\varphi^2M+8M\partial_u\phi_1+10(\partial_u\phi_1)^2-2\partial_u\partial_\varphi N\right) \\
&-2\partial_\varphi M\partial_\varphi\phi_1+4\partial_\varphi\phi_1\partial_uN+\frac{5}{2}\phi_1^2\partial_uM+3\phi_2\partial_uM=0. \qquad (14)
\end{aligned}
$$

The constraints from the supplementary equations are

$$\partial_u M = 0, \tag{15}$$

$$\partial_u N = \frac{1}{2}\partial_\varphi M, \tag{16}$$

which are the same as in the pure Einstein case. This is well expected, as the α corrections are in the higher orders. In the end, there is no propagating degree of freedom at all in this theory in three dimensions. The whole effect of the higher dimensional Gauss-Bonnet terms is a kind of deformation of Einstein gravity.

III. ASYMPTOTIC STRUCTURE OF EINSTEIN-GAUSS-BONNET THEORY IN FOUR DIMENSIONS

We now turn to the more realistic case of four dimensions. The action is given by (1.1) with $p=4$. The derivation of the equations of motion is quite similar to the three dimensional case, with the additional contribution from the Gauss-Bonnet term, which is detailed in Appendix A. The Einstein equation is obtained as

$$G_{\mu\nu} - \alpha T_{\mu\nu} = 0, \tag{17}$$

where the modification to $T_{\mu\nu}$ (5) from the Gauss-Bonnet term is

$$-4R_{\mu\rho\nu\sigma}\nabla^\rho\nabla^\sigma\phi + 4G_{\mu\nu}\nabla^2\phi - 4R^\rho_\mu\nabla_\nu\nabla_\rho\phi - 4R^\rho_\nu\nabla_\mu\nabla_\rho\phi + 4g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma\phi + 2R\nabla_\mu\nabla_\nu\phi, \tag{18}$$

and the scalar equation is

$$G^{\mu\nu}\nabla_\mu\nabla_\nu\phi + R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \nabla^2\phi(\nabla\phi)^2 - (\nabla^2\phi)^2 + 2\nabla_\rho\nabla_\sigma\phi\nabla^\sigma\phi\nabla^\rho\phi + \nabla_\rho\nabla_\sigma\phi\nabla^\sigma\nabla^\rho\phi - \frac{1}{8}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right) = 0. \tag{19}$$

A. Bondi gauge

In four dimensions, we choose the Bondi gauge fixing ansatz [50]

$$\mathrm{d}s^2 = \left[\frac{V}{r}\mathrm{e}^{2\beta} + U^2r^2\mathrm{e}^{2\gamma}\right]\mathrm{d}u^2 - 2\mathrm{e}^{2\beta}\mathrm{d}u\mathrm{d}r - 2Ur^2\mathrm{e}^{2\gamma}\mathrm{d}u\mathrm{d}\theta + r^2\left[\mathrm{e}^{2\gamma}\mathrm{d}\theta^2 + \mathrm{e}^{-2\gamma}\sin^2\theta\mathrm{d}\phi^2\right], \tag{20}$$

in (u,r,θ,φ) coordinates. The metric ansatz involves four functions (V,U,β,γ) of (u,r,θ) that are to be determined by the equations of motion. These functions and the scalar field are φ-independent, and hence, the metric has manifest global Killing direction ∂_φ. This is the "axisymmetric isolated system" introduced in [50].[1)] Following [50] closely, the falloff conditions for the functions (β,γ,U,V) in the metric for asymptotic flatness are given by

$$\beta = O(r^{-1}),\quad \gamma = O(r^{-1}),\quad U = O(r^{-2}),\quad V = -r + O(1). \tag{21}$$

Considering the metric of the Minkowski vacuum

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{22}$$

we have two branches of the scalar solution

$$\phi = 0, \quad \text{or} \quad \phi = \log\frac{r}{r_0}. \tag{23}$$

The first gives the true vacuum with the maximal spacetime symmetry preserved; the second solution is nearly Minkowski, since the scalar does not preserve the full symmetry. Both are valid solutions, with one *not* encompassing the other. Analogous emergence of logarithmic dependence for the scalar also occurs in the AdS vacuum for some critical Einstein-Horndeski gravity, where the scalar breaks the full conformal symmetry of the AdS to the subgroup of the Poincare together with the scaling invariance [56]. However, ours is the first example in the Minkowski vacuum. The necessary falloff condition of the scalar field consistent with the metric falloffs is either

$$\phi = O(r^{-1}), \quad \text{or} \quad \phi = \log\frac{r}{r_0} + O(r^{-1}). \tag{24}$$

