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Thermodynamic extremality relations in massive gravity *

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Abstract: A universal relation between the leading correction to the entropy and extremality was proposed in the work of Goon and Penco. In this paper, we extend this work to massive gravity and investigate thermodynamic extremality relations in a topologically higher-dimensional black hole. A rescaled cosmological constant is added to the action of the massive gravity as a perturbative correction. This correction modifies the extremality bound of the black hole and leads to shifts in the mass, entropy, etc. Regarding the cosmological constant as a variable related to pressure, we obtain the thermodynamic extremality relations between the mass and entropy, pressure, charge, and parameters c_i by accurate calculations. Finally, these relations are verified by a triple product identity, which shows that the universal relation exists in black holes.

Keywords: thermodynamic extremality relations, massive gravity, perturbative corrections

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I. INTRODUCTION

The string landscapes formed by effective quantum field theories are broad and complex. However, there are some theories that appear to be self-consistent but are not compatible with string theory. Thus, the swampland program was proposed [1-4]. Its aim is to find the subset of infinite space in effective field theories that arises at low energies from quantum gravity theories with specific constraints. These constraints were first proposed in [1]. As one of the constraints, the weak gravity conjecture (WGC) has attracted much attention. It asserts that, for the lightest charged particle along the direction of a basis vector in charge space, the charge-to-mass ratio is larger than those for extremal black holes [2]. This conjecture shows that extremal black holes are allowed to decay.

A proof of the WGC is that it is mathematically equivalent to a certain property of a black hole entropy. In [5], the authors introduced the higher-derivative operators to the action to compute the shift in the entropy. Using these operators, the extremality condition of the black hole is modified, and the mass and entropy are shifted. These authors derived the relation between the ratio of charge-to-mass and the entropy shift, $q/m-1\propto\Delta S$, where $\Delta S>0$. The charge-to-mass ratio approaches unity asymptotically with increasing mass. Thus, a large extremal black hole is unstable and decays to a smaller extremal black hole with charge-to-mass ratios greater than unity. This phenomenon satisfies the requirement of the WGC. Subsequently, WGC behavior was found in a four-dimensional rotating dyonic black hole and other spacetimes [6, 7]. Other studies of the WGC have been reported in [8-23]; see also the references therein.

In a recent study [24], Goon and Penco derived a universal extremality relation using perturbative corrections to the free energy of generic thermodynamic systems. This relation takes the form

$$\frac{\partial M_{\rm ext}(\vec{Q},\epsilon)}{\partial\epsilon}=\lim_{M\to M_{\rm ext}(\vec{Q},\epsilon)}-T\left(\frac{\partial S(M,\vec{Q},\epsilon)}{\partial\epsilon}\right)_{M,\vec{Q}},\tag{1}$$

where $M_{\rm ext}(\vec{Q},\epsilon)$ and $S(M,\vec{Q},\epsilon)$ are the extremality mass and entropy, respectively. Both of them are $\epsilon-$dependent, and ϵ is a control parameter for the free energy. $\vec{Q}$ are additional quantities in thermodynamic systems, other than the mass. The above relation can be interpreted as a comparison between states in the classical and corrected theories. Meanwhile, an approximation relation $\Delta M_{\rm ext}(\vec{Q})\approx -T_0(M,\vec{Q})\,\Delta S(M,\vec{Q})|_{M\approx M^0_{\rm ext}(\vec{Q})}$ was proposed, where $\Delta M_{\rm ext}(\vec{Q})$ and $\Delta S(M,\vec{Q})$ are the leading order corrections to the extremal bound and to the entropy of a state with fixed mass and $\vec{Q}$, respectively. $M^0_{\rm ext}$ is the mass in the

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extremal case without corrections. The result shows that the mass of the perturbed extremal black hole is less than that of the unperturbed one with the same quantum numbers, if $\Delta S > 0$, which implies that the perturbation decreases the mass of the extremal black hole. Therefore, WGC-like behavior exists in the extremal black hole. In particular, the Goon-Penco relation (1) was verified in an AdS-Reissner-Nordstr öm black hole by rescaling the cosmological constant as a perturbative correction. The approximation relation was also verified using higher-derivative operators introduced in the action.

