Equal-time kinetic equations in a rotational field*

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Abstract: We investigate quantum kinetic theory for a massive fermion system under a rotational field. From the Dirac equation in rotating frame we derive the complete set of kinetic equations for the spin components of the 8- and 7-dimensional Wigner functions. While the particles are no longer on a mass shell in the general case due to the rotation–spin coupling, there are always only two independent components, which can be taken as the number and spin densities. With help from the off-shell constraint we obtain the closed transport equations for the two independent components in the classical limit and at the quantum level. The classical rotation–orbital coupling controls the dynamical evolution of the number density, but the quantum rotation–spin coupling explicitly changes the spin density.

Keywords: equal-time transport, rotation field, kinetic theory

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I. INDRODUCTION

Classical transport equations are often used to study the dynamical evolution of multi-particle systems in phase space. In high energy nuclear physics, they, together with the hydrodynamic equations which are momentum moments of the transport equations and statistical approaches which are the equilibrium limits of the transport equations, successfully describe the non-equilibrium properties, space-time evolution and equilibrium distributions of the particles created in heavy ion collisions [1]. Besides the classical motion, quantum anomalous transport is also widely investigated in many fields such as astrophysics [2] and condensed matter physics [3]. In high energy nuclear collisions, the chiral magnetic effect [4, 5] and chiral vortical effect [6-9] induced by the spin of chiral fermions in electromagnetic fields and rotational fields have recently been deeply studied, both experimentally [10, 11] and theoretically [12–17]. Different from the electromagnetic field which rapidly decays in time, the angular momentum conservation during the evolution of the collisions may lead to a more visible rotational effect on the final state particles. The other advantage of the rotation is that it may become stronger in intermediate energy nuclear collisions at high baryon density due to the stopping power effect [18, 19]. In this paper, we aim to derive a group of quantum kinetic equations which can be directly solved as an initial value problem in applications.

The vortical field $\boldsymbol{\omega}$ of a system can be either generated self-consistently by the curl of the medium velocity $\boldsymbol{\omega}=\nabla\times\boldsymbol{v}$ or considered as an external field, depending on the particles we describe in the kinetic equations. For light quarks which are constituents of the medium, the quark vorticity is just the rotation of the medium, but for heavy flavors which are considered as a probe of the medium, the vorticity in kinetic equations can be treated as an external field. In this paper we consider the latter. We will neglect the collision terms among particles, in order to focus on the coupling between particles and the external rotational field. This means that we treat the particles quantum mechanically but use classical approximations for the field.

In the general case, a moving particle in a medium is not on the mass shell due to the interaction with the surrounding constituents, especially for a massive particle [20, 21]. Considering this off-shell effect, the Wigner function $W(x,p)$ defined in 8-dimensional phase space is not directly related to physical distributions which are controlled by the equal-time Wigner function $W(x,\mathbf{p})$ defined in 7-dimensional phase space [22]. Therefore, one should consider the equal-time hierarchy constructed by the energy moments of the 8-dimensional Wigner function [23, 24]. We will calculate such equal-time

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quantum kinetic equations for fermions in an external rotational field.

The paper is organized as follows. We review the Dirac equation and its non-relativistic limit in a rotational field in Section II, and then derive the kinetic equations for the 8-dimensional Wigner function $W(x,p)$ and its spin components in Section III. By taking the energy integration of the 8-dimensional kinetic equations we obtain the constraint and transport equations for the spin components of the equal-time Wigner function $W_0(x,\boldsymbol{p})$ in Section IV. By taking a semi-classical expansion, our main result on the classical and quantum transport equations for the number density and spin density is shown in Section V. We briefly discuss quarkonium polarization in heavy ion collisions in Section VI and summarize the work in Section VII.

II. DIRAC EQUATION IN A ROTATIONAL FIELD

The starting point for deriving a relativistic kinetic theory for quarks in Wigner function formalism is the Dirac equation controlled by the Lagrangian density with a rotational field. The covariant kinetic equations in curved space for chiral fermions in a rotational field are systematically calculated in Refs. [25, 26]. If we want to derive a covariant kinetic theory in flat space, we can start with the free Dirac equation. The point is then how to include the rotation in the kinetic equations; see for instances the discussion in Refs. [27, 28]. One often-used way to describe particle motion in a rotational field in textbooks is to choose the rotating frame. In this frame the system under a rotational field is equivalently regarded as a system at rest. Many new physics phenomena like the well-known Coriolis force and centrifugal force are introduced in this frame. The solution to the Dirac equation in this frame has been obtained and the chiral symmetry restoration is strongly enhanced by the rotation [29]. In this paper, we choose the rotating frame to establish the kinetic equations for quarks moving in an external rotational field. We will see very clear physics of the transport equation in this frame: the quark motion in phase space is controlled by the Coriolis force and centrifugal force. Since we have taken a specific frame, the Lorentz covariance is broken, so the obtained 8-dimensional kinetic equations are not covariant.

To avoid confusion, we use in the following the indices $\{\mu,\nu,\lambda,\sigma\}$ and $\{\alpha,\beta,\gamma,\delta\}$ to separately describe Lorentz vectors and tensors in curved and flat space, known respectively as coordinate and non-coordinate basis. The Lagrangian density for fermions under the mean-field approximation in the non-coordinate basis has the following form:

$$\mathcal{L}=\sqrt{-g}\bar{\psi}(\mathrm{i}\gamma^{\alpha}\partial_{\alpha}-m)\psi, \tag{1}$$

where $\sqrt{-g}$ is related to the coordinate we choose. Considering that in the coordinate basis the tangent space T_pM and cotangent space T_p^*M are expanded in ∂_μ and dx^μ, the coordinate transformation between the two spaces can be expressed as

$$\hat{e}_{\alpha}=e_{\alpha}^{\ \mu}\partial_{\mu},\qquad e_{\alpha}^{\ \mu}\in GL(m,\mathbb{R}), \tag{2}$$

where $\{\hat{e}_\alpha\}$ is required to be orthonormal with respect to $g\,(=g_{\mu\nu}dx^\mu\otimes dx^\nu)$, which means the relation $g(\hat{e}_\alpha,\hat{e}_\beta)=e_\alpha^{\ \mu}e_\beta^{\ \nu}g_{\mu\nu}=\eta_{\alpha\beta}$ or inversely $g_{\mu\nu}=e^\alpha_{\ \mu}e^\beta_{\ \nu}\eta_{\alpha\beta}$. With the requirement of local Lorentz invariance, the Lagrangian density in coordinate basis becomes

$$\mathcal{L}=\sqrt{-g}\bar{\psi}\left[\mathrm{i}\gamma^{\alpha}e_{\alpha}^{\ \mu}\left(\partial_{\mu}+\frac{\mathrm{i}}{2}\Xi_{\mu}^{\ \alpha\ \beta}\Sigma_{\alpha\beta}\right)-m\right]\psi \tag{3}$$

with the affine connection $\Xi_{\mu}^{\ \alpha\ \beta}=\eta^{\beta\gamma}e^{\alpha}_{\ \nu}(\partial_\mu e^{\nu}_{\ \gamma}+e_\gamma^{\ \sigma}\Gamma^{\nu}_{\ \mu\sigma})$. The Lagrangian density (3) is the starting point for deriving the covariant kinetic equations in curved space [25, 26].