Similar to the three-dimensional case, the equations of motion are organized as follows: $G_{rr} - \alpha T_{rr} = 0$, $G_{r\theta} - \alpha T_{r\theta} = 0$, and $G_{\theta\theta}g^{\theta\theta} + G_{\varphi\varphi}g^{\varphi\varphi} - \alpha T_{\theta\theta}g^{\theta\theta} - \alpha T_{\varphi\varphi}g^{\varphi\varphi} = 0$ are the main equations. The scalar equation and $G_{\theta\theta} - \alpha T_{\theta\theta} = 0$ are the standard equations; $G_{u\theta} - \alpha T_{u\theta} = 0$ and $G_{uu} - \alpha T_{uu} = 0$ are supplementary; and $G_{ru} - \alpha T_{ru} = 0$ is trivial. $G_{r\varphi} - \alpha T_{r\varphi} = 0$, $G_{\theta\varphi} - \alpha T_{\theta\varphi} = 0$, and $G_{u\varphi} - \alpha T_{u\varphi} = 0$ are trivial because the system is φ-independent.

B. Solution space with an ordinary vacuum

Suppose that γ and ϕ are given in a series expansion as initial data[2)]

1) In the present work, our main purpose is to demonstrate the effects of the Gauss-Bonnet term in the asymptotic analysis. For simplicity, we adopt the axisymmetric condition. However we do not expect any principle difficulties in the study of the general four dimensional asymptotic flatness solutions by choosing Sachs' gauge fixing ansatz.

2) To avoid logarithm terms in the metric, we turn off the order $O(r^{-2})$ in γ.

$$\gamma = \frac{c(u,\theta)}{r} + \sum_{a=3}^{\infty} \frac{\gamma_a(u,\theta)}{r^a}, \tag{25}$$

$$\phi = \sum_{a=1}^{\infty} \frac{\phi_a(u,\theta)}{r^a}. \tag{26}$$

The unknown functions β, U, V are solved in asymptotic form as

$$\beta = -\frac{c^2}{4r^2} + \frac{4\alpha\phi_1\partial_u\phi_1}{3r^3} + O(r^{-4}), \tag{27}$$

$$\begin{aligned} U =& -\frac{2\cot\theta c + \partial_\theta c}{r^2} + \frac{N(u,\varphi)}{r^3} + \frac{1}{2r^4}\Big[5\cot\theta c^3 - 3cN \\ &+ 6\cot\theta\gamma_3 + \frac{5}{2}c^2\partial_\theta c + 3\partial_\theta c + \alpha\Big(16\cot\theta\phi_1\partial_u c \\ &-\frac{20}{3}\partial_\theta\phi_1\partial_u\phi_1 + 8\phi_1\partial_u\partial_\theta c + \frac{4}{3}\phi_1\partial_u\partial_\theta\phi_1\Big)\Big] + O(r^{-5}), \end{aligned} \tag{28}$$

$$\begin{aligned} V =& -r + M(u,\theta) + \frac{1}{2r}\Big[\cot\theta N - \frac{1}{2}c^2(5+11\cos 2\theta)\csc^2\theta \\ &-5(\partial_\theta c)^2 + \partial_\theta N - c(19\cot\theta\partial_\theta c + 3\partial_\theta^2 c) \\ &+ 8\alpha(\partial_u\phi_1)^2\Big] + O(r^{-2}), \end{aligned} \tag{29}$$

where $N(u,\theta)$ and $M(u,\theta)$ are integration constants. Clearly, the coupling α emerges just one order after the integration constants. They are from the non-minimal coupled scalar rather than the four dimensional Gauss-Bonnet term.