To further explore the WGC behavior and the Goon-Penco relation, researchers have studied the thermodynamic corrections in specific spacetimes by introducing higher-derivative operators or perturbative parameters [25, 26]. The Goon-Penco relation was confirmed, and other extremality relations were obtained. In [25], Cremonini et al. computed the four-derivative corrections to thermodynamic quantities in the higher-dimensional AdS-Reissner-Nordström black hole and found the extremality relation between the mass and charge,

$$\lim_{T\to 0}\left(\frac{\partial M}{\partial \epsilon}\right)_{Q,T} = \lim_{T\to 0} -\Phi\left(\frac{\partial Q}{\partial \epsilon}\right)_{M,T}. \tag{2}$$

Extending this work to rotating anti-de Sitter spacetimes, Liu et al. derived the extremality relation between the mass and angular momentum in the BTZ and Kerr anti-de Sitter spacetimes [26],

$$\left(\frac{\partial M_{\rm ext}}{\partial \epsilon}\right)_{J,l} = \lim_{M\to M_{\rm ext}} -\Omega\left(\frac{\partial J}{\partial \epsilon}\right)_{M,S,l}. \tag{3}$$

Relations (2) and (3) are extensions of the Goon-Penco relation (1). These relations will shed light on theories of quantum gravity.

In this paper, we extend the work of [24] to massive gravity and investigate the extremality relations between the mass and pressure, entropy, charge, and parameters c_i of a charged topological black hole in higher-dimensional spacetime. Einstein's general relativity (GR) is a low energy effective theory. The UV completeness requires that GR be modified to meet physical descriptions in the high energy region. Massive gravity is a straightforward modification to GR. We introduce a perturbative correction by adding a rescaled cosmological constant to the action of massive gravity. This scenario is different from that in [24], where the cosmological constant was directly rescaled in the action and consistent with that in [26]. In our investigation, the cosmological constant is regarded as a variable related to pressure [27-31]. Its conjugate quantity is a thermodynamic volume. The black hole mass is naturally interpreted as an enthalpy. The first reason for this is that the cosmological constant, as a variable, can reconcile the inconsistency between the first law of thermodynamics of black holes and the Smarr relation, derived from the scaling method. The second reason is that physical constants, such as the gauge coupling constants, Newtonian constant, or cosmological constant, which are vacuum expectation values, are not fixed and vary in the more fundamental theories [32].

The rest of this paper is organized as follows. In the next section, the solution of the higher-dimensional black hole in massive gravity is given, and its thermodynamic properties are discussed. In section III, we introduce a perturbative correction to the action and derive the extremality relations between the mass and pressure, entropy, charge, and parameters c_i. Section IV is devoted to our discussion and conclusion.

II. BLACK HOLE SOLUTION IN MASSIVE GRAVITY

The action for an $(n+2)$-dimensional massive gravity is [33]

$$\mathcal{S} = \frac{1}{16\pi}\int {\rm d}x^{n+2}\sqrt{-g}\left[R + \frac{n(n+1)}{l^2} - \frac{F^2}{4} + m^2\sum_{i=1}^{4} c_i u_i(g,f)\right], \tag{4}$$

where the terms including m^2 represent the massive potential associated with the graviton mass, f is a fixed symmetric tensor called the reference metric, c_i are constants, and u_i are symmetric polynomials of the eigenvalues of the $(n+2)\times(n+2)$ matrix $\mathcal{K}^{\mu}_{\nu} = \sqrt{f^{\mu\alpha}g_{\alpha\nu}}$:

$$\begin{aligned}
u_1 =& [\mathcal{K}], \quad u_2 = [\mathcal{K}]^2 - [\mathcal{K}^2],\\
u_3 =& [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],\\
u_4 =& [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4].
\end{aligned} \tag{5}$$

The square root in $\mathcal{K}$ denotes $(\sqrt{A})^{\mu}_{\nu}(\sqrt{A})^{\nu}_{\lambda} = A^{\mu}_{\lambda}$ and $[\mathcal{K}] = \mathcal{K}^{\mu}_{\mu}$.

