Universality of entropy principle for a general diffeomorphism-covariant purely gravitational theory*

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Abstract: Thermodynamics plays an important role in gravitational theories. It is a principle that is independent of gravitational dynamics, and there is still no rigorous proof to show that it is consistent with the dynamical principle. We consider a self-gravitating perfect fluid system with the general diffeomorphism-covariant purely gravitational theory. Based on the Noether charge method proposed by Iyer and Wald, considering static off/on-shell variational configurations, which satisfy the gravitational constraint equation, we rigorously prove that the extrema of the total entropy of a perfect fluid inside a compact region for a fixed total particle number demands that the static configuration is an on-shell solution after we introduce some appropriate boundary conditions, i.e., it also satisfies the spatial gravitational equations. This means that the entropy principle of the fluid stores the same information as the gravitational equation in a static configuration. Our proof is universal and holds for any diffeomorphism-covariant purely gravitational theories, such as Einstein gravity, $f(R)$ gravity, Lovelock gravity, f(Gauss-Bonnet) gravity and Einstein-Weyl gravity. Our result indicates the consistency between ordinary thermodynamics and gravitational dynamics.

Keywords: entropy principle, self-gravitating perfect fluid, thermodynamics

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I. INTRODUCTION

Black holes became important and fundamental objects in gravitational theories after the proposal of general relativity. In the past few decades, many studies on general relativity have implied that black holes can be regarded as thermodynamical systems. The four laws of black hole mechanics in general relativity have been constructed in Refs. [1-3]. The discovery of Hawking radiation provided a natural interpretation of the laws of black hole mechanics as ordinary laws of thermodynamics [4-6]. Since then, black hole thermodynamics has attracted considerable attention, and researchers believe that they can provide us with a deeper understanding of gravitational theories.

It is generally believed that the gravitational equation is the basic equation of nature. Therefore, traditionally, people always study how to construct the laws of thermodynamics based on gravitational dynamics. For instance, Wald generally derived the first law of black hole thermodynamics for the general diffeomorphism-covariant gravitational theory based on the Noether charge method [7, 8], and Wall also discussed the second law for general higher curvature gravity [9]. However, some people believe that the thermodynamical relations are the more fundamental assumption, and that the gravitational equations should be derived from thermodynamics. From this point of view, Jacobson considered that the gravitational equations are the equations of state, and the Einstein equation can be derived from the first thermodynamic law on the local Rindler horizons [10]. This idea has been accepted by an increasing number of people in the past few years [11-13]. All these discussions show the consistency between gravitational thermodynamics and gravitational dynamics.

In spacetime without an event horizon, there also exists some matter field, such as a self-gravitating perfect fluid, which satisfies the ordinary thermodynamical laws. In contrast to black hole thermodynamics, the local thermodynamical quantities of this fluid, such as entropy

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density s, energy density ρ, and local temperature T, are well defined. From the viewpoint of gravitational dynamics, the on-shell static distribution of these local quantities for perfect fluids can be determined by the gravitational equations. Moreover, from the viewpoint of thermodynamics, as a static self-gravitating fluid can be regarded as a thermodynamical system, the on-shell distribution can also be obtained with the extrema of the total entropy of the perfect fluid in the off/on-shell static variation. In general, thermodynamics of a fluid in spacetime and gravitational dynamics are two independent principles. However, if we believe that both of them are reliable, they should give the same distribution of the fluid. Therefore, Cocke proposed the entropy principle for a self-gravitating fluid. It states that under a few natural conditions, the extrema of total entropy of a perfect fluid are equivalent to the Einstein equation of a static self-gravitating fluid system [14]. For the spherical radiation system, Sorkin, Wald, and Zhang have shown that the Tolman-Oppenheimer-Volkoff equation can be derived from the Einstein constraint equation (time-time component of the gravitational equations) with the extrema of the total entropy of a perfect fluid inside a compact region [15]. Gao extended their work to an arbitrary perfect fluid in a static spherical geometry [16], for which they only used some thermodynamical relations. After that, this principle has been widely studied by researchers [17-22]. Their results show consistency between ordinary thermodynamics of fluids and gravitational dynamics.