The standard equations control the time evolution of the initial data γ and ϕ. In particular, the time evolution of every order of the scalar field has been constrained. That means there is no news function associated to the scalar field. Hence, the scalar field does not have a propagating degree of freedom similar to the three dimensional case. We list the first two orders of the scalar equation

$$(\partial_u\phi_1)^2 + \phi_1\partial_u^2\phi_1 = 0, \tag{30}$$

$$\begin{aligned} &2\phi_1\partial_u^2\phi_2 + 6\partial_u\phi_1\partial_u\phi_2 - \partial_\theta^2\phi_1\partial_u\phi_1 - \cot\theta\partial_\theta\phi_1\partial_u\phi_1 + 4\partial_\theta\phi_1\partial_u\partial_\theta\phi_1 + \phi_1^2\partial_u^2\phi_1 + 6\phi_2\partial_u^2\phi_1 + 6\phi_1\partial_u\phi_1 + 4\phi_1(\partial_u\phi_1)^2 \\ &+ \phi_1\left[\partial_u\partial_\theta^2 c + 3\cot\theta\partial_u\partial_\theta c - 2\partial_u c - 2(\partial_u c)^2 - \partial_u M\right] = 0. \end{aligned} \tag{31}$$

The first order of the standard equation from the Einstein equation is

$$\begin{aligned} \partial_u\gamma_3 =& \frac{1}{8}\Big[3(\partial_\theta c)^2 + c(5\cot\theta\partial_\theta c + 3\partial_\theta^2 c) - 2c^2\csc^2\theta \\ &\times(3+\cos 2\theta) + 2cM + \cot\theta N - \partial_\theta N - 16\alpha\phi_1\partial_u^2 c\Big]. \end{aligned} \tag{32}$$

In the Newman-Penrose variables, γ_3 is related to Ψ_0^0 or $\bar{\Psi}_0^0$ [57]. Since its time evolution involves α, the effect of the higher dimensional Gauss-Bonnet term arises, starting from the first radiating source, i.e., quadrupole, in the multipole expansion [58]. This can be seen more precisely on a linearized level from the logarithm vacuum case, which we will present in the next subsection.

The supplementary equations yield

$$\partial_u N = \frac{1}{3}\left[7\partial_\theta c\partial_u c + c(16\cot\theta\partial_u c + 3\partial_u\partial_\theta c) - \partial_\theta M\right]. \tag{33}$$

$$\partial_u m = -2(\partial_u c)^2, \quad m \equiv M - \frac{1}{\sin\theta}\partial_\theta(2\cos\theta c + \sin\theta\partial_\theta c). \tag{34}$$

The latter is the mass-loss formula in this theory. It is the same as that in the pure Einstein case [50] , which is expected, as the corrections from the Gauss-Bonnet term are in the higher orders.

C. Solution space with the logarithm vacuum

One intriguing feature of the theory is that the scalar admits a logarithmic dependence in the Minkowski vacuum, such that the full Lorentz group breaks down for any matter coupled to the scalar. We would like to analyze its solution space here. Suppose that γ and ϕ are given in series expansions as initial data:

$$\gamma = \frac{c(u,\theta)}{r} + \sum_{a=3}^{\infty} \frac{\gamma_a(u,\theta)}{r^a}, \tag{35}$$

$$\phi = \log\frac{r}{r_0} + \sum_{a=1}^{\infty} \frac{\phi_a(u,\theta)}{r^a}. \tag{36}$$

We can solve the unknown functions β, U, V in asymptotic form as

$$\beta = -\frac{c^2}{4r^2} + \frac{1}{4r^4}\Big[-3c\gamma_3 + \alpha\Big(4c\cot\theta(\partial_\theta c + \partial_\theta\phi_1) + c^2(\csc^2\theta + 3\cot^2\theta) - \phi_1^2 - 2\phi_2 + (\partial_\theta c)^2 + 2\partial_\theta c\partial_\theta\phi_1 + (\partial_\theta\phi_1)^2 + 2\alpha\partial_u\phi_1 - 8\alpha(\partial_u\phi_1)^3\Big)\Big] + O(r^{-5}), \tag{37}$$

$$U = -\frac{2\cot\theta c + \partial_\theta c}{r^2} + \frac{N(u,\varphi)}{r^3} + \frac{1}{2r^4}\Big\{5\cot\theta c^3 - 3cN + 6\cot\theta\gamma_3 + \frac{5}{2}c^2\partial_\theta c + 3\partial_\theta c + \alpha\Big[14\partial_\theta c\partial_u c\partial_u\phi_1 - 4\partial_\theta c\partial_u c - 4\partial_\theta\phi_1\partial_u c - 4\partial_\theta c\partial_u\phi_1 - 2\partial_\theta M\partial_u\phi_1 - 6\partial_u N\partial_u\phi_1 - 2c(4\cot\theta\partial_u c - 16\cot\theta\partial_u c\partial_u\phi_1 + 4\cot\theta\partial_u\phi_1 - 3\partial_u\partial_\theta c\partial_u\phi_1)\Big]\Big\} + O(r^{-5}), \tag{38}$$