We now consider a system under rotation with a constant angular velocity denoted by $\boldsymbol{\omega}$. The local velocity of this rotating frame is given by $\boldsymbol{v}=\boldsymbol{\omega}\times\boldsymbol{x}$, and the space-time metric is written as

$$g_{\mu\nu}=\begin{pmatrix}1-\boldsymbol{v}^2 & -v_1 & -v_2 & -v_3\\ -v_1 & -1 & 0 & 0\\ -v_2 & 0 & -1 & 0\\ -v_3 & 0 & 0 & -1\end{pmatrix},$$
$$g^{\mu\nu}=\begin{pmatrix}1 & -v_1 & -v_2 & -v_3\\ -v_1 & -1+v_1^2 & v_1v_2 & v_1v_3\\ -v_2 & v_1v_2 & -1+v_2^2 & v_2v_3\\ -v_3 & v_1v_3 & v_2v_3 & -1+v_3^2\end{pmatrix}, \tag{4}$$

where we have introduced a specific tetrad [29],

$$e^{\alpha}_{\ \mu}=\delta^{\alpha}_{\ \mu}+\delta^{\alpha}_{i}\delta^{0}_{\mu}v_i,\quad e_{\alpha}^{\ \mu}=\delta_{\alpha}^{\ \mu}-\delta^{0}_{\alpha}\delta^{\mu}_{i}v_i, \tag{5}$$

and $\boldsymbol{v}$ is the velocity of the coordinate transformation. It is worth noticing that the choice of the tetrad is not unique, since the degrees of freedom of a n-dimensional metric is $(n+1)n/2$ and of the tetrad n^2. After plunging the chosen tetrad into the Lagrangian, we obtain in the flat space

$$\mathcal{L}=\bar{\psi}\left[\mathrm{i}\gamma^{\mu}\partial_{\mu}+\gamma^{0}\boldsymbol{\omega}\cdot(\boldsymbol{x}\times(-\mathrm{i}\nabla)+\boldsymbol{s})-m\right]\psi \tag{6}$$

with $\boldsymbol{s}=-(1/2)\gamma^0\gamma^5\boldsymbol{\gamma}=(1/2)\mathrm{diag}(\boldsymbol{\sigma},\boldsymbol{\sigma})$. Under the choice of the space-time metric (4), the higher orders of the rotational field, namely the terms $\sim\boldsymbol{\omega}^2,\boldsymbol{\omega}^3$, vanish automatically, and only the linear term $\sim\boldsymbol{\omega}$ appears in the Lagrangian density. From the structure of the Lagrangian, the

rotational field $\boldsymbol{\omega}$ serves as a chemical potential coupled to the total angular momentum $\hat{\boldsymbol{J}} = \boldsymbol{x}\times\hat{\boldsymbol{p}} + \hat{\boldsymbol{s}}$ which is conserved during the evolution of the system.

With the known Lagrangian density it is easy to derive the Dirac equation for quarks in the rotational field,

$$\left[\mathrm{i}\gamma^\mu\partial_\mu + \gamma_0\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}} - m\right]\psi = 0, \tag{7}$$

which is exactly the same as the result discussing the inertial effect of Dirac field [30], when the acceleration is taken to be zero. The exact solution for this Dirac equation is given in [29]. The Dirac equation (7) in the rotating frame is the starting point for us to derive kinetic equations in the next section.

The Schrödinger equation for the corresponding non-relativistic system under a rotational field can be obtained by considering the limit of the Dirac equation in a standard way, as shown in Appendix A.

III. 8-DIMENSIONAL KINETIC EQUATIONS

The core ingredient in describing the transport phenomena of a non-equilibrium system is the distribution function in phase space. The Wigner function is the quantum analogue to the classical distribution function, and has been widely adopted in the investigation of quantum transport phenomena. The Wigner function $W(x,p)$ is the ensemble average of the Wigner operator, and the Wigner operator is the four-dimensional Wigner transformation of the density matrix. Without considering gauge interaction, the Wigner function for fermions is defined as [31]

$$W(x,p) = \int \frac{\mathrm{d}^4y}{(2\pi)^4}\sqrt{-g(x)}\mathrm{e}^{\mathrm{i}p\cdot y}\left\langle\psi\left(x+\frac{y}{2}\right)\bar{\psi}\left(x-\frac{y}{2}\right)\right\rangle, \tag{8}$$

where the quark field satisfies the Dirac Eq. (7).

The 8-dimensional kinetic equation in the Wigner function formalism is derived by calculating the first-order derivatives of the density matrix and using the Dirac equations for the fields ψ and $\bar{\psi}$. After a straightforward but tedious calculation, we obtain the equation of motion for the Wigner function in phase space which is equivalent to the equation of motion for the field in coordinate space,

$$\left[\gamma^\mu K_\mu + \frac{\hbar}{2}\gamma^5\gamma^\mu\omega_\mu - m\right]W(x,p) = 0, \tag{9}$$

with the definitions of $K_\mu = \Pi_\mu + (\mathrm{i}\hbar/2)D_\mu$ and $\omega_\mu = (0,\boldsymbol{\omega})$, where the extended momentum and derivative operators in phase space are defined as

$$\begin{aligned}
\Pi_\mu &= (p_0+\pi_0, \boldsymbol{p}),\\
\pi_0 &= \boldsymbol{\omega}\cdot\left(\boldsymbol{l} + \frac{\hbar^2}{4}\nabla\times\nabla_p\right) + \mu_B,\\
D_\mu &= (d_t, \nabla),\\
d_t &= \partial_t - \boldsymbol{\omega}\cdot\left(\boldsymbol{x}\times\nabla + \boldsymbol{p}\times\nabla_p\right)
\end{aligned} \tag{10}$$

with the orbital angular momentum $\boldsymbol{l} = \boldsymbol{x}\times\boldsymbol{p}$. Since the kinetic equations are valid only in the rotating frame, the rotational effect changes only the particle energy from p_0 to $p_0+\pi_0$ and time derivative from ∂_t to d_t, and the vector momentum $\boldsymbol{p}$ and space derivative ∇ are not modified. In comparison with nuclear collisions at extremely high energy, the rotational effect will become more important in heavy ion collisions at intermediate energy where the baryon density becomes high. Aiming at a kinetic theory in such case, we have included here the baryon chemical potential μ_B which shifts the particle energy. In order to semi-classically solve the kinetic equations below, we have displayed the $\hbar$-dependence explicitly. It is clear that the highest order quantum correction in the operators comes from the term $\sim\hbar^2$.

Very different from the classical distribution which is a scalar function, the Wigner function in the quantum case is a 4×4 matrix in spin space, including in the general case 16 independent components. It is convenient to choose the 16 matrices $1, \mathrm{i}\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}/2$ as the basis for an expansion of the Wigner function in spin space,

$$W = \frac{1}{4}\left(F + \mathrm{i}\gamma^5 P + \gamma^\mu V_\mu + \gamma^\mu\gamma^5 A_\mu + \frac{1}{2}\sigma^{\mu\nu}S_{\mu\nu}\right). \tag{11}$$

All the components $\Gamma_\alpha = \{F, P, V_\mu, A_\mu, S_{\mu\nu}\}$ are real functions, since the basis elements transform under hermitian conjugation like the Wigner function itself, $W^+ = \gamma_0 W\gamma_0$. The components can be interpreted as phase-space densities; their physics meanings become clear in the equal-time formalism which will be discussed below.

The expansion (11) decomposes the kinetic equation into 5 coupled equations for the 5 spinor components Γ_α. Since these components are real and the operators P_μ and D_μ are self-adjoint, one can separate the real and imaginary parts of these 5 complex equations,

$$\begin{aligned}
&2\Pi^\mu V_\mu + \hbar\omega^\mu A_\mu = 2mF,\\
&\hbar D^\mu A_\mu = 2mP,\\
&4\Pi_\mu F - 2\hbar D^\nu S_{\nu\mu} - \hbar\epsilon_{\mu\nu\alpha\beta}\omega^\nu S^{\alpha\beta} = 4mV_\mu,\\
&-\hbar D_\mu P + \epsilon_{\mu\nu\alpha\beta}\Pi^\nu S^{\alpha\beta} - \hbar\omega_\mu F = 2mA_\mu,\\
&\hbar(D_\mu V_\nu - D_\nu V_\mu) + 2\epsilon_{\mu\nu\alpha\beta}\Pi^\alpha A^\beta + \hbar\epsilon_{\mu\nu\alpha\beta}\omega^\alpha V^\beta = 2mS_{\mu\nu}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\hbar D^\mu V_\mu = 0,\\
&2\Pi^\mu A_\mu + \hbar\omega^\mu V_\mu = 0,\\
&\hbar D_\mu F + 2\Pi^\nu S_{\nu\mu} - \hbar\omega_\mu P = 0,\\
&4\Pi_\mu P + \hbar\epsilon_{\mu\nu\alpha\beta} D^\nu S^{\alpha\beta} + 2\hbar\omega^\nu S_{\mu\nu} = 0,\\
&2(\Pi_\mu V_\nu - \Pi_\nu V_\mu) - \hbar\epsilon_{\mu\nu\alpha\beta} D^\alpha A^\beta + \hbar(\omega_\mu A_\nu - \omega_\nu A_\mu) = 0. \quad (13)
\end{aligned}$$

These equations can be divided into two groups. Those equations with explicit p_0-dependence appearing in Π_μ form the constraint group which links the Wigner function W and $p_0 W$, and the others with explicit derivative ∂_μ appearing in D_μ form the transport group which describes the evolution of W in phase space. These will be discussed in more detail in the equal-time formalism.