The solution of the charged black hole with the spacetime metric and reference metric is given by [34]

$$ {\rm d}s^2 = -f(r){\rm d}t^2 + \frac{1}{f(r)}{\rm d}r^2 + r^2 h_{ij}{\rm d}x^i{\rm d}x^j, \tag{6}$$

$$f_{\mu\nu} = {\rm diag}(0,0,c_0^2 h_{ij}), \tag{7}$$

where

$$\begin{aligned}
f(r) =& k + \frac{r^2}{l^2} - \frac{16\pi M}{n\Omega_n r^{n-1}} + \frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r^{2(n-1)}} + \frac{c_0 c_1 m^2 r}{n}\\
&+ c_0^2 c_2 m^2 + \frac{(n-1)c_0^3 c_3 m^2}{r} + \frac{(n-1)(n-2)c_0^4 c_4 m^2}{r^2},
\end{aligned} \tag{8}$$

l^2 is related to the cosmological constant Λ as $l^2=-\frac{n(n+1)}{2\Lambda}$. M and Q are the mass and charge of the M black hole, respectively. Ω_n is the volume spanned by coordinates x^i, and c_0 is a positive integral constant. $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is the line element for an Einstein space with the constant curvature $n(n-1)k$. $k=1$, 0, or -1 denotes spherical, Ricci flat, or hyperbolic topology black hole horizons, respectively. The thermodynamics in the extended phase space of massive gravity have been studied in [35-41]. The event horizon r_+ is determined by $f(r)=0$. A general formula for the Hawking temperature can be given as $T=\frac{\kappa}{2\pi}$, where $\kappa=-\frac{1}{2}\lim_{r\to r_+}\sqrt{\frac{-g^{11}}{g^{00}}}\frac{\partial\ln(-g^{00})}{\partial r}$ is the surface gravity. For this black hole, the Hawking temperature is

$$\begin{aligned}T=&\frac{f'(r_+)}{4\pi}=\frac{1}{4\pi r_+}\left[\frac{(n+1)r_+^2}{l^2}+\frac{(16\pi Q)^2}{2n\Omega_n^2r_+^{2(n-1)}}+c_0c_1m^2r_+\right.\\&+(n-1)c_0^2c_2m^2+(n-1)k+\frac{(n-1)(n-2)c_0^3c_3m^2}{r_+}\\&\left.+\frac{(n-1)(n-2)(n-3)c_0^4c_4m^2}{r_+^2}\right].\end{aligned}\tag{9}$$

The mass expressed by the horizon radius and charge is

$$\begin{aligned}M=&\frac{n\Omega_nr_+^{n-1}}{16\pi}\left[k+\frac{r_+^2}{l^2}+\frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2r_+^{2(n-1)}}\right.\\&+\frac{c_0c_1m^2r_+}{n}+c_0^2c_2m^2+\frac{(n-1)c_0^3c_3m^2}{r_+}\\&\left.+\frac{(n-1)(n-2)c_0^4c_4m^2}{r_+^2}\right].\end{aligned}\tag{10}$$

The cosmological constant was seen as a fixed constant in the past. In this paper, it is regarded as a variable related to pressure, $P=-\frac{\Lambda}{8\pi}=\frac{n(n+1)}{16\pi l^2}$, and its conjugate quantity is a thermodynamic volume V. The entropy,volume, and electric potential at the event horizon are given by

$$\begin{aligned}S=&\frac{\Omega_nr_+^n}{4},\quad V=\frac{\Omega_nr_+^{n+1}}{n+1},\\\Phi_e=&\frac{16\pi Q}{(n-1)\Omega_nr_+^{n-1}},\end{aligned}\tag{11}$$

respectively. Because of the appearance of pressure, the mass is no longer interpreted as the internal energy but as an enthalpy. c_1, c_2, c_3, and c_4 are seen as extensive parameters for the mass. Their conjugate quantities are

$$\begin{aligned}\Phi_1=&\frac{\Omega_nc_0m^2r_+^n}{16\pi},\\\Phi_2=&\frac{n\Omega_nc_0^2m^2r_+^{n-1}}{16\pi},\\\Phi_3=&\frac{n(n-1)\Omega_nc_0^3m^2r_+^{n-2}}{16\pi},\\\Phi_4=&\frac{n(n-1)(n-2)\Omega_nc_0^4m^2r_+^{n-3}}{16\pi},\end{aligned}\tag{12}$$

respectively. It is easy to verify that these thermodynamic quantities obey the first law of thermodynamics,

$$\mathrm{d}M=T\mathrm{d}S+V\mathrm{d}P+\Phi_e\mathrm{d}Q+\sum_{i=1}^4\Phi_i\mathrm{d}c_i.\tag{13}$$

When the cosmological constant is fixed, the term $V\mathrm{d}P$ disappears, and the mass is interpreted as the internal energy. When a perturbative correction is introduced, the related thermodynamic quantities are shifted, which is discussed in the next section.