$$V = -r + M(u,\theta) + \frac{1}{2r}\Big[\cot\theta N - \frac{1}{2}c^2(5 + 11\cos 2\theta)\csc^2\theta - 5(\partial_\theta c)^2 + \partial_\theta N - c(19\cot\theta\partial_\theta c + 3\partial_\theta^2 c) - 2\alpha + 8\alpha(\partial_u\phi_1)^2\Big] + O(r^{-2}). \tag{39}$$

The coupling α emerges again one order after the integration constants. At this order, it is from the non-minimally coupled scalar. The α^2 terms in β indicate the nonlinear scalar-gravity coupling.

The time evolution of every order of the scalar field is also constrained. There is no news function associated with the scalar field. The first two orders of the scalar equation are

$$\partial_u\phi_1 + (\partial_u\phi_1)^2 - (\partial_u c)^2 - \frac{1}{2} = 0, \tag{40}$$

$$4\phi_2\partial_u^2\phi_1 - 4\partial_u\phi_2 - 8\partial_u\phi_1\partial_u\phi_2 - 3M - 2\phi_1 + 3\cot\theta\partial_\theta c + \cot\theta\partial_\theta\phi_1 + \partial_\theta^2 c + \partial_\theta^2\phi_1 - 2\cot\theta\partial_\theta c\partial_u c + 2\cot\theta\partial_\theta\phi_1\partial_u c - 2\partial_\theta^2 c\partial_u c - 2\partial_\theta^2\phi_1\partial_u c - 4\phi_1(\partial_u c)^2 - 6\phi_1\partial_u\phi_1 - 12\cot\theta\partial_\theta c\partial_u\phi_1 - 4\cot\theta\partial_\theta\phi_1\partial_u\phi_1 - 4\partial_\theta^2 c\partial_u\phi_1 - 4\partial_\theta^2\phi_1\partial_u\phi_1 + 8\phi_1(\partial_u\phi_1)^2 + 4\partial_\theta c\partial_u\partial_\theta\phi_1 + 4\partial_\theta\phi_1\partial_u\partial_\theta\phi_1 - 2c + 8c\partial_u\phi_1 + c\partial_u c(8\csc^2\theta - 4\partial_u\phi_1) + 8\cot\theta c\partial_u\partial_\theta\phi_1 - 2c^2\partial_u^2\phi_1 + 2\phi_1^2\partial_u^2\phi_1 = 0. \tag{41}$$

The first order of the standard equation from the Einstein equation is

$$\partial_u\gamma_3 = \frac{1}{8}\Big[3(\partial_\theta c)^2 + c(5\cot\theta\partial_\theta c + 3\partial_\theta^2 c) - 2c^2\csc^2\theta(3 + \cos 2\theta) + 2cM + \cot\theta N - \partial_\theta N - 8\alpha\partial_u c + 16\alpha\partial_u\phi_1\partial_u c\Big]. \tag{42}$$

The constraints from the supplementary equations are

$$\partial_u N = \frac{1}{3}\left[7\partial_\theta c\partial_u c + c(16\cot\theta\partial_u c + 3\partial_u\partial_\theta c) - \partial_\theta M\right]. \tag{43}$$

$$\partial_u m = -2(\partial_u c)^2, \quad m \equiv M - \frac{1}{\sin\theta}\partial_\theta(2\cos\theta c + \sin\theta\partial_\theta c). \tag{44}$$

The mass-loss formula is the same as that for the pure Einstein case [50].