Similar to the Klein–Gordon equation for the wave function $\psi(x)$ which describes the plane-wave solution of the Dirac equation satisfying the on-shell condition $p^2 - m^2 = 0$, we can obtain the phase-space version of the Klein–Gordon equation for the Wigner function $W(x,p)$ by acting on the kinetic equation (9) with the operator $\gamma^\mu K_\mu + (\hbar/2)\gamma^5\gamma^\mu\omega_\mu + m$, which leads to

$$\left[K^\mu K_\mu - \frac{\mathrm{i}}{2}\left[K_\mu, K_\nu\right]\sigma^{\mu\nu} - \hbar\gamma^5 K^\mu\omega_\mu + \frac{\hbar^2}{4}\omega^\mu\omega_\mu - m^2\right]W(x,p) = 0. \quad (14)$$

We will see in the following that this equation controls whether the particle is on the mass shell.

IV. EQUAL-TIME KINETIC EQUATIONS

From the definition (8), it is easy to see that the Wigner function at any time is related to the fields at all times. Therefore, the 8-dimensional kinetic equations in general case cannot be solved as an initial value problem, and we should go to the equal-time formalism of the kinetic theory, by doing energy integration of the 8-dimensional equations [23]. The equal-time Wigner function is defined as

$$W_0(x,\boldsymbol{p}) = \int \frac{\mathrm{d}^3\boldsymbol{y}}{(2\pi)^3} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{y}} \left\langle \psi\left(\boldsymbol{x} + \frac{\boldsymbol{y}}{2}, t\right)\psi^+\left(\boldsymbol{x} - \frac{\boldsymbol{y}}{2}, t\right)\right\rangle. \quad (15)$$

It is clear that the 8- and 7-dimensional Wigner functions are related to each other through the energy integration,

$$W_0(x,\boldsymbol{p}) = \int \mathrm{d}p_0 W(x,p)\gamma^0. \quad (16)$$

This indicates that the equal-time Wigner function is the zeroth-order energy moment of the 8-dimensional Wigner function. This is the reason why we label the equal-time Wigner function using the subscript 0. In the general case, particles moving in a medium are not on the mass shell, and the 8-dimensional Wigner function is equivalent to the collection of all the energy moments [24]

$$W_n(x,\boldsymbol{p}) = \int \mathrm{d}p_0 p_0^n W(x,p)\gamma^0 \quad (17)$$

with $n = 0, 1, 2, \ldots$. Only in the quasi-particle approximation where particles are on the shell and the 8-dimensional Wigner function satisfies the on-shell condition $W(x,p)(p^2 - m^2) = 0$ do the two Wigner functions become equivalent to each other.

Similar to the 8-dimensional scenario, the equal-time Wigner function is decomposed into 8 components in spin space,

$$\begin{aligned}
W_0 = \frac{1}{4}(&f_0 + \gamma_5 f_1 - \mathrm{i}\gamma_0\gamma_5 f_2 + \gamma_0 f_3 + \gamma_5\gamma_0\boldsymbol{\gamma}\cdot\mathbf{g}_0\\
&+ \gamma_0\boldsymbol{\gamma}\cdot\mathbf{g}_1 - \mathrm{i}\boldsymbol{\gamma}\cdot\mathbf{g}_2 - \gamma_5\boldsymbol{\gamma}\cdot\mathbf{g}_3), \quad (18)
\end{aligned}$$

where the equal-time components $f_i(x,\boldsymbol{p})$ and $\mathbf{g}_i(x,\boldsymbol{p})$ (i = 0, 1, 2, 3) are the zeroth-order energy moments of the corresponding 8-dimensional components $\Gamma_\alpha(x,p)$.

By taking p_0-integration of the 8-dimensional equations (12) and (13), one obtains two groups of equal-time kinetic equations,

$$\begin{aligned}
&\hbar(d_t f_0 + \nabla\cdot\mathbf{g}_1) = 0,\\
&\hbar(d_t f_1 + \nabla\cdot\mathbf{g}_0) = -2m f_2,\\
&\hbar d_t f_2 + 2\boldsymbol{p}\cdot\mathbf{g}_3 = 2m f_1,\\
&\hbar d_t f_3 - 2\boldsymbol{p}\cdot\mathbf{g}_2 = 0,\\
&\hbar(d_t\mathbf{g}_0 + \nabla f_1) - 2\boldsymbol{p}\times\mathbf{g}_1 + \hbar\boldsymbol{\omega}\times\mathbf{g}_0 = 0,\\
&\hbar(d_t\mathbf{g}_1 + \nabla f_0) - 2\boldsymbol{p}\times\mathbf{g}_0 + \hbar\boldsymbol{\omega}\times\mathbf{g}_1 = -2m\mathbf{g}_2,\\
&\hbar(d_t\mathbf{g}_2 + \nabla\times\mathbf{g}_3) + 2\boldsymbol{p}f_3 + \hbar\boldsymbol{\omega}\times\mathbf{g}_2 = 2m\mathbf{g}_1,\\
&\hbar(d_t\mathbf{g}_3 - \nabla\times\mathbf{g}_2) - 2\boldsymbol{p}f_2 + \hbar\boldsymbol{\omega}\times\mathbf{g}_3 = 0, \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
&2\int \mathrm{d}p_0 p_0 F = \hbar\nabla\cdot\mathbf{g}_2 - 2\pi_0 f_3 + 2m f_0 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_3,\\
&2\int \mathrm{d}p_0 p_0 P = -\hbar\nabla\cdot\mathbf{g}_3 - 2\pi_0 f_2 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_2,\\
&2\int \mathrm{d}p_0 p_0 V_0 = 2\boldsymbol{p}\cdot\mathbf{g}_1 - 2\pi_0 f_0 + 2m f_3 - \hbar\boldsymbol{\omega}\cdot\mathbf{g}_0,\\
&2\int \mathrm{d}p_0 p_0 A_0 = -2\boldsymbol{p}\cdot\mathbf{g}_0 + 2\pi_0 f_1 + \hbar\boldsymbol{\omega}\cdot\mathbf{g}_1,\\
&2\int \mathrm{d}p_0 p_0 \mathbf{V} = \hbar\nabla\times\mathbf{g}_0 - 2\boldsymbol{p}f_0 + 2\pi_0\mathbf{g}_1 - \hbar\boldsymbol{\omega}f_1,\\
&2\int \mathrm{d}p_0 p_0 \mathbf{A} = -\hbar\nabla\times\mathbf{g}_1 - 2\boldsymbol{p}f_1 + 2\pi_0\mathbf{g}_0 + \hbar\boldsymbol{\omega}f_0 - 2m\mathbf{g}_3,
\end{aligned}$$

$$
\begin{aligned}
&2\int \mathrm{d}p_0 p_0 S^{0i}\mathbf{e}_i = \hbar\nabla f_3 - 2\boldsymbol{p}\times\mathbf{g}_3 + 2\pi_0\mathbf{g}_2 + \hbar\boldsymbol{\omega} f_2,\\
&\int \mathrm{d}p_0 p_0 \epsilon^{ijk} S_{jk}\mathbf{e}_i = \hbar\nabla f_2 - 2\pi_0\mathbf{g}_3 - 2\boldsymbol{p}\times\mathbf{g}_2 - \hbar\boldsymbol{\omega} f_3 + 2m\mathbf{g}_0\,.
\end{aligned}
\tag{20}
$$

The kinetic equations (19) and (20) form, respectively, the transport and constraint groups. The former is an extension of the Boltzmann equation, describing the phase-space evolution of the 8 equal-time distributions in a rotational field. The latter is an extension of the on-shell condition $f(x,p)(p^2-m^2)=0$ associated with the Boltzmann equation. Since particles are generally not on the mass shell, the off-shell constraints cannot be neglected arbitrarily, and only the two groups together form a complete description of the quantum system. This was firstly pointed out by Zhuang and Heinz for a QED system [23, 24].