III. EXTREMALITY RELATIONS IN MASSIVE GRAVITY

In this section, we derive the extremality relations between the mass and entropy, charge, pressure, and parameters c_i by adding a rescaled cosmological constant to the action as a perturbative correction. The rescaled parameter is ϵ. Here, the black hole is designated as an extremal one.

We first introduce the correction

$$\Delta\mathcal{S}=\frac{1}{16\pi}\int\mathrm{d}x^{n+2}\sqrt{-g}\frac{n(n+1)\epsilon}{l^2},\tag{14}$$

to the action (4). The corrected action is $\mathcal{S}+\Delta\mathcal{S}$. The action (4) is recovered when $\epsilon=0$. A black hole solution is obtained from the corrected action and takes the form of Eqs. (6) and (8), but there is a shift. With the correction, the Hawking temperature is also shifted, and it is given by

$$\begin{aligned}T=&\frac{1}{4\pi r_+}\left[\frac{(n+1)r_+^2\epsilon}{l^2}+\frac{(n+1)r_+^2}{l^2}+\frac{(16\pi Q)^2}{2n\Omega_n^2r_+^{2(n-1)}}+c_0c_1m^2r_+\right.\\&+(n-1)c_0^2c_2m^2+(n-1)k+\frac{(n-1)(n-2)c_0^3c_3m^2}{r_+}\\&\left.+\frac{(n-1)(n-2)(n-3)c_0^4c_4m^2}{r_+^2}\right].\end{aligned}\tag{15}$$

The corrected mass is

$$M=\frac{n\Omega_n r_+^{n-1}}{16\pi}\left[\frac{r_+^2\epsilon}{l^2}+k+\frac{r_+^2}{l^2}+\frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r_+^{2(n-1)}}+\frac{c_0c_1m^2r_+}{n}+c_0^2c_2m^2+\frac{(n-1)c_0^3c_3m^2}{r_+}+\frac{(n-1)(n-2)c_0^4c_4m^2}{r_+^2}\right], \tag{16}$$

which is a function of parameters r_+, ϵ, Q, l, c_1, c_2, c_3, and c_4. Our interest is focused on the thermodynamic extremality relation. The Hawking temperature (15) in the extremal case is zero, which leads to a solution $r_+=r_+(\epsilon)$. Inserting this solution into the above equation yields an expression regarding the mass, $M_{\rm ext}=M_{\rm ext}(\epsilon)$. Carrying out the differential on $M_{\rm ext}(\epsilon)$, we have

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4}=\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \tag{17}$$

Because the expression of the differential expressed by ϵ is very complex, we adopted the expression of r_+in the above derivation. In fact, this relation can also be derived by the following calculation. For convenience, we use c to denote all parameters Q, l, c_1, c_2, c_3, c_4, except for r_+ and ϵ. From Eq. (16), the differential of M to ϵ is obtained as follows:

$$\begin{aligned}\left(\frac{\partial M}{\partial\epsilon}\right)_c&=\left(\frac{\partial M}{\partial r_+}\right)_{c,\epsilon}\left(\frac{\partial r_+}{\partial\epsilon}\right)_c+\left(\frac{\partial M}{\partial\epsilon}\right)_{c,r_+}\\&=\left(\frac{\partial M}{\partial S}\right)_{c,\epsilon}\left(\frac{\partial S}{\partial r_+}\right)_{c,\epsilon}\left(\frac{\partial r_+}{\partial\epsilon}\right)_c+\left(\frac{\partial M}{\partial\epsilon}\right)_{c,r_+}\\&=\frac{1}{4}T\Omega_n r_+^{n-1}\left(\frac{\partial r_+}{\partial\epsilon}\right)_c+\left(\frac{\partial M}{\partial\epsilon}\right)_{c,r_+}.\end{aligned} \tag{18}$$

In the extremal case, the first term in the last line of the above equation disappears, and the mass can be rewritten as $M=M_{\rm ext}$. Therefore, Eq. (17) is readily recovered.