Most recently, the entropy principle was generally proved in static spacetime without spherical symmetry for Einstein-Maxwell gravity [23, 24], Lovelock gravity [25], and $f(R)$ gravity [26]. Although all of them showed the validity of the entropy principle, there is still a lack of a general proof to show that it is valid for all gravitational theories. When string effects or quantum corrections are taken into account, the effective gravitational action should be corrected by the powers of the curvature tensor and its derivatives, and the Einstein-Hilbert action is just the first term of the effective action [27-32]. Moreover, there are some studies in literature in which the Einstein gravity was modified by introducing higher-curvature corrections, such as the $f(R)$ term and f(Gauss-Bonnet) term, to deal with problems in cosmology and astrophysics [33-35]. Thus, it is natural for us to ask whether the entropy principle is satisfied in these modified gravitational theories. We can see that all of these modified theories can be described by a diffeomorphism-invariant action. Therefore, in the following, we would like to prove the equivalence of the extrema of the total entropy of a perfect fluid inside a compact region in the off/on-shell static configurations and the gravitational equations for a general diffeomorphism-covariant purely gravitational theory in which the Lagrangian is constructed by a metric and its derivatives only. To prove equivalence, we first present a theorem related to the entropy principle:

Theorem: Consider a variation related to a one-parameter family of the off/on-shell static configurations which satisfy the gravitational constraint equation (the off-shell configuration does not obey the spatial gravitational equation) in n-dimensional spacetime for a general diffeomorphism-covariant purely gravitational theory coupled to a self-gravitating perfect fluid. Denote the $(n-1)$-dimensional hypersurface Σ to be a moment for static observers. Choose C to be a compact region inside the hypersurface Σ with boundary ∂C. Assume that the fluid velocity coincides with the static Killing vector and that the local temperature T of the fluid obeys Tolman's law. After introducing the appropriate boundary conditions, which keep the quasi-local conserved charge of the static Killing vector inside C fixed, the extrema of the total entropy of the perfect fluid inside C for a fixed total particle number in this off/on-shell variation demand that the static configuration is an on-shell solution (i.e., it also satisfies the spatial gravitational equations).

Our paper is organized as follows. In Sec. II, we discuss dynamical and thermodynamical features of the off/on-shell static configuration for a general diffeomorphism-covariant gravitational theory. In Sec. III, we prove the **Theorem** of the entropy principle based on the Noether charge method proposed by Iyer and Wald [8]. Finally, the conclusions are presented in Sec. IV.

II. STATIC CONFIGURATION OF A SELF-GRAVITATING PERFECT FLUID

In this section, we start by discussing the properties of the static configuration in a general n-dimensional diffeomorphism-covariant purely gravitational theory sourced by a self-gravitating perfect fluid. As mentioned above, we consider a static configuration in n-dimensional spacetime. In this situation, there exists a static Killing vector field ξ^a satisfying $\mathcal{L}_\xi g_{ab} = \nabla_{(a}\xi_{b)} = 0$. The integral curves of ξ^a are the worldlines of static observers in spacetime. The velocity vector field is given by $u^a = \chi^{-1}\xi^a$, in which $\chi = \sqrt{-\xi^a\xi_a}$ is the red-shift factor for the static observers and u^a is also the normal vector to Σ. The induced metric h^{ab} in Σ is given by

$$h^{ab} = g^{ab} + \chi^{-2}\xi^a\xi^b\,. \tag{1}$$

The Lagrangian n-form in this theory is given by

$$\boldsymbol{L} = \boldsymbol{L}_{\rm grav} + \boldsymbol{L}_{\rm fluid}\,, \tag{2}$$

in which $\boldsymbol{L}_{\rm grav}$ and $\boldsymbol{L}_{\rm mt}$ are the gravitational part and fluid part of the Lagrangian, respectively. The gravitational part of the Lagrangian is a function of the metric g_{ab}, Riemann tensor R_{bcde} and its higher-order derivative

$\nabla_{a_1}\cdots\nabla_{a_k}R_{bcde}$. Using the relationship

$$2\nabla_{[a}\nabla_{b]}T_{c_1\cdots c_k}=\sum_{i=1}^{i=k}R_{abc_i}{}^{d}T_{c_1\cdots d\cdots c_k} \tag{3}$$

to exchange the indices of the derivative operators, it is not hard to verify that the Lagrangian $\boldsymbol{L}$ can be reexpressed as

$$\boldsymbol{L}_{\rm grav}=\boldsymbol{\epsilon}\mathcal{L}_{\rm grav}(g_{ab},R_{bcde},\cdots,\nabla_{(a_1}\cdots\nabla_{a_k)}R_{bcde},\cdots), \tag{4}$$