The constraints play a tremendous role in calculating some of the physical distributions. Let's consider the energy density as an example. From the energy-moment tensor,

$$
T_{\mu\nu}(x) = \mathrm{i}\left\langle\bar{\psi}(x)\gamma_\mu\partial_\nu\psi(x) - \bar{\psi}(x)\gamma^0\epsilon^{ijk}\omega_j x_k\delta_{\mu i}\partial_\nu\psi(x)\right\rangle, \tag{21}
$$

the energy distribution in phase space is the first-order energy moment of the 8-dimensional component V_0,

$$
\varepsilon(x,\boldsymbol{p}) = T_{00}(x,\boldsymbol{p}) = \int \mathrm{d}p_0 p_0 V_0(x,p). \tag{22}
$$

Without the constraints (20) which link the zeroth- and first-order energy moments, there is no way to calculate the energy distribution in kinetic theory. With help from the constraints (20), ϵ is a combination of the equal-time spin components,

$$
\varepsilon = \boldsymbol{p}\cdot\mathbf{g}_1 - \pi_0 f_0 + m f_3 - \frac{\hbar}{2}\boldsymbol{\omega}\cdot\mathbf{g}_0, \tag{23}
$$

where the components $f_1, f_3, \mathbf{g}_0$ and $\mathbf{g}_1$ are controlled by the transport Eq. (19).

V. SEMI-CLASSICAL EXPANSION

The equal-time kinetic equations can directly be solved for some non-perturbative problems like pair production in electromagnetic fields [22, 32]. As a systematical method the semi-classical expansion is widely used in 8-dimensional [33–35] and 7-dimensional [23, 24] kinetic theories for massive [20, 36] and massless [37, 38] fermions. We discuss in this section the semi-classical expansion of the equal-time kinetic Eqs. (19) and (20). Considering the fact that the rotational field appears in the covariant kinetic Eq. (9) only up to the second order in $\hbar$, the equal-time kinetic equations at zeroth, first and second order of $\hbar$ already include the entire quantum effect, since the higher-order kinetic equations will not contain any new term in comparison with the lower-order equations.

We now take the $\hbar$ expansion for the 8- and 7-dimensional Wigner functions $W(x,p)$ and $W_0(x,\boldsymbol{p})$ and the operator Π_μ,

$$
\begin{aligned}
W &= W^{(0)} + \hbar W^{(1)} + \hbar^2 W^{(2)} + \cdots,\\
W_0 &= W_0^{(0)} + \hbar W_0^{(1)} + \hbar^2 W_0^{(2)} + \cdots,\\
\Pi_\mu &= \Pi_\mu^{(0)} + \hbar^2\Pi_\mu^{(2)},\\
\Pi_\mu^{(0)} &= (p_0 + \boldsymbol{\omega}\cdot\boldsymbol{l} + \mu_B, \mathbf{p}),\\
\Pi_\mu^{(2)} &= \left(\boldsymbol{\omega}\cdot(\nabla\times\nabla_p)/4, \mathbf{0}\right).
\end{aligned}
\tag{24}
$$

Note that the other operator D_μ contains only the classical part.

We first consider the Klein–Gordon Eq. (14) at the zeroth order in $\hbar$,

$$
\left[\Pi_\mu^{(0)}\Pi^{(0)\mu} - m^2\right] W^{(0)}(x,p) = 0. \tag{25}
$$

This is just the on-shell condition for classical particles,

$$
p_0 = E_p^{\pm} = \pm\epsilon_p - (\boldsymbol{\omega}\cdot\boldsymbol{l} + \mu_B) \tag{26}
$$

with $\epsilon_p = \sqrt{m^2+\boldsymbol{p}^2}$. Different from the kinetic theory for QED where the electromagnetic fields do not affect the free-particle shell [20], the rotational field here changes the shell from ϵ_p to E_p due to the interaction of the orbital angular momentum with the rotational field. The reason is clear: the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are derivatives of the gauge potential but $\boldsymbol{\omega}$ appears directly in the effective gauge potential $\boldsymbol{\omega}\times\boldsymbol{x}$ [39]. The derivative leads to the appearance of $\boldsymbol{E}$ and $\boldsymbol{B}$ at least at the first order in $\hbar$, but $\boldsymbol{\omega}$ starts to contribute at the zeroth order.

Considering the two elementary solutions of the classical Wigner function, corresponding to the positive and negative energies,

$$
\begin{aligned}
W^{(0)}(x,p) = &W^{(0)+}(x,p)\delta(p_0 - E_p^+)\\
&+ W^{(0)-}(x,p)\delta(p_0 - E_p^-),
\end{aligned}
\tag{27}
$$

the constraint equations (20) reduce the number of independent spin components from 8 to 2. The independent components can be chosen to be f_0 and $\mathbf{g}_0$, and the others can be expressed in terms of them explicitly,

$$
\begin{aligned}
f_1^{(0)\pm} &= \pm\frac{1}{\epsilon_p}\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm},\\
f_2^{(0)\pm} &= 0,\\
f_3^{(0)\pm} &= \pm\frac{m}{\epsilon_p}f_0^{(0)\pm},\\
\mathbf{g}_1^{(0)\pm} &= \pm\frac{\boldsymbol{p}}{\epsilon_p}f_0^{(0)\pm},\\
\mathbf{g}_2^{(0)\pm} &= \frac{1}{m}\boldsymbol{p}\times\mathbf{g}_0^{(0)\pm},\\
\mathbf{g}_3^{(0)\pm} &= \pm\frac{1}{m\epsilon_p}\left[\epsilon_p^2\mathbf{g}_0^{(0)\pm}-\boldsymbol{p}(\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm})\right].
\end{aligned}
\tag{28}
$$

It is now important to understand the physics of the spin components at quasi-particle level. Expressing the charge current and total angular momentum tensor in terms of the equal-time Wigner function, it is clear that the independent components f_0 and $\mathbf{g}_0$ are, respectively, the particle number density and spin density, and $\mathbf{g}_1$ is the number current density [22]. Taking the classical relation $f_1=\boldsymbol{p}/|\boldsymbol{p}|\cdot\mathbf{g}_0$ for massless fermions, f_1 can be interpreted as the helicity density. The components f_3 and f_2 describe the contribution from spontaneous chiral symmetry breaking and isospin symmetry breaking to the particle mass [21]. From the non-relativistic limit $\mathbf{g}_3\to\mathbf{g}_0$ and the comparison of the term $-m/(2m)\boldsymbol{\sigma}\cdot\boldsymbol{\omega}$ in the Schrödinger equation (A4) in a rotational field for particles with effective charge m with the term $-e/(2m)\boldsymbol{\sigma}\cdot\boldsymbol{B}$ in the Schrödinger equation in QED for particles with charge e, $\mathbf{g}_3$ which is known as the magnetic moment density [22] in electromagnetic fields can be understood as the rotational moment density. Considering the classical relation $\mathbf{g}_2=\boldsymbol{p}\times\mathbf{g}_0/m$, $\mathbf{g}_2$ describes the spin property in the direction perpendicular to the particle momentum. Using the above classical relations, the energy density in the quasi-particle approximation is simply expressed in terms of the number distributions with positive and negative energy,

$$
\varepsilon(x,\boldsymbol{p})=E_p^+f_0^{(0)+}(x,\boldsymbol{p})+E_p^-f_0^{(0)-}(x,\boldsymbol{p}). \tag{29}
$$

Since any derivative is multiplied by a factor of $\hbar$, the classical limit of the transport equations (19) cannot describe the phase-space evolution of the classical components but shows again some of the relations appearing in the classical constraints (28). To describe the dynamical evolution of the equal-time Wigner function, we should go to the first order of the transport equations (19),

$$
\begin{aligned}
&d_tf_0^{(0)}+\nabla\cdot\mathbf{g}_1^{(0)}=0,\\
&d_tf_1^{(0)}+\nabla\cdot\mathbf{g}_0^{(0)}+2mf_2^{(1)}=0,\\
&d_tf_2^{(0)}+2\boldsymbol{p}\cdot\mathbf{g}_3^{(1)}-2mf_1^{(1)}=0,\\
&d_tf_3^{(0)}-2\boldsymbol{p}\cdot\mathbf{g}_2^{(1)}=0,\\
&d_t\mathbf{g}_0^{(0)}+\nabla f_1^{(0)}-2\boldsymbol{p}\times\mathbf{g}_1^{(1)}+\boldsymbol{\omega}\times\mathbf{g}_0^{(1)}=0,\\
&d_t\mathbf{g}_1^{(0)}+\nabla f_0^{(0)}-2\boldsymbol{p}\times\mathbf{g}_0^{(1)}+\boldsymbol{\omega}\times\mathbf{g}_1^{(0)}+2m\mathbf{g}_2^{(1)}=0,\\
&d_t\mathbf{g}_2^{(0)}+\nabla\times\mathbf{g}_3^{(0)}+2\boldsymbol{p}f_3^{(1)}+\boldsymbol{\omega}\times\mathbf{g}_2^{(0)}-2m\mathbf{g}_1^{(1)}=0,\\
&d_t\mathbf{g}_3^{(0)}-\nabla\times\mathbf{g}_2^{(0)}-2\boldsymbol{p}f_2^{(1)}+\boldsymbol{\omega}\times\mathbf{g}_3^{(0)}=0.
\end{aligned}
\tag{30}
$$