To reveal the α correction in the radiating source, we linearize the theory, for which we drop all the quadratic terms in the solutions. Then, the evolution equations are reduced to

$$\partial_u M = \frac{1}{\sin\theta}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin^2\theta\partial_u c\right)\right], \tag{45}$$

$$\partial_u N = -\frac{1}{3}\partial_\theta M, \tag{46}$$

$$\partial_u\gamma_3 = -\frac{1}{8}\sin\theta\partial_\theta\frac{N}{\sin\theta} - \alpha\partial_u c. \tag{47}$$

The α correction is now only from the scalar background $\log\frac{r}{r_0}$ term. The multipole expansion is encoded in the expansion of γ [58]. The quadrupole in Eq. (2.46) of [58] corresponds to $\gamma_3 = a_2(u)\sin^2\theta$, where the subscript 2 denotes the second order of the second associated Legendre function. The function c can be solved from the above evolution equations. The solution is $c = c_2(u)\sin^2\theta$, where $c_2(u)$ satisfies

$$c_2 - \alpha\partial_u^2 c_2 = \partial_u^2 a_2. \tag{48}$$

Suppose that a_2 is a periodic function, e.g., $a_2 = A\sin u + B\cos u$. Then the response of c_2 will have an α correction $c_2 = \frac{\partial_u^2 a_2}{1+\alpha}$. By setting $\alpha = 0$, we just recover the Einstein gravity result $c = \partial_u^2 a_2\sin^2\theta$. For the same type of gravitational source, the new theory (1.1) is indeed distinguishable from Einstein gravity. Since the c function has a direct connection to the Weyl tensor [57], we can expect a direct experimental test of the α corrections.

IV. CONCLUSION AND DISCUSSION

In this paper, the asymptotic structures of three and four dimensional EGB gravity have been studied in the Bondi-Sachs framework. It was shown from the solution space that, in both dimensions, there is no scalar propagator. The α corrections were discussed in detail from the perspective of both the gravitational solution and radiating sources.

There are several open questions in the theory (1.1) that should be addressed in the future. There is no scalar propagator in the theory, but there are differential couplings between gravity and the scalar field. The absence of the scalar propagator is likely to be consistent with observations; thus, it is of interest to know how to construct a gravity-scalar vertex without a scalar propagator [59]. A second interesting point is from the holography. In three dimensions, asymptotically flat gravitational theory has a holographic dual description [53, 60]. It would be very meaningful to explore the dual theory of the three dimensional EGB gravity. Another question worth mentioning is from the recent proposal of a triangle equivalence [61]. Since the change in the c function has α corrections for the same type of gravitational source, the gravitational memory receives the α correction [62]. In the context of the triangle relation, it is a very interesting question as to whether the soft graviton theorem and the asymptotic symmetry have α corrections as well.

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APPENDIX A: USEFUL RELATIONS

We list some useful relations that may help readers who are less familiar with the variational principle involving the Gauss-Bonnet term.

The Bianchi identity is given by

$$\nabla_\mu R_{\nu\sigma\rho}{}^{\nu} + \nabla_\nu R_{\sigma\mu\rho}{}^{\nu} + \nabla_\sigma R_{\mu\nu\rho}{}^{\nu} = 0. \tag{A1}$$

The commutator of ∇:

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)S^{\rho\sigma} = R^{\rho}{}_{\tau\mu\nu}S^{\tau\sigma} + R^{\sigma}{}_{\tau\mu\nu}S^{\rho\tau}. \tag{A2}$$

Variations of some relevant quantities are as follows:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{A3}$$

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\nabla^\sigma\delta g_{\mu\nu} - \frac{1}{2}g_{\mu\tau}\nabla_\nu\delta g^{\sigma\tau} - \frac{1}{2}g_{\nu\tau}\nabla_\mu\delta g^{\sigma\tau}, \tag{A4}$$

$$g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\nabla^\sigma\delta g^{\mu\nu} - \nabla_\mu\delta g^{\sigma\mu}, \tag{A5}$$

$$\delta R^{\sigma}{}_{\mu\rho\nu} = \nabla_\rho\delta\Gamma^\sigma_{\mu\nu} - \nabla_\nu\delta\Gamma^\sigma_{\mu\rho}, \tag{A6}$$

$$\begin{aligned}\delta R_{\mu\nu} =& \frac{1}{2}\Big(g_{\sigma\rho}\nabla_\mu\nabla_\nu\delta g^{\sigma\rho} - g_{\sigma\nu}\nabla_\rho\nabla_\mu\delta g^{\rho\sigma}\\ &-g_{\sigma\mu}\nabla_\rho\nabla_\nu\delta g^{\rho\sigma} - \nabla^2\delta g_{\mu\nu}\Big),\end{aligned} \tag{A7}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\left(g_{\sigma\rho}\nabla^\mu\delta g^{\sigma\rho} - \nabla_\nu\delta g^{\mu\nu}\right), \tag{A8}$$