The entropy S, pressure P, charge Q, c_1, c_2, c_3, and c_4 are usually regarded as a complete set of extensive parameters for the mass. Their conjugate quantities can be derived from the mass and take the same form as those given in section II, except for the temperature and volume. We first verify the extremality relation between the mass and entropy.

The expression for ϵ is obtained from Eq. (16) and takes the form

$$\begin{aligned}\epsilon=&\left[\frac{16\pi M}{n\Omega_n r_+^{n+1}}-\frac{k}{r_+^2}-\frac{(16\pi Q)^2}{2n(n-1)\Omega_n^2 r_+^{2n}}-\frac{c_0c_1m^2}{nr_+}-\frac{c_0^2c_2m^2}{r_+^2}\right.\\&\left.-\frac{(n-1)c_0^3c_3m^2}{r_+^3}-\frac{(n-1)(n-2)c_0^4c_4m^2}{r_+^4}\right]l^2-1.\end{aligned} \tag{19}$$

Using the relation between the entropy and horizon radius given in Eq. (11), the above equation is a function $\epsilon(S)$, and $\dfrac{\partial r_+}{\partial S}=\dfrac{4}{n\Omega_n r_+^{n-1}}$. Carrying out the differential calculation on this function yields

$$\begin{aligned}\left(\frac{\partial\epsilon}{\partial S}\right)_{M,Q,l,c_1,c_2,c_3,c_4}=&\frac{4l^2}{n\Omega_n r_+^{n-1}}\left[-\frac{(n+1)16\pi M}{n\Omega_n r_+^{n+2}}+\frac{2k}{r_+^3}+\frac{(16\pi Q)^2}{(n-1)\Omega_n^2 r_+^{2n+1}}+\frac{c_0c_1m^2}{nr_+^2}+\frac{2c_0^2c_2m^2}{r_+^3}\right.\\&\left.+\frac{3(n-1)c_0^3c_3m^2}{r_+^4}+\frac{4(n-1)(n-2)c_0^4c_4m^2}{r_+^5}\right].\end{aligned} \tag{20}$$

To evaluate the value, we insert the expression of the mass into the above equation and obtain

$$\begin{aligned}\left(\frac{\partial\epsilon}{\partial S}\right)_{M,Q,l,c_1,c_2,c_3,c_4}=&\frac{4l^2}{n\Omega_n r_+^{n-1}}\left[-\frac{(n-1)k}{r_+^3}-\frac{(n+1)(1+\epsilon)}{r_+l^2}+\frac{(16\pi Q)^2}{2n\Omega_n^2 r_+^{2(n+1)}}-\frac{c_0c_1m^2}{r_+^2}-\frac{(n-1)c_0^2c_2m^2}{r_+^3}\right.\\&\left.-\frac{(n-1)(n-2)c_0^3c_3m^2}{r_+^4}-\frac{(n-1)(n-2)(n-3)c_0^4c_4m^2}{r_+^5}\right].\end{aligned} \tag{21}$$

Combining the inverse of the above differential with the expression of the temperature given in Eq. (15), we have

$$T\left(\frac{\partial S}{\partial\epsilon}\right)_{M,Q,l,c_1,c_2,c_3,c_4}=-\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \tag{22}$$

Compared with relation (17), it is easy to see that

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4}=\lim_{M\to M_{\rm ext}}-T\left(\frac{\partial S}{\partial\epsilon}\right)_{M,Q,l,c_1,c_2,c_3,c_4}, \tag{23}$$

where S is a function of M, Q, l, c_1, c_2, c_3, c_4, and ϵ.

Therefore, the Goon-Penco relation is verified in the higher-dimensional black hole.