This is exactly the expression of the Lagrangian considered in [8] for the general diffeomorphism-covariant gravitational theory. In our discussion, we use boldface symbols to denote the differential forms in spacetime. The gravitational equations are given by variations of the Lagrangian and we can express them by [8]

$$\mathcal{E}_{ab}=H_{ab}-T_{ab} \tag{5}$$

with

$$H_{ab}=A_{ab}+P_{acde}R_b{}^{cde}-2\nabla^c\nabla^dP_{acdb}-\frac{1}{2}g_{ab}\mathcal{L}_{\rm grav} \tag{6}$$

and the stress-energy tensor of the perfect fluid

$$T_{ab}=\rho u_au_b+p(g_{ab}+u_au_b), \tag{7}$$

in which ρ and p are the energy density and pressure of the fluid, respectively, and we have denoted

$$A_{ab}=\frac{\partial\mathcal{L}_{\rm grav}}{\partial g^{ab}},\quad P^{abcd}=\frac{\delta\mathcal{L}_{\rm grav}}{\delta R_{abcd}}. \tag{8}$$

The expression of the stress-energy tensor implies that static observers are also comoving observers of the fluid.

The **Theorem** of the entropy principle shows that we need to consider an off/on-shell static configuration which satisfies the gravitational constraint equation (time-time component of the equation of motion)

$$\rho=H_{uu}=H_{ab}u^au^b. \tag{9}$$

Therefore, “off-shell” refers specifically to the off-shell static configuration which does not obey the spatial gravitational equations

$$ph_{ab}\neq\hat{H}_{ab}=h_{ac}h_{bd}H^{cd}. \tag{10}$$

In the following, we consider the thermodynamics of the self-gravitating perfect fluid which satisfies Tolman's law $T\chi=T_0$ in a static configuration, where T_0 is a constant and can be regarded as the red-shift temperature. Without loss of generality, we shall set $T_0=1$, such that

$$T=\chi^{-1}. \tag{11}$$

The entropy density s is a function of the energy density ρ and the particle number density n, i.e., $s=s(\rho,n)$. From the familiar first law for region C, one can derive the local first law and the Gibbs-Duhem relation of the fluid [16],

$$\begin{aligned}\mathrm{d}\rho&=T\mathrm{d}s+\mu\mathrm{d}n,\\ \rho&=Ts-p+\mu n,\end{aligned} \tag{12}$$

where μ is the chemical potential corresponding to the particle number density n. From the local first law, we can see that the entropy of a perfect fluid can be treated as a function of the energy density ρ and particle number density n, i.e., $s=s(\rho,n)$. The conservation law $\nabla_aT^{ab}=0$ for a perfect fluid results in

$$\mathrm{d}p+\chi^{-1}(\rho+p)\mathrm{d}\chi=0. \tag{13}$$

Together with the local first law and the Gibbs-Duhem relation in Eq. (12), we can further obtain the result that

$$\mu\chi=\text{constant}. \tag{14}$$

III. NOETHER CHARGE METHOD AND PROOF OF THE ENTROPY PRINCIPLE

In this section, we would like to prove the entropy principle based on the Neother charge method proposed by Iyer and Wald [8]. We consider a one-family $\phi(\lambda)$ of the off/on-shell static field configurations, as described in the previous section, in which we denote $\phi(\lambda)$ to metric $g_{ab}(\lambda)$ and the shelf-gravitating perfect fluid with $\rho(\lambda),p(\lambda)$. That is to say, $\phi(\lambda)$ satisfies the thermodynamical properties of the fluid, as described in the previous section, as well as the gravitational constraint equation

$$H_{uu}(\lambda)=\rho(\lambda). \tag{15}$$

For the off-shell configuration $\phi(\lambda)$, the spatial gravitational equations are not satisfied, i.e.,

$$\hat{H}_{ab}(\lambda)\neq p(\lambda)h_{ab}(\lambda). \tag{16}$$

In the following, we will define the notations

$$\chi = \chi(0), \qquad \delta\chi = \frac{\partial\chi}{\partial\lambda}\Big|_{\lambda=0} \tag{17}$$

to denote the background quantity and its variation in the family $\phi(\lambda)$. Considering the diffeomorphism covariance of the theory, we can choose a gauge to fix the static Killing vector $\xi^a(\lambda)$ under the variation in the static configuration $\phi(\lambda)$, i.e., $\delta\xi^a = 0$. For each static configuration $\phi(\lambda)$, we have