$$\begin{aligned}\delta G^{\mu\nu} =& \frac{1}{2}\left(g_{\sigma\rho}\nabla^\mu\nabla^\nu\delta g^{\sigma\rho} - \nabla_\sigma\nabla^\mu\delta g^{\sigma\nu} - \nabla_\sigma\nabla^\nu\delta g^{\sigma\mu} + \nabla^2\delta g^{\mu\nu}\right)\\ &+ R^\mu_\sigma\delta g^{\nu\sigma} + R^\nu_\sigma\delta g^{\mu\sigma} - \frac{1}{2}R\delta g^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R_{\sigma\rho}\delta g^{\sigma\rho}\\ &- \frac{1}{2}g^{\mu\nu}g_{\sigma\rho}\nabla^2\delta g^{\sigma\rho} + \frac{1}{2}g^{\mu\nu}\nabla_\rho\nabla_\sigma\delta g^{\rho\sigma},\end{aligned} \tag{A9}$$

$$\delta R^2 = 2RR_{\rho\sigma}\delta g^{\rho\sigma} + 2R\left(g_{\sigma\rho}\nabla^2\delta g^{\sigma\rho} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right), \tag{A10}$$

$$\delta(R^{\sigma\mu\rho\nu}R_{\sigma\mu\rho\nu}) = 4R_{\sigma\mu\rho\nu}\nabla^\nu\nabla^\mu\delta g^{\rho\sigma} + 2R_{\sigma\mu\rho\nu}R^{\sigma}{}_{\tau}{}^{\rho\nu}\delta g^{\mu\tau}, \tag{A11}$$

$$\begin{aligned}\delta(R^{\mu\nu}R_{\mu\nu}) =& R_{\rho\sigma}\nabla^2\delta g^{\rho\sigma} - R_{\mu\rho}\nabla_\sigma\nabla^\mu\delta g^{\rho\sigma}\\ &+ g_{\rho\sigma}R^{\mu\nu}\nabla_\mu\nabla_\nu\delta g^{\rho\sigma} - R^\mu_\sigma\nabla_\mu\nabla_\rho\delta g^{\rho\sigma}\\ &+ R_{\mu\nu}R^\nu_\sigma\delta g^{\mu\sigma} + R_{\sigma\mu\rho\nu}R^{\mu\nu}\delta g^{\sigma\rho},\end{aligned} \tag{A12}$$

$$\begin{aligned}g^{\mu\nu}\delta(\nabla_\mu\nabla_\nu\phi) =& g^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi - \frac{1}{2}g_{\mu\nu}\nabla_\sigma\phi\nabla^\sigma\delta g^{\mu\nu}\\ &+ \nabla_\sigma\phi\nabla_\mu\delta g^{\sigma\mu},\end{aligned} \tag{A13}$$

$$\begin{aligned}\delta\mathcal{G} =& 2R_{\sigma\mu\tau\nu}R_\rho{}^{\mu\tau\nu}\delta g^{\sigma\rho} + 2RR_{\rho\sigma}\delta g^{\rho\sigma} - 4R_{\rho\nu}R^\nu_\sigma\delta g^{\rho\sigma}\\ &- 4R_{\sigma\mu\rho\nu}R^{\mu\nu}\delta g^{\sigma\rho}4R_{\sigma\mu\rho}{}^{\nu}\nabla_\nu\nabla^\mu\delta g^{\rho\sigma} + 4R_{\nu\sigma\rho}{}^{\nu}\nabla_\mu\nabla^\mu\delta g^{\rho\sigma}\\ &+ 4R_{\mu\nu\rho}{}^{\nu}\nabla_\sigma\nabla^\mu\delta g^{\rho\sigma} - 4g_{\rho\sigma}G^{\mu\nu}\nabla_\mu\nabla_\nu\delta g^{\rho\sigma}\\ &+ 4G^\mu_\rho\nabla_\mu\nabla_\sigma\delta g^{\rho\sigma}.\end{aligned} \tag{A14}$$