Eliminating the first-order components $f_i^{(1)}$ and $\mathbf{g}_i^{(1)}$ by simple algebra and taking into account the classical relations (28) (see the details in Appendix B) we obtain the transport equations for the two independent components $f_0^{(0)}$ and $\mathbf{g}_0^{(0)}$,

$$
\begin{aligned}
&\left[\partial_t+\left(\pm\frac{\boldsymbol{p}}{\epsilon_p}+\boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla-(\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right]f_0^{(0)\pm}=0,\\
&\left[\partial_t+\left(\pm\frac{\boldsymbol{p}}{\epsilon_p}+\boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla-(\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right]\mathbf{g}_0^{(0)\pm}=-\boldsymbol{\omega}\times\mathbf{g}_0^{(0)\pm}.
\end{aligned}
\tag{31}
$$

The two equations are both in the Boltzmann form. The particle velocity appearing in the free-streaming terms is modified by the rotation induced linear velocity $\boldsymbol{x}\times\boldsymbol{\omega}$, and the classical part of the rotational potential $-\boldsymbol{\omega}\cdot\boldsymbol{l}$ in the Dirac equation leads to a mean-field force (Coriolis force) $-\nabla(-\boldsymbol{\omega}\cdot\boldsymbol{l})=-\boldsymbol{\omega}\times\boldsymbol{p}$. For the spin density $\mathbf{g}_0$, there is an extra term $\boldsymbol{\omega}\times\mathbf{g}_0$ indicating the spin-rotation interaction, similar to the term $\boldsymbol{B}\times\mathbf{g}_0$ in spinor QED. From the transport equations we obtain the equations of motion of the system,

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \pm\frac{\boldsymbol{p}}{\epsilon_p}+\boldsymbol{x}\times\boldsymbol{\omega},\\
\dot{\boldsymbol{p}} &= -\boldsymbol{\omega}\times\boldsymbol{p}.
\end{aligned}
\tag{32}
$$

Considering positive energy, the total force acting on the particles

$$
\boldsymbol{F}=\epsilon_p\ddot{\boldsymbol{x}}=-\boldsymbol{\omega}\times\boldsymbol{p}-\epsilon_p\boldsymbol{\omega}\times(\boldsymbol{x}\times\boldsymbol{\omega}) \tag{33}
$$

contains both the Coriolis force and centrifugal force. Note that the spin evolution equation in (31) is the equal-time phase-space version of a generalized Bargmann–Michel–Telegdi equation [40, 41] on spin precession.

In order to investigate spin-induced anomalous phenomena in a rotational field, one needs to go beyond the classical limit and derive quantum transport equations. To this end, we consider the Klein–Gordon equation (14) again to see if quantum particles are still on a mass shell. At the first order in $\hbar$, the whole operator acting on the Wigner function becomes

$$\mathrm{i}D^{\mu}\Pi^{(0)}_{\mu}-(i/2)(\boldsymbol{\omega}\times\boldsymbol{p})\cdot[\boldsymbol{\gamma},\gamma_0]+\gamma_5\boldsymbol{p}\cdot\boldsymbol{\omega}, \tag{34}$$

which is γ-matrix dependent. Therefore, there is no longer a common mass shell for all the spin components, $\left(p_0^2-\mathcal{E}_p^2\right)W^{(1)}(x,p)\neq 0$. To confirm this conclusion, we tried the quasi-particle solution of the first-order constraint equations, but the procedure fails: we cannot find a new mass shell $p_0=\pm\mathcal{E}_p$ (see the detailed calculation in Appendix C). The case here is very different from the chiral limit where massless particles are always on a shell at any order of $\hbar$ [42]; massive particles cannot be on the shell when quantum effect is included. How about a spin-dependent on-shell condition? Unfortunately, this condition fails again, $\left(p_0^2-\mathcal{E}_{p\alpha}^2\right)\Gamma^{(1)}_{\alpha}(x,p)\neq 0$. Neither a common on-shell nor a component-dependent on-shell condition can be the solution of the constraint equations for massive fermions [20]. The quantum effects in a general kinetic theory are essentially reflected in two aspects: one is the spin, and the other is the off-shell constraint.

Without the on-shell condition, the constraint equations (20) at the first order in $\hbar$ become

$$\begin{aligned}
\pm 2\epsilon_p f_3^{(1)}+2\Delta E_{p3} &= \nabla\cdot\mathbf{g}_2^{(0)}+2mf_0^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_3^{(0)},\\
\pm 2\epsilon_p f_2^{(1)}+2\Delta E_{p2} &= -\nabla\cdot\mathbf{g}_3^{(0)}-\boldsymbol{\omega}\cdot\mathbf{g}_2^{(0)},\\
\pm 2\epsilon_p f_0^{(1)}+2\Delta E_{p0} &= 2\boldsymbol{p}\cdot\mathbf{g}_1^{(1)}+2mf_3^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_0^{(0)},\\
\pm 2\epsilon_p f_1^{(1)}+2\Delta E_{p1} &= 2\boldsymbol{p}\cdot\mathbf{g}_0^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_1^{(0)},\\
\pm 2\epsilon_p \mathbf{g}_1^{(1)}+2\Delta \mathbf{E}_{p1} &= \nabla\times\mathbf{g}_0^{(0)}+2\boldsymbol{p}f_0^{(1)}-\boldsymbol{\omega}f_1^{(0)},\\
\pm 2\epsilon_p \mathbf{g}_0^{(1)}+2\Delta \mathbf{E}_{p0} &= \nabla\times\mathbf{g}_1^{(0)}+2\boldsymbol{p}f_1^{(1)}-\boldsymbol{\omega}f_0^{(0)}+2m\mathbf{g}_3^{(1)},\\
\pm 2\epsilon_p \mathbf{g}_2^{(1)}+2\Delta \mathbf{E}_{p2} &= -\nabla f_3^{(0)}+2\boldsymbol{p}\times\mathbf{g}_3^{(1)}-\boldsymbol{\omega}f_2^{(0)},\\
\pm 2\epsilon_p \mathbf{g}_3^{(1)}+2\Delta \mathbf{E}_{p3} &= \nabla f_2^{(0)}-2\boldsymbol{p}\times\mathbf{g}_2^{(1)}-\boldsymbol{\omega}f_3^{(0)}+2m\mathbf{g}_0^{(1)}
\end{aligned} \tag{35}$$

with the energy shifts

$$\Delta E_{p\alpha}=2\int \mathrm{d}p_0(p_0-E_p)\Gamma^{(1)}_{\alpha}. \tag{36}$$

To close the equal-time constraint equations (35) which are related to the 8-dimensional components through the energy shifts (36), we semi-classically expand the 8-dimensional kinetic equations (12) and (13). At classical level, the vector component is proportional to $\Pi^{(0)}_{\mu}$, and both the vector and axial-vector are on the mass shell,

$$\begin{aligned}
V^{(0)}_{\mu} &= \Pi^{(0)}_{\mu}f^{(0)}\delta(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2),\\
A^{(0)}_{\mu} &= g^{(0)}_{\mu}\delta(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2),
\end{aligned} \tag{37}$$

where $f(x,p)$ and $g_{\mu}(x,p)$ are arbitrary Lorentz scalar and vector distributions. After a straightforward but a little bit tedious algebra, the vector and axial-vector at first order in $\hbar$ can be decomposed into

$$\begin{aligned}
V^{(1)}_{\mu} &= \Pi^{(0)}_{\mu}f^{(1)}\delta(\Pi^{(0)}_{\rho}\Pi_{(0)\rho}-m^2)\\
&\quad+\Pi^{(0)}_{\mu}\omega^{\nu}A^{(0)}_{\nu}\delta'(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2),\\
A^{(1)}_{\mu} &= g^{(1)}_{\mu}\delta(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2)\\
&\quad+\Pi^{(0)}_{\mu}\omega^{\nu}V^{(0)}_{\nu}\delta'(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2)\\
&\quad-\omega_{\mu}\Pi^{(0)\nu}V^{(0)}_{\nu}\delta'(\Pi^{(0)}_{\rho}\Pi^{(0)\rho}-m^2),
\end{aligned} \tag{38}$$

where δ' means the derivative of the δ-function.

