Thermodynamics for higher dimensional rotating black holes with variable Newton constant*

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Abstract: The extensivity for the thermodynamics of general D-dimensional rotating black holes with or without a cosmological constant can be proved analytically, provided that the effective number of microscopic degrees of freedom and the chemical potential are given respectively as $N = L^{D-2}/G$, $\mu = GTI_D/L^{D-2}$, where G is the variable Newton constant, I_D is the Euclidean action, and L is a constant length scale. In the cases without a cosmological constant, i.e., the Myers-Perry black holes, the physical mass and the intensive variables can be expressed as explicit macro state functions in the extensive variables in a simple and compact form, which allows for an analytical calculation of the heat capacity. The results indicate that the Myers-Perry black holes with zero, one, and k equal rotation parameters are all thermodynamically unstable.

Keywords: black hole thermodynamics, higher dimensional spacetime, Euler relation, thermodynamic instability, extensive thermodynamics

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I. INTRODUCTION

This work is a continuation of the recent works [1-3] on black hole thermodynamics and the cases of general rotating black holes in D-dimensional Einstein gravity. The formalism introduced in [1-3] involves a variable Newton constant G, which enters the expression

$$N = \frac{L^{D-2}}{G} \tag{1}$$

for the new thermodynamic variable N. For asymptotically AdS black holes, N is proportional to the central charge of the dual CFT, while for non-AdS black holes, N may be simply understood as the effective number of microscopic degrees of freedom for the black holes. The thermodynamic conjugate of N is given by the chemical potential

$$\mu = \frac{GTI_D}{L^{D-2}}, \tag{2}$$

where I_D is the Euclidean action, and L is a constant length scale (which may be identified as the largest radius of the event horizon during a thermodynamic process of interest). This formalism coincides neither with the traditional formalism [4-7] nor with the so-called extended phase space formalism [8-14] but is closely related with Visser's holographic thermodynamics [15]. The idea to introduce a chemical potential in black hole thermodynamics was already presented in [16-20], while a variable Newton constant in black hole thermodynamics was also presented in [21]. However, the major points of concern are completely different. The motivation of the works [1-3] is mainly to introduce a formalism in which the first law and the Euler relation hold simultaneously. This makes the thermodynamics extensive, and the thermodynamic potential and the intensive variables behave as appropriate homogeneous functions in the extensive variables. In this spirit, the ideas of the works [22, 23] are quite close to ours but with a different set of extensive variables. The chemical potential introduced in [22, 23] is conjugate to the so-called topological charge, while the chemical potential introduced in [1, 2] is conjugate to the central charge of the dual CFT for AdS black holes, and the formalism is extended to non-AdS cases without a holographic dual in [3]. The wider applicability of the variable Newton constant formalism may be a signature for its universality, and one of the purposes of the present work is to illustrate the power and strength of this formal-

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ism in the cases of general rotating black hole solutions in higher dimensional Einstein gravity, regardless of the value and sign of the cosmological constant.

The black hole solutions to be analyzed in this paper was first obtained in [24]. The solutions have $k \equiv [(D-1)/2]$ independent rotation parameters a_i in k orthogonal 2-planes. For a vanishing cosmological constant, the solutions degenerate into Myers-Perry black holes [25]. Using the explicit expression for the chemical potential, it will be shown that the Hawking-Page (HP) transition [26] appears only in the asymptotically AdS cases and only if the radius of the event horizon approaches the AdS radius. Moreover, by introducing the N, μ variables, the physical mass as well as all the intensive variables can, in principle, be expressed as macro state functions in the extensive variables with appropriate homogeneity behaviors. These macro state functions are made explicit in the cases with a vanishing cosmological constant, i.e., the Myers-Perry cases. I will also analytically calculate the heat capacities and discuss the thermodynamic instabilities for the Myers-Perry black holes.

II. GENERAL ROTATING BLACK HOLES IN HIGHER DIMENSIONS

The general D-dimensional rotating black hole solutions with the cosmological constant were first obtained in [24]. In Boyer-Linquist coordinates, the metrics are given by

$$\begin{aligned} \mathrm{d}s^2 =& -W\left(1-\lambda r^2\right)\mathrm{d}\tau^2 + \frac{2Gm}{U}\left(W\mathrm{d}\tau - \sum_{i=1}^{k}\frac{a_i\mu_i^2\,\mathrm{d}\varphi_i}{\Xi_i}\right)^2 \\ &+\sum_{i=1}^{k}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2\,\mathrm{d}\varphi_i^2 + \frac{U\,\mathrm{d}r^2}{V-2Gm} + \sum_{i=1}^{k+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mathrm{d}\mu_i^2 \\ &+\frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{k+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i\,\mathrm{d}\mu_i\right)^2, \end{aligned} \tag{3}$$

where $k \equiv [(D-1)/2]$, $\epsilon \equiv (