In this paper, the cosmological constant is regarded as a variable related to pressure. The entropy, pressure, charge, c_1, c_2, c_3, and c_4 are usually regarded as extensive parameters for the mass. Because the entropy satisfies the thermodynamic extremality relation, it is natural to ask whether other extensive quantities also satisfy corresponding relations. The goal of the following investigation is to determine these relations. Let us first derive the extremality relation between the mass and pressure. The pressure can be expressed by the constant l^2 as $P=\frac{n(n+1)}{16\pi l^2}$. Then, $\frac{\partial P}{\partial l^2}=-\frac{16\pi l^4}{n(n+1)}$. Using Eqs. (16) and (19), we get the differential of ϵ with respect to the pressure,

$$\left(\frac{\partial\epsilon}{\partial P}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4}=\frac{-16\pi l^2(1+\epsilon)}{n(n+1)}. \tag{24}$$

The perturbation parameter ϵ exists in the above differential relation as an explicit function. The reason for this is that the perturbation correction is introduced by adding the rescaled cosmological constant to the action, and this constant is related to the pressure. Because of the shift in the mass, the thermodynamic volume is also shifted, and its expression is different from that given given in Eq. (11). The volume is

$$V=\frac{\epsilon+1}{n+1}\Omega_n r_+^{n+1}. \tag{25}$$

Using Eq. (25) and the inverse of the differential of ϵ to P yields

$$V\left(\frac{\partial P}{\partial\epsilon}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4}=-\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \tag{26}$$

Comparing the above equation with Eq. (17), we obtain the extremality relation between the mass and pressure,

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4}=\lim_{M\to M_{\rm ext}}-V\left(\frac{\partial P}{\partial\epsilon}\right)_{M,r_+,Q,c_1,c_2,c_3,c_4}, \tag{27}$$

where P is a function of M, r_+, Q, c_1, c_2, c_3, c_4, and ϵ. This relation is an extension of the Goon-Penco relation.

We continue to investigate the extremality relation between the mass and charge. The calculation process is similar. From Eq. (19), the differential of ϵ with respect to Q takes the form

$$\left(\frac{\partial\epsilon}{\partial Q}\right)_{M,r_+,l,c_1,c_2,c_3,c_4}=-\frac{(16\pi)^2Ql^2}{n(n-1)\Omega_n^2 r_+^{2n}}. \tag{28}$$

Multiplying the electric potential $\Phi_e=\frac{16\pi Q}{(n-1)\Omega_n r_+^{n-1}}$ by the inverse of the above differential yields

$$\Phi_e\left(\frac{\partial Q}{\partial\epsilon}\right)_{M,r_+,l,c_1,c_2,c_3,c_4}=-\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \tag{29}$$

Obviously, there is a minus sign difference between Eqs. (17) and (29). Therefore,

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4}=\lim_{M\to M_{\rm ext}}-\Phi_e\left(\frac{\partial Q}{\partial\epsilon}\right)_{M,r_+,l,c_1,c_2,c_3,c_4}, \tag{30}$$

which is the extremality relation between the mass and charge. Now, Q is a function of M, r_+, l, c_1, c_2, c_3, c_4, and ϵ. This relation is also an extension of the Goon-Penco relation.

For the extremality relations between the mass and parameters c_1, c_2, c_3, and c_4, the calculations are parallel. Their differential relations are

$$\left(\frac{\partial\epsilon}{\partial c_1}\right)_{M,r_+,Q,l,c_2,c_3,c_4}=-\frac{c_0m^2l^2}{nr_+}, \tag{31}$$

$$\left(\frac{\partial\epsilon}{\partial c_2}\right)_{M,r_+,Q,l,c_1,c_3,c_4}=-\frac{c_0^2m^2l^2}{r_+^2}, \tag{32}$$

$$\left(\frac{\partial\epsilon}{\partial c_3}\right)_{M,r_+,Q,l,c_1,c_2,c_4}=-\frac{(n-1)c_0^3m^2l^2}{r_+^3}, \tag{33}$$

$$\left(\frac{\partial\epsilon}{\partial c_4}\right)_{M,r_+,Q,l,c_1,c_2,c_3}=-\frac{(n-1)(n-2)c_0m^2l^2}{r_+^4}. \tag{34}$$