Taking together the first order transport and constraint equations (30) and (35) for the equal-time components and (38) for the covariant components, we determine uniquely the energy shifts

$$\begin{aligned}
\Delta E^{\pm}_{p0} &= -\frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{g}_0^{(0)\pm},\\
\Delta E^{\pm}_{p1} &= \mp\frac{\boldsymbol{\omega}\cdot\boldsymbol{p}}{2\epsilon_p}f_0^{(0)\pm},\\
\Delta E^{\pm}_{p2} &= -\frac{1}{2m}\boldsymbol{\omega}\cdot(\boldsymbol{p}\times\mathbf{g}_0^{(0)\pm}),\\
\Delta E^{\pm}_{p3} &= \mp\frac{1}{2m\epsilon_p}\left[\epsilon_p^2\boldsymbol{\omega}\cdot\mathbf{g}_0^{(0)\pm}-(\boldsymbol{\omega}\cdot\boldsymbol{p}(\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm})\right],\\
\Delta \mathbf{E}^{\pm}_{p0} &= -\frac{1}{2\epsilon_p^2}\left[m^2\boldsymbol{\omega}+\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\omega})\right]f_0^{(0)\pm},\\
\Delta \mathbf{E}^{\pm}_{p1} &= \mp\frac{\boldsymbol{\omega}}{2\epsilon_p}\boldsymbol{p}\cdot\mathbf{g}_0^{(0)},\\
\Delta \mathbf{E}^{\pm}_{p2} &= \frac{\boldsymbol{p}^2(\mathbf{p}\times\boldsymbol{\omega})}{2m\epsilon_p^2}f_0^{(0)\pm},\\
\Delta \mathbf{E}^{\pm}_{p3} &= \mp\frac{1}{2m\epsilon_p}\left[\epsilon_p^2\boldsymbol{\omega}-\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\omega})\right]f_0^{(0)\pm}.
\end{aligned} \tag{39}$$

All the energy shifts will disappear when the external field is turned off. The reason is, without the coupling between the total angular momentum and the rotational field, particles will keep at the classical shell. Note that there are two solutions for any energy shift, corresponding to the two classical shells $E^{\pm}_p$.

The transport and constraint equations (30) and (35) not only fix the quantum correction from the off-shell effect to the classical mass shell, but also reduce the number of independent spin components at quantum level. Again there are only two independent components. Similar to the classical limit, we can still choose the number density $f_0^{(1)}$ and spin density $\mathbf{g}_0^{(1)}$ as the independent components, and the others are determined by them self-consistently,

$$
\begin{aligned}
f_1^{(1)\pm} &= \pm\frac{1}{\epsilon_p}\boldsymbol{p}\cdot\mathbf{g}_0^{(1)\pm},\\
f_2^{(1)\pm} &= -\frac{1}{2m\epsilon_p^2}\left[\epsilon_p^2\nabla\cdot\mathbf{g}_0^{(0)\pm}-(\boldsymbol{p}\cdot\nabla)(\boldsymbol{p}\cdot\mathbf{g}_0^{(0)\pm})\right],\\
f_3^{(1)\pm} &= \pm\frac{m}{\epsilon_p}f_0^{(1)\pm}\mp\frac{1}{2m\epsilon_p}\boldsymbol{p}\cdot(\nabla\times\mathbf{g}_0^{(0)\pm}),\\
\mathbf{g}_1^{(1)\pm} &= \pm\frac{\boldsymbol{p}}{\epsilon_p}f_0^{(1)\pm}\pm\frac{1}{2\epsilon_p}\nabla\times\mathbf{g}_0^{(0)\pm},\\
\mathbf{g}_2^{(1)\pm} &= \frac{1}{m}\boldsymbol{p}\times\mathbf{g}_0^{(1)\pm}-\frac{m}{2\epsilon_p^2}\nabla f_0^{(0)\pm}+\frac{1}{2m\epsilon_p^2}\boldsymbol{p}\times(\boldsymbol{p}\times\nabla)f_0^{(0)\pm},\\
\mathbf{g}_3^{(1)\pm} &= \pm\frac{1}{m\epsilon_p}\left[\epsilon_p^2\mathbf{g}_0^{(1)}-\boldsymbol{p}(\boldsymbol{p}\cdot\mathbf{g}_0^{(1)})\right]\pm\frac{\boldsymbol{p}\times\nabla}{2m\epsilon_p}f_0^{(0)\pm}\\
&\quad+\frac{1}{2m\epsilon_p^2}\left[\boldsymbol{p}^2\boldsymbol{\omega}-\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\omega})\right]f_0^{(0)\pm}.
\end{aligned}
\tag{40}
$$

There are here three kinds of quantum corrections. The first is a direct analogy to the classical relations shown in (28), by simply replacing the classical components $f_0^{(0)}$ and $\mathbf{g}_0^{(0)}$ by the first-order components $f_0^{(1)}$ and $\mathbf{g}_0^{(1)}$. The second comes from the derivative of the classical components, remembering that a derivative in kinetic equations is always accompanied by a factor of $\hbar$. The third correction is from the interaction with the external field which appears only in the rotational moment $\mathbf{g}_3$.

The dynamical evolution of the equal-time Wigner function $W_0(x,\mathbf{p})$ at the first order in $\hbar$ is controlled by the transport equations (19) at the second order in $\hbar$,

$$
\begin{aligned}
&d_t f_0^{(1)}+\nabla\cdot\mathbf{g}_1^{(1)}=0,\\
&d_t f_1^{(1)}+\nabla\cdot\mathbf{g}_0^{(1)}+2mf_2^{(2)}=0,\\
&d_t f_2^{(1)}+2\boldsymbol{p}\cdot\mathbf{g}_3^{(2)}-2mf_1^{(2)}=0,\\
&d_t f_3^{(1)}-2\boldsymbol{p}\cdot\mathbf{g}_2^{(2)}=0,\\
&d_t\mathbf{g}_0^{(1)}+\nabla f_1^{(1)}-2\boldsymbol{p}\times\mathbf{g}_1^{(2)}+\boldsymbol{\omega}\times\mathbf{g}_0^{(1)}=0,\\
&d_t\mathbf{g}_1^{(1)}+\nabla f_0^{(1)}-2\boldsymbol{p}\times\mathbf{g}_0^{(2)}+\boldsymbol{\omega}\times\mathbf{g}_1^{(1)}+2m\mathbf{g}_2^{(2)}=0,\\
&d_t\mathbf{g}_2^{(1)}+\nabla\times\mathbf{g}_3^{(1)}+2\boldsymbol{p}f_3^{(2)}+\boldsymbol{\omega}\times\mathbf{g}_2^{(1)}-2m\mathbf{g}_1^{(2)}=0,\\
&d_t\mathbf{g}_3^{(1)}-\nabla\times\mathbf{g}_2^{(1)}-2\boldsymbol{p}f_2^{(2)}+\boldsymbol{\omega}\times\mathbf{g}_3^{(1)}=0.
\end{aligned}
\tag{41}
$$

By eliminating the second-order components and taking into account the classical and first-order kinetic equations (28), (30) and (35), we obtain the transport equations for the two independent quantum distribution functions, namely the number density $f_0^{(1)}$ and spin density $\mathbf{g}_0^{(1)}$,

$$
\begin{aligned}
&\left[\partial_t+\left(\pm\frac{\boldsymbol{p}}{\epsilon_p}+\boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla-(\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right]f_0^{(1)\pm}=0,\\
&\left[\partial_t+\left(\pm\frac{\boldsymbol{p}}{\epsilon_p}+\boldsymbol{x}\times\boldsymbol{\omega}\right)\cdot\nabla-(\boldsymbol{\omega}\times\boldsymbol{p})\cdot\nabla_p\right]\mathbf{g}_0^{(1)\pm}\\
&=-\boldsymbol{\omega}\times\mathbf{g}_0^{(1)}-\frac{1}{2\epsilon_p^4}\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{\omega})(\boldsymbol{p}\cdot\nabla)f_0^{(0)\pm}.
\end{aligned}
\tag{42}
$$

While the number density satisfies the same transport equation as the classical equivalent, the coupling between the two independent components leads to a new term on the right-hand side of the quantum transport equation for the spin density.

Following the method we used to derive transport equations in the classical case and to the first order in $\hbar$, it is not a problem to obtain transport equations for the second-order components of the Wigner function. As has been mentioned above, the rotational field appears only up to the second order of $\hbar$ in the kinetic equations, so there should be no more new information when going beyond the second order.