The first line of (A14) equals $\frac{1}{2}g_{\sigma\rho}\mathcal{G}\delta g^{\sigma\rho}$ in four dimensions. Thus, they will not contribute to the equations of motion. When performing integration by parts, the second line and the third line vanish automatically for the pure Gauss-Bonnet term. However, it will contribute when the scalar field is coupled to the Gauss-Bonnet term, e.g., $\phi\mathcal{G}$. The second line and the third line can be reorganized as follows:

$$
\begin{aligned}
4R_{\sigma\mu\rho\nu}\nabla^{\nu}\nabla^{\mu}\delta g^{\rho\sigma}&+4R_{\nu\sigma\rho}{}^{\nu}\nabla_{\mu}\nabla^{\mu}\delta g^{\rho\sigma}+4R^{\mu}_{\rho}\nabla_{\sigma}\nabla_{\mu}\delta g^{\rho\sigma}\\
&-4g_{\rho\sigma}G^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\delta g^{\rho\sigma}+4R^{\mu}_{\rho}\nabla_{\mu}\nabla_{\sigma}\delta g^{\rho\sigma}\\
&-2R\nabla_{\sigma}\nabla_{\rho}\delta g^{\rho\sigma}.
\end{aligned}
\tag{A15}
$$

The point of such reorganization is to make the indexes of the two covariant derivatives in every term symmetric. When integrating by parts for the pure Gauss-Bonnet term, both covariant derivatives are identically zero.