$$g^{ab}(\lambda) = -\chi(\lambda)^{-2}\xi^a\xi^b + h^{ab}(\lambda). \tag{18}$$

Then, we have

$$\delta g^{ab} = 2\chi^{-3}\delta\chi\xi^a\xi^b + \delta h^{ab}. \tag{19}$$

In this family, the variation of the gravitational part of the Lagrangian gives

$$\delta \boldsymbol{L}_{\text{grav}} = \boldsymbol{E}^{\text{grav}}_{ab}\delta g^{ab} + d\boldsymbol{\Theta}^{\text{grav}}(g,\delta g), \tag{20}$$

in which

$$\boldsymbol{E}^{\text{grav}}_{ab} = \frac{1}{2}\boldsymbol{\epsilon}H_{ab}, \tag{21}$$

denotes the gravitational part of the equation of motion, and $\boldsymbol{\Theta}^{\text{grav}}(g,\delta g)$ is the symplectic potential. After completing all indices, Eq. (21) is expressed as $(E^{\text{grav}}_{ab})_{a_1\cdots a_n} = \epsilon_{a_1\cdots a_n}H_{ab}$.

For any vector field ζ^a, we can define the Noether current $(n-1)$-form as

$$\boldsymbol{J}^{\text{grav}}_{\zeta} = \boldsymbol{\Theta}^{\text{grav}}(g,\mathcal{L}_\zeta g) - \zeta\cdot\boldsymbol{L}_{\text{grav}}. \tag{22}$$

It has been shown in [8] that it can be expressed as

$$\boldsymbol{J}^{\text{grav}}_{\zeta} = \boldsymbol{C}^{\text{grav}}_{\zeta} + d\boldsymbol{Q}^{\text{grav}}_{\zeta}, \tag{23}$$

in which $C^{\text{grav}}_{\zeta} = \zeta\cdot\boldsymbol{C}^{\text{grav}}$ with

$$\boldsymbol{C}^{\text{grav}}_{aa_1\cdots a_{n-1}} = \boldsymbol{\epsilon}_{ba_1\cdots a_{n-1}}H_a{}^b \tag{24}$$

is the constraint of the gravitational theory and $\boldsymbol{Q}^{\text{grav}}_{\zeta}$ is a Neother charge $(n-2)$-form of the vector field ζ^a.

Using the two expressions (22) and (23) of the Noether current, we have

$$\delta\boldsymbol{C}^{\text{grav}}_{\zeta} + \zeta\cdot\boldsymbol{E}^{\text{grav}}_{ab}\delta g^{ab} = d\left[\zeta\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g) - \delta\boldsymbol{Q}^{\text{grav}}_{\zeta}\right] + \boldsymbol{\omega}^{\text{grav}}(g,\delta g,\mathcal{L}_\zeta g) \tag{25}$$

where

$$\omega^{\text{grav}}(g,\delta_1 g,\delta_2 g) = \delta_1\boldsymbol{\Theta}^{\text{grav}}(g,\delta_2 g) - \delta_2\boldsymbol{\Theta}^{\text{grav}}(g,\delta_1 g) \tag{26}$$

is the symplectic current $(n-1)$-form.

After replacing ζ^a by ξ^a and noting that the configuration is static, such that $\mathcal{L}_\xi g_{ab} = 0$, we have $\omega(g,\delta g,\mathcal{L}_\xi g) = 0$. Then, integration of Eq. (25) with respect to C yields

$$\int_C \delta\boldsymbol{C}^{\text{grav}}_{\xi} + \int_C \xi\cdot\boldsymbol{E}^{\text{grav}}_{ab}\delta g^{ab} = \int_{\partial C}\left[\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g) - \delta\boldsymbol{Q}^{\text{grav}}_{\xi}\right]. \tag{27}$$

For the first term in Eq. (27), using Eqs. (24) and (15), we have

$$\int_C \delta\boldsymbol{C}^{\text{grav}}_{\xi} = -\int_C \delta(\chi\tilde{\boldsymbol{\epsilon}}H_{uu}) = -\int_C \delta(\chi\tilde{\boldsymbol{\epsilon}}\rho), \tag{28}$$

where we have denoted $\tilde{\boldsymbol{\epsilon}}(\lambda)$ as the volume element of Σ in the static configuration $\phi(\lambda)$. Substituting Eq. (19) into the second term in Eq. (27), we have

$$\int_C \xi\cdot\boldsymbol{E}^{\text{grav}}_{ab}\delta g^{ab} = \int_C \tilde{\boldsymbol{\epsilon}}H_{uu}\delta\chi + \frac{1}{2}\int_C \chi\tilde{\boldsymbol{\epsilon}}H_{ab}\delta h^{ab} = \int_C \tilde{\boldsymbol{\epsilon}}\rho\delta\chi + \frac{1}{2}\int_C \chi\tilde{\boldsymbol{\epsilon}}H_{ab}\delta h^{ab}. \tag{29}$$