VI. QUARKONIUM POLARIZATION

When we define the rotation as an external field, it means that the heavy quark motion is separated from the surrounding thermal matter. This is quite different from the light quarks, which share the thermal vorticity with the medium. Like Lambda hyperon polarization, which is dominated by s quarks and is a hot topic in recent heavy ion collisions, quarkonium polarization is already measured in nuclear collisions at LHC energy [43]. The equal-time Wigner function under rotation can shed light on the study of quarkonium polarization.

At the quark level, the quarkonium polarization comes from the heavy quark polarization. For the ensemble of heavy quarks, the density operator in phase space is a Wigner transformation of the operator in coordinate space,

$$
\hat{\rho}_q=\sum_{s_z,\boldsymbol{x},\boldsymbol{p}}w_q(s_z,\boldsymbol{x},\boldsymbol{p})\sum_{\boldsymbol{y}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{y}}\left|s_z,\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\right\rangle\left\langle s_z,\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\right|,
\tag{43}
$$

where $s_z,\boldsymbol{x}$ and $\boldsymbol{p}$ are the quark spin, coordinate and momentum, $|s_z,\boldsymbol{x}\rangle$ is the quark state, and w_q is the quark distribution function. For the ensemble of heavy quark pairs, the density operator in the non-coupling representation can be expressed as

$$
\begin{aligned}
\hat{\rho}_{q\bar{q}} &= \sum_{s_{1z},s_{2z},\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{p}_1,\boldsymbol{p}_2}w_q(s_{1z},\boldsymbol{x}_1,\boldsymbol{p}_1)w_{\bar{q}}(s_{2z},\boldsymbol{x}_2,\boldsymbol{p}_2)\\
&\quad\times\sum_{\boldsymbol{y}_1,\boldsymbol{y}_2}\mathrm{e}^{\mathrm{i}\boldsymbol{p}_1\cdot\boldsymbol{y}_1}\mathrm{e}^{\mathrm{i}\boldsymbol{p}_2\cdot\boldsymbol{y}_2}\times\left|s_{1z},s_{2z},\boldsymbol{x}_1+\frac{\boldsymbol{y}_1}{2},\boldsymbol{x}_2+\frac{\boldsymbol{y}_2}{2}\right\rangle\\
&\quad\times\left\langle s_{1z},s_{2z},\boldsymbol{x}_1-\frac{\boldsymbol{y}_1}{2},\boldsymbol{x}_2-\frac{\boldsymbol{y}}{2}\right|,
\end{aligned}
\tag{44}
$$

where $s_{1z},\boldsymbol{x}_1,\boldsymbol{p}_1$ and $s_{2z},\boldsymbol{x}_2,\boldsymbol{p}_2$ are the quark and antiquark spin, coordinate and momentum, and $|s_{1z},s_{2z},\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = |s_{1z},\boldsymbol{x}_1\rangle|s_{2z},\boldsymbol{x}_2\rangle$ is the pair state in non-coupling representation.

Taking replacement of quark variables $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{y}_1,\boldsymbol{y}_2$ by pair variables $\boldsymbol{X},\boldsymbol{x},\boldsymbol{P},\boldsymbol{p},\boldsymbol{Y},\boldsymbol{y}$, the Wigner transformation of the diagonal element of the pair density operator in the coupled representation constructed by the pair state $|S,S_z,\boldsymbol{X},\boldsymbol{x}\rangle$ can be factorized as

$$\begin{aligned}\rho(S,S_z,\boldsymbol{X},\boldsymbol{P},\boldsymbol{x},\boldsymbol{p}) &= \sum_{\boldsymbol{Y},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}\boldsymbol{P}\cdot\boldsymbol{Y}}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{y}}\Big\langle S,S_z,\boldsymbol{X}+\frac{\boldsymbol{Y}}{2},\boldsymbol{x} \\ &\quad +\frac{\boldsymbol{y}}{2}|\hat{\rho}_{q\bar{q}}|S,S_z,\boldsymbol{X}-\frac{\boldsymbol{Y}}{2},\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\Big\rangle \\ &= \sum_{s_{1z},s_{2z}} w_q(s_{1z},\boldsymbol{x}_1,\boldsymbol{p}_1)w_{\bar{q}}(s_{2z},\boldsymbol{x}_2,\boldsymbol{p}_2) \\ &\quad \times W(S,S_z,\boldsymbol{X},\boldsymbol{P},\boldsymbol{x},\boldsymbol{p}),\end{aligned} \tag{45}$$

where W is the coalescence probability for two quarks to form a meson state,

$$\begin{aligned}W(S,S_z,\boldsymbol{X},\boldsymbol{P},\boldsymbol{x},\boldsymbol{p}) &= \sum_{\boldsymbol{y},\boldsymbol{Y}} \Psi^*\Big(S,S_z,\boldsymbol{X}+\frac{\boldsymbol{Y}}{2},\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\Big) \\ &\quad \times\Psi\Big(S,S_z,\boldsymbol{X}-\frac{\boldsymbol{Y}}{2},\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\Big) \\ &\quad \times\psi\Big(s_{1z},\boldsymbol{x}_1+\frac{\boldsymbol{y}_1}{2}\Big)\psi\Big(s_{2z},\boldsymbol{x}_2+\frac{\boldsymbol{y}_2}{2}\Big) \\ &\quad \times\psi^*\Big(s_{1z},\boldsymbol{x}_1-\frac{\boldsymbol{y}_1}{2}\Big)\psi^*\Big(s_{2z},\boldsymbol{x}_2-\frac{\boldsymbol{y}_2}{2}\Big)\end{aligned} \tag{46}$$

with the quark and meson wave functions $\psi(s_{iz},\boldsymbol{x}_i)$ and $\Psi(S,S_z,\boldsymbol{X},\boldsymbol{x})$.

$\rho(S,S_z,\boldsymbol{X},\boldsymbol{P},\boldsymbol{x},\boldsymbol{p})$ is the statistical probability for the $q\bar{q}$ pair to be in the meson state. By integrating out the center-of-mass coordinate $\boldsymbol{X}$ and the relative coordinate and momenrum $\boldsymbol{x}$ and $\boldsymbol{p}$, one obtains the meson polarization probability as a function of the meson momentum $\boldsymbol{P}$, $\rho(S,S_z,\boldsymbol{P})$. In the calculation on quarkonium polarization, the quarkonium wave function $\Psi(S,S_z,\boldsymbol{X},\boldsymbol{x})$ is controlled by the two-body Dirac equation [44– 46], and the key point is the heavy quark potential between the quark q and antiquark $\bar{q}$ [47]. The heavy quark wave function $\psi(s_z,\boldsymbol{x})$ is the solution of the one-body stationary Dirac equation (A1) shown in Appendix A. The quark distribution function w_q is controlled by the number distribution f_0 and helicity distribution f_1,

$$w_q(s_z,\boldsymbol{x},\boldsymbol{p}) \equiv f_\chi(\boldsymbol{x},\boldsymbol{p}) = f_0(\boldsymbol{x},\boldsymbol{p}) + \chi f_1(\boldsymbol{x},\boldsymbol{p}) \tag{47}$$

with $\chi = \pm$. To the zeroth and first order in $\hbar$, the transport equations for the number density f_0 and spin density $\mathbf{g}_0$ and the relation between f_1 and $\mathbf{g}_0$ are as shown in Section V. The particle equilibrium distribution is controlled by the detailed balance for the collision terms, namely the lose term and gain term cancel each other when the system is in equilibrium state [38]. For quarks moving in an external rotational field, the quark equilibrium distribution f_χ is the Fermi–Dirac function

$$f_\chi = \frac{1}{\mathrm{e}^{(\sqrt{m^2+\boldsymbol{p}^2}-\omega(l_z+\chi s_z))/T}+1} \tag{48}$$

with medium temperature T, where we have chosen the rotational field along the $z-$axis. We will study J/ψ polarization in detail in a separate paper.

VII. SUMMARY AND OUTLOOK

We investigated the quantum kinetic theory for a massive fermion system under a rotational field in Wigner function formalism. We derived two groups of 8-dimensional kinetic equations (12) and (13) and their 7-dimensional (equal time) version (19) and (20); one is the constraint group which describes the off-shell effect in quantum case, and the other is the transport group which is the quantum analogy to the classical Boltzmann equation. For the structure of a quantum kinetic theory, the off-shell constraint is essentially important. It provides the physical interpretation for all the equal-time spin components, reduces the number of independent distribution functions, and closes the transport equations for the number density and spin density in classical limit and at quantum level.