References

[1] D. Lovelock, J. Math. Phys. **12**, 498-501 (1971)
[2] D.G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985)
[3] D.L. Wiltshire, Phys. Lett. B **169**, 36 (1986)
[4] R.G. Cai and K.S. Soh, Phys. Rev. D **59**, 044013 (1999), arXiv:gr-qc/9808067 [gr-qc]
[5] R.G. Cai, Phys. Rev. D **65**, 084014 (2002), arXiv:hep-th/0109133 [hep-th]
[6] G.W. Horndeski, Int. J. Theor. Phys. **10**, 363-384 (1974)
[7] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D **80**, 064015 (2009), arXiv:0906.1967 [gr-qc]
[8] K. Van Acoleyen and J. Van Doorsselaere, Phys. Rev. D **83**, 084025 (2011), arXiv:1102.0487 [gr-qc]
[9] C. Deffayet, X. Gao, D. A. Steer *et al.*, Phys. Rev. D **84**, 064039 (2011), arXiv:1103.3260[hep-th]
[10] D. Glavan and C. Lin, Phys. Rev. Lett. **124**(8), 081301 (2020), arXiv:1905.03601 [gr-qc]
[11] C. Lin and Z. Lalak, arXiv:1911.12026 [gr-qc]
[12] R. Konoplya and A. Zinhailo, arXiv:2003.01188 [gr-qc]
[13] M. Guo and P.C. Li, Eur. Phys. J. C **80**, 588 (2020), arXiv:2003.02523 [gr-qc]
[14] P.G.S. Fernandes, Phys. Lett. B **805**, 135468 (2020), arXiv:2003.05491 [gr-qc]
[15] R.A. Konoplya and A. Zhidenko, Phys. Rev. D **101**, 084038 (2020), arXiv:2003.07788 [gr-qc]
[16] S.W. Wei and Y.X. Liu, arXiv:2003.07769 [gr-qc]
[17] A. Casalino, A. Colleaux, M. Rinaldi *et al.*, arXiv: 2003.07068 [gr-qc]
[18] R. Kumar and S. G. Ghosh, JCAP **20**, 053 (2020), arXiv:2003.08927 [gr-qc]
[19] K. Hegde, A. N. Kumara, C.L.A. Rizwan *et al.*, arXiv: 2003.08778 [gr-qc]
[20] D.D. Doneva and S.S. Yazadjiev, arXiv:2003.10284 [gr-qc]
[21] S.G. Ghosh and S.D. Maharaj, Phys. Dark Univ. **30**, 100687 (2020), arXiv:2003.09841 [gr-qc]
[22] C.-Y. Zhang, P.-C. Li, and M. Guo, arXiv:2003.13068 [hep-th]
[23] S. A. Hosseini Mansoori, arXiv:2003.13382 [gr-qc]
[24] S.-W. Wei and Y.-X. Liu, Phys. Rev. D **101**, 104081 (2020), arXiv:2003.14275 [gr-qc]
[25] M. Churilova, arXiv:2004.00513 [gr-qc]
[26] S. U. Islam, R. Kumar, and S. G. Ghosh, arXiv:2004.01038 [gr-qc]
[27] A. K. Mishra, arXiv:2004.01243 [gr-qc]
[28] S.-L. Li, P. Wu, and H. Yu, arXiv:2004.02080 [gr-qc]
[29] M. Heydari-Fard, M. Heydari-Fard, and H. Sepangi, arXiv:2004.02140 [gr-qc]
[30] R.A. Konoplya and A.F. Zinhailo, arXiv:2004.02248 [gr-qc]
[31] R. Kumar and S.G. Ghosh, arXiv:2003.08927 [gr-qc]
[32] S.-J. Yang, J.-J. Wan, J. Chen *et al.*, arXiv:2004.07934 [gr-qc]
[33] P.G. Fernandes, P. Carrilho, T. Clifton *et al.*, Phys. Rev. D **102**, 024025 (2020), arXiv:2004.08362 [gr-qc]
[34] F.-W. Shu, arXiv:2004.09339 [gr-qc]
[35] A. Casalino and L. Sebastiani, arXiv:2004.10229 [gr-qc]
[36] X. Zeng, H. Zhang, and H. Zhang, arXiv:2004.12074 [gr-qc]
[37] X. Ge and S. Sin, Eur. Phys. J. C **80**, 595 (2020), arXiv:2004.12191 [hep-th]
[38] R. Kumar, S. U. Islam, and S. G. Ghosh, arXiv:2004.12970 [gr-qc]
[39] R. A. Hennigar, D. Kubiznak, R. B. Mann *et al.*, arXiv:2004.12995 [gr-qc]
[40] J. Arrechea, A. Delhom, and A. Jiménez-Cano, arXiv:2004.12998 [gr-qc]
[41] M. Gurses, T. C. Sisman, and B. Tekin, Eur. Phys. J. C **80**, 647 (2020), arXiv:2004.03390 [gr-qc]
[42] H. Lü and Y. Pang, arXiv: 2003.11552 [gr-qc]
[43] R.G. Cai, L.M. Cao, and N. Ohta, JHEP **04**, 082 (2010), arXiv:0911.4379 [hep-th]
[44] R.G. Cai, Phys. Lett. B **733**, 183 (2014), arXiv:1405.1246 [hep-th]
[45] T. Kobayashi, arXiv: 2003.12771 [gr-qc]
[46] R.A. Hennigar, D. Kubiznak, R.B. Mann *et al.*, JHEP **07**, 027 (2020), arXiv:2004.09472 [gr-qc]
[47] R.B. Mann and S.F. Ross, Class. Quant. Grav. **10**, 1405 (1993), arXiv:gr-qc/9208004
[48] S. Nojiri and S.D. Odintsov, EPL **130**, 10004 (2020), arXiv:2004.01404 [hep-th]
[49] W.-Y. Ai, arXiv: 2004.02858 [gr-qc]
[50] H. Bondi, M.G. J. van der Burg, and A.W. K. Metzner, Proc. Roy. Soc. Lond. A **269**, 21-52 (1962)
[51] R.K. Sachs, Proc. Roy. Soc. Lond. A **270**, 103-126 (1962)
[52] G. Barnich and G. Compère, Class. Quant. Grav. **24**, F15-F23 (2007), arXiv:gr-qc/0610130
[53] G. Barnich and C. Troessaert, JHEP **05**, 062 (2010), arXiv:1001.1541 [hep-th]
[54] G. Barnich, P.-H. Lambert, and P. Mao, Class. Quant. Grav. **32**, 245001 (2015), arXiv:1503.00856 [gr-qc]
[55] H. Lü, P. Mao, and J.-B. Wu, JHEP **11**, 005 (2019), arXiv:1909.00970 [hep-th]
[56] Y.Z. Li, H. Lü, and H.Y. Zhang, Eur. Phys. J. C **79**, 592 (2019), arXiv:1812.05123 [hep-th]
[57] E. Conde and P. Mao, JHEP **05**, 060 (2017), arXiv:1612.08294 [hep-th]
[58] A.I. Janis and E. T. Newman, J. Math. Phys. **6**, 902-914 (1965)
[59] J. Bonifacio, K. Hinterbichler, and L.A. Johnson, Phys. Rev. D **102**, 024029 (2020), arXiv:2004.10716 [hep-th]
[60] A. Bagchi, S. Detournay, R. Fareghbal *et al.*, Phys. Rev. Lett. **110**, 141302 (2013), arXiv:1208.4372 [hep-th]
[61] A. Strominger, arXiv: 1703.05448 [hep-th]
[62] J. Frauendiener, Class. Quant. Grav. **9**, 1639-1641 (1992)