The conjugate quantities of c_1, c_2, c_3, and c_4 are

$$\Phi_1=\frac{\Omega_n c_0m^2r_+^n}{16\pi},$$

$$\Phi_2=\frac{n\Omega_n c_0^2m^2r_+^{n-1}}{16\pi},$$

$$\Phi_3=\frac{n(n-1)\Omega_n c_0^3m^2r_+^{n-2}}{16\pi},$$

$$\Phi_4=\frac{n(n-1)(n-2)\Omega_n c_0^4m^2r_+^{n-3}}{16\pi},$$

respectively. Using these quantities, it is not difficult to obtain

$$\left(\Phi_1\frac{\partial c_1}{\partial\epsilon}\right)_{M,r_+,Q,l,c_2,c_3,c_4}=\left(\Phi_2\frac{\partial c_2}{\partial\epsilon}\right)_{M,r_+,Q,l,c_1,c_3,c_4}=\left(\Phi_3\frac{\partial c_3}{\partial\epsilon}\right)_{M,r_+,Q,l,c_1,c_2,c_4}=\left(\Phi_4\frac{\partial c_4}{\partial\epsilon}\right)_{M,r_+,Q,l,c_1,c_2,c_3}=-\frac{n\Omega_n r_+^{n+1}}{16\pi l^2}. \tag{35}$$

Thus, the extremality relations between the mass and extensive parameters c_i are

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{Q,l,c_1,c_2,c_3,c_4}=\lim_{M\to M_{\rm ext}}-\Phi_i\left(\frac{\partial c_i}{\partial\epsilon}\right)_{M,r_+,Q,l,c_j,c_k,c_u}, \tag{36}$$

where $i,j,k,u=1,2,3,4$, and $i\neq j\neq k\neq u$. Therefore, the Goon-Penco relation is extended to the case of the extensive parameters c_i of the higher-dimensional black hole.

In the above investigation, the thermodynamic extremality relations between the mass and entropy, pressure, charge, and parameters c_i were obtained by accurate calculations. They are expressed as Eqs. (22), (27), (30), and (36), respectively. The values of these relations are equal. In fact, these relations can be derived uniformly using the triple product identity

$$\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T}\left(\frac{\partial X^i}{\partial\epsilon}\right)_{M,T}\left(\frac{\partial\epsilon}{\partial M}\right)_{T,X^i}=-1, \tag{37}$$

which yields

$$\left(\frac{\partial M}{\partial\epsilon}\right)_{T,X^i}=-\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T}\left(\frac{\partial X^i}{\partial\epsilon}\right)_{M,T}=-\Phi_i\left(\frac{\partial X^i}{\partial\epsilon}\right)_{M,T}. \tag{38}$$

In the above derivation, $\left(\frac{\partial M}{\partial X^i}\right)_{\epsilon,T}$ were identified as Φ_i, which are the conjugate quantities of X^i. Here, X^i are chosen as S, Q, P, c_1, c_2, c_3, and c_4. M and T are the corrected mass and temperature given in (16) and (15), respectively. In the extremal case, $T\to 0$ and $M\to M_{\rm ext}$. The above relation becomes

$$\left(\frac{\partial M_{\rm ext}}{\partial\epsilon}\right)_{M,X^i}=\lim_{M\to M_{\rm ext}}-\Phi_i\left(\frac{\partial X^i}{\partial\epsilon}\right)_{M,X^j}, \tag{39}$$

where $X^i\neq X^j$, and X^j are parameters S, Q, P, c_1, c_2, c_3, or c_4, except for X^i. This relation implies that the universal extremality relation exists in black holes. The relation (39) is easily reduced to (22), (27), (30), and (36) when X^i are the entropy, charge, parameters c_i, and pressure, respectively. In the calculation, because of the shift in the mass, the expression of the volume $V=\frac{\epsilon+1}{n+1}\Omega_n r_+^{n+1}$ is different from that given in Eq. (11). In [26], the authors derived the extremality relation between the mass and angular momentum in BTZ and Kerr anti-de Sitter spacetimes and suggested that a general formula of the extremality relation existed in black holes. Our result provides verification of this conjecture.

IV. CONCLUSION

In this paper, we extended the work of Goon and Penco to massive gravity and investigated the thermodynamic extremality relations in a higher-dimensional black hole. The extremality relations between the mass and pressure, entropy, charge, and parameters c_i were derived by accurate calculations. The values of these extremality relations are equal, which may be due to the first law of thermodynamics. In the calculation, the cosmological constant was treated as a variable related to pressure. A perturbative correction was introduced by adding the rescaled cosmological constant to the action, but this addition does not affect the form of the extremality relation between the mass and pressure.

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