Combining the above results, we can further obtain

$$\frac{1}{2}\int_C \chi\tilde{\boldsymbol{\epsilon}}H_{ab}\delta h^{ab} - \int_C \chi\delta(\tilde{\boldsymbol{\epsilon}}\rho) = \int_{\partial C}\left[\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g) - \delta\boldsymbol{Q}^{\text{grav}}_{\xi}\right]. \tag{30}$$

For the off-shell configuration $\phi = \phi(0)$, the first term of the left side in Eq. (30) is not equal to ph_{ab}.

Next, we would like to evaluate the variation in the total entropy of a perfect fluid inside C when the total particle number is fixed. The total entropy of a perfect fluid is given by

$$S = \int_C \tilde{\boldsymbol{\epsilon}}s(\rho,n). \tag{31}$$

The variation of the total entropy yields

$$\delta S = \int_C\left[s\delta\tilde{\boldsymbol{\epsilon}} + \left(\frac{\partial s}{\partial\rho}\delta\rho + \frac{\partial S}{\partial n}\delta n\right)\tilde{\boldsymbol{\epsilon}}\right]. \tag{32}$$

From the local first law $\mathrm{d}s=\chi\mathrm{d}\rho-\chi\mu\mathrm{d}n$, we have

$$\frac{\partial s}{\partial\rho}=\chi,\quad \frac{\partial s}{\partial n}=-\chi\mu. \tag{33}$$

Then, Eq. (32) becomes

$$\delta S=\int_C[s\delta\tilde{\boldsymbol{\epsilon}}+\chi(\delta\rho-\mu\delta n)\tilde{\boldsymbol{\epsilon}}]. \tag{34}$$

From the assumption that the total number of particles

$$N(\lambda)=\int_C\tilde{\boldsymbol{\epsilon}}(\lambda)n(\lambda) \tag{35}$$

are fixed inside C, we have

$$\int_C n\delta\tilde{\boldsymbol{\epsilon}}=-\int_C\tilde{\boldsymbol{\epsilon}}\delta n. \tag{36}$$

Together with the fact that $\mu\chi=\text{constant}$, Eq. (34) reduces to

$$\begin{aligned}\delta S&=\int_C\left[(s+\chi\mu n)\delta\tilde{\boldsymbol{\epsilon}}+\tilde{\boldsymbol{\epsilon}}\chi\delta\rho\right]\\&=\int_C\left[\chi\delta(\tilde{\boldsymbol{\epsilon}}\rho)-\frac{1}{2}\tilde{\boldsymbol{\epsilon}}\chi p h_{ab}\delta h^{ab}\right]\\&=\int_C\tilde{\boldsymbol{\epsilon}}\frac{\chi}{2}(\hat{H}_{ab}-ph_{ab})\delta h^{ab}+\int_{\partial C}\left[\delta\boldsymbol{Q}_\xi^{\text{grav}}-\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g)\right].\end{aligned} \tag{37}$$

In the second step, we used the Gibbs-Duhem relation in Eq. (12) and $\delta\tilde{\boldsymbol{\epsilon}}=-(1/2)\tilde{\boldsymbol{\epsilon}}h_{ab}\delta h^{ab}$. In the last step, we used the variational identity in Eq. (30). The second part of the righthand side of Eq. (37) is only a boundary quantity for ∂C. Indeed, this boundary term corresponds to the variations in the quasi-local conserved charge corresponding to the static Killing vector ξ^a in the enclosed region C, which is defined as

$$Q(\xi)=\int_{\partial C}\left[\Delta\boldsymbol{Q}_\xi^{\text{grav}}-\xi\cdot\int_{\lambda_0}^{\lambda}\mathrm{d}\lambda\boldsymbol{\Theta}^{\text{grav}}(g(\lambda),g'(\lambda))\right], \tag{38}$$

with

$$\Delta\boldsymbol{Q}_\xi^{\text{grav}}=\boldsymbol{Q}_\xi^{\text{grav}}(g(\lambda))-\boldsymbol{Q}_\xi^{\text{grav}}(g(\lambda_0)). \tag{39}$$

in which $\phi(\lambda_0)$ of the one-parameter family $\phi(\lambda)$ is a vacuum solution in the diffeomorphism-covariant purely gravitational theory [36]. Then, neglecting this term amounts to the variations in the quasi-local conserved charge vanishing, i.e.,

$$\delta Q(\xi)=\int_{\partial C}\left[\delta\boldsymbol{Q}_\xi^{\text{grav}}-\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g)\right]=0. \tag{40}$$