The interaction between the external rotational field and total angular momentum significantly changes the transport properties of the particles. The classical rotation–orbital coupling controls the dynamical evolution of the number distribution f_0. It adds a linear velocity $\boldsymbol{x}\times\boldsymbol{\omega}$ to the particle velocity, and the induced Coriolis force $\boldsymbol{p}\times\boldsymbol{\omega}$ behaves as a mean field force acting on the particles, see Eqs. (31) – (33). Apart from the classical coupling, the quantum rotation–spin coupling changes the spin distribution $\mathbf{g}_0$. While the two distributions, f_0 and $\mathbf{g}_0$, are independent in the classical limit, the number density influences the spin density at the quantum level.

An important application of the obtained transport equations for heavy quarks is the quarkonium distribution in high energy nuclear collisions. The rotation introduces a specific direction in coordinate space. The particles moving in the plane perpendicular to the rotation are strongly accelerated by the rotation, but those moving along the rotation direction are not affected. Therefore, the final state distribution, especially the collective flow, will largely be changed by the rotation, even at the classical level. In the quantum case, the rotation –spin coupling will induce spin polarization of

heavy quarks and may lead to quarkonium polarization in heavy ion collisions.

APPENDIX A: SCHRÖDINGER EQUATION

Considering the stationary solution of the Dirac equation (7), $\psi(x)=\psi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}Et}$, the stationary wave function $\psi(\boldsymbol{x})$ satisfies the equation

$$\left[\left(\gamma_0\boldsymbol{\gamma}\cdot\hat{\boldsymbol{p}}-\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}\right)+m\gamma_0\right]\psi=E\psi. \tag{A1}$$

To move to the familiar non-relativistic expression, we separate the quark energy into the mass and the kinetic energy, $E=m+\epsilon$, write the stationary wave function in a two-component form, $\psi(\boldsymbol{x})=(\phi(\boldsymbol{x}),\chi(\boldsymbol{x}))^T$, and take the Pauli –Dirac representation for the γ-matrix, $\gamma_0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}$ and $\boldsymbol{\gamma}=\begin{pmatrix} 0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}$. The two-component Dirac equation is then written as

$$\begin{aligned}&\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\chi-\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}\phi=\epsilon\phi,\\ &\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\phi-\left(\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}+2m\right)\chi=\epsilon\chi\end{aligned} \tag{A2}$$

with the total angular momentum $\hat{\boldsymbol{J}}=\boldsymbol{x}\times\hat{\boldsymbol{p}}+\boldsymbol{\sigma}/2$ in its two-dimensional form. From the second equation, the small component χ can be expressed as

$$\chi=\frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}}{2m+\epsilon+\boldsymbol{\omega}\cdot\boldsymbol{J}}\phi\approx\frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}}{2m}\phi \tag{A3}$$

to the first order in $1/m$. Substituting this into the first equation leads to the Schrödinger equation for the large component ϕ,

$$\left[\frac{\hat{\boldsymbol{p}}^2}{2m}-\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}\right]\phi=\epsilon\phi \tag{A4}$$

which is the same as obtained using a non-relativistic Galilean transformation [39].

To the second order in $1/m$, the small component χ becomes

$$\chi=\frac{1}{2m}\left(1-\frac{\epsilon+\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}}{2m}\right)\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\phi, \tag{A5}$$

Taking the commutation relations between x_i and $\hat{p}_j$ and between σ_i and σ_j and employing the above Schrödinger equation to the first order in $1/m$, we obtain the Schrödinger equation to the second order,

$$\left[\frac{\hat{\boldsymbol{p}}^2}{2m}-\frac{\hat{\boldsymbol{p}}^4}{8m^3}-\boldsymbol{\omega}\cdot\hat{\boldsymbol{J}}\right]\phi=\epsilon\phi. \tag{A6}$$

The only relativistic correction is to the kinetic energy.

APPENDIX B: ELIMINATING THE FIRST -ORDER COMPONENTS

Taking the transport equations for the classical components $f_0^{(0)},f_1^{(0)}$ and $\mathbf{g}_3^{(0)}$ in Eq. (30),

$$\begin{aligned}&d_tf_0^{(0)}+\nabla\cdot\mathbf{g}_1^{(0)}=0,\\ &d_tf_1^{(0)}+\nabla\cdot\mathbf{g}_0^{(0)}+2mf_2^{(1)}=0,\\ &d_t\mathbf{g}_3^{(0)}-\nabla\times\mathbf{g}_2^{(0)}-2\boldsymbol{p}f_2^{(1)}+\boldsymbol{\omega}\times\mathbf{g}_3^{(0)}=0,\end{aligned} \tag{B1}$$

and multiplying the second equation by $\boldsymbol{p}$ and the third equation by m, one can eliminating the first-order component f_2^{91} by subtracting the two equations. Then employing the classical relations among $\mathbf{g}_0^{(0)},\mathbf{g}_2^{(2)}$ and $\mathbf{g}_3^{(0)}$ shown in Eq. (28), one obtains the transport equations (31) for $f_0^{(0)}$ and $\mathbf{g}_0^{(0)}$.

APPENDIX C: OFF-SHELL EFFECT ON THE FIRST-ORDER COMPONENTS

To see clearly why it is impossible for the first-order Wigner function to be on a mass shell, we try a on-shell solution $\left(p_0^2-\mathcal{E}_p^2\right)W^{(1)}(x,p)=0$ of the constraint equations (20),

$$\begin{aligned}\pm2\mathcal{E}_pf_3^{(1)}&=\nabla\cdot\mathbf{g}_2^{(0)}+2mf_0^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_3^{(0)},\\ \pm2\mathcal{E}_pf_2^{(1)}&=-\nabla\cdot\mathbf{g}_3^{(0)}-\boldsymbol{\omega}\cdot\mathbf{g}_2^{(0)},\\ \pm2\mathcal{E}_pf_0^{(1)}&=2\boldsymbol{p}\cdot\mathbf{g}_1^{(1)}+2mf_3^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_0^{(0)},\\ \pm2\mathcal{E}_pf_1^{(1)}&=2\boldsymbol{p}\cdot\mathbf{g}_0^{(1)}-\boldsymbol{\omega}\cdot\mathbf{g}_1^{(0)},\\ \pm2\mathcal{E}_p\mathbf{g}_1^{(1)}&=\nabla\times\mathbf{g}_0^{(0)}+2\boldsymbol{p}f_0^{(1)}-\boldsymbol{\omega}f_1^{(0)},\\ \pm2\mathcal{E}_p\mathbf{g}_0^{(1)}&=\nabla\times\mathbf{g}_1^{(0)}+2\boldsymbol{p}f_1^{(1)}-\boldsymbol{\omega}f_0^{(0)}+2m\mathbf{g}_3^{(1)},\\ \pm2\mathcal{E}_p\mathbf{g}_2^{(1)}&=-\nabla f_3^{(0)}+2\boldsymbol{p}\times\mathbf{g}_3^{(1)}-\boldsymbol{\omega}f_2^{(0)},\\ \pm2\mathcal{E}_p\mathbf{g}_3^{(1)}&=\nabla f_2^{(0)}-2\boldsymbol{p}\times\mathbf{g}_2^{(1)}-\boldsymbol{\omega}f_3^{(0)}+2m\mathbf{g}_0^{(1)}.\end{aligned} \tag{C1}$$

To be specific, we choose the first, third, and fifth equations which construct a set of closed equations for the first-order components $f_0^{(1)},f_3^{(1)}$ and $\mathbf{g}_1^{(1)}$. After some simple algebraic calculus to eliminate the first order components, we obtain

$$-(\boldsymbol{\omega}\cdot\boldsymbol{p})f_1^{(0)}-m\boldsymbol{\omega}\cdot\mathbf{g}_3^{(0)}=0. \tag{C2}$$

Taking into account the classical relations shown in Eq. (28), it leads to

$$\epsilon_p(\boldsymbol{\omega}\cdot\mathbf{g}_0^{(0)}) = 0. \tag{C3}$$

Since we are not interested in the trivial solution $\mathbf{g}_0 \equiv 0$ in general case, the rotation $\boldsymbol{\omega}$ should disappear to guarantee the above result. This means that the first - Wigner function in a rotational field cannot be on a mass shell.

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