For the Einstein gravity in the static spherically symmetric spacetime [16] with the line element

$$\mathrm{d}s^2=g_{tt}\mathrm{d}t^2+\left(1-\frac{2m(r)}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2. \tag{41}$$

Let C be in the compact region $r\leqslant R$. With a straightforward calculation, $\delta Q(\xi)=0$ implies that the total mass $M=m(R)$ within R is fixed at the boundary $r=R$. The boundary condition for fixing the quasi-local conserved charge $Q(\xi)$ is dependent on the explicit theories considered. For instance, in Einstein gravity or Lovelock gravity, the induced metric and its derivative at the boundary ∂C need to be fixed [16, 25]; in $f(R)$ gravity, we need to fix the induced metric as well as the scalar curvature R and its derivative at the boundary ∂C [26].

For a usual thermodynamic system, the entropy principle is satisfied when the system is isolated. However, for self-gravitating cases, the phrase "isolated system" becomes more ambiguous as the gravitational theory is diffeomorphism invariant. For a quasi-local system, i.e., C is a finite compact region, and the boundary condition of the isolated system should be quasi-locally imposed. By analogy to the usual cases, we should impose a boundary condition such that the variation inside the compact region C does not affect the dynamics outside C, i.e., the variation in spacetime will not affect the on-shell solution without the fluid outside region C. That is to say, for any element of the one-parameter, their geometries outside C only differ by a diffeomorphism. Under the above condition, using the on-shell variational identity (25) outside C, it is easy to determine

$$\begin{aligned}\delta Q(\xi)&=\int_{\partial C}\left[\delta\boldsymbol{Q}_\xi^{\text{grav}}-\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g)\right]\\&=\int_{\infty}\left[\delta\boldsymbol{Q}_\xi^{\text{grav}}-\xi\cdot\boldsymbol{\Theta}^{\text{grav}}(g,\delta g)\right],\end{aligned} \tag{42}$$

where "∞" denotes a $(n-2)$-sphere at asymptotical infinity. If the spacetime is asymptotically flat, $Q(\xi)$ can be regarded as the mass M of spacetime. As we assume that the variation inside the isolated region will not affect the on-shell geometry outside, it is natural to impose a condition such that the total mass of the spacetime is fixed under the variation, i.e., we have $\delta Q(\xi)=0$. Then, the second part of the righthand side of Eq. (37) vanishes and we have

$$\delta S=\frac{1}{2}\int_C\tilde{\boldsymbol{\epsilon}}\chi(\hat{H}_{ab}-ph_{ab})\delta h^{ab}. \tag{43}$$

From the above result, we show that for the off-shell static configuration $\hat{H}_{ab} \neq ph_{ab}$, the variation in the total entropy is nonvanishing. In other words, the extrema of the total entropy of a perfect fluid inside an isolated region C for a fixed total particle number demand that the static configuration is an on-shell solution. This completes the proof of the **Theorem** of the entropy principle.

IV. CONCLUSION

Our proof shows the equivalence of the extrema of the total entropy of a perfect fluid in the off/on-shell static configurations inside a compact region C and the dynamical equations of a static self-gravitating perfect fluid for a general diffeomorphism-covariant gravitational theory, although they are derived from two different and independent principles. The result is universal and suitable for any diffeomorphism-covariant purely gravitational theories only imposing the static condition of spacetime. Our work provides strong evidence to show that as two independent basic principles, ordinary thermodynamics and dynamics in gravitational theories are consistent.

It is worth noting that our result is only valid for purely gravitational theories minimally coupled to the self-gravitating perfect fluid. Is it still possible to extend the entropy principle to some more general diffeomorphism-covariant theories, for example, when there are non-minimal coupling interactions between matter and gravity. For these cases, we first need to derive the concrete expressions of the Neother charge $\boldsymbol{Q}_\xi$ and constraint $\boldsymbol{C}_\xi$, and discuss the interactions between the self-gravitating fluid and these non-minimally coupling matters. This is an interesting question and needs careful investigation in the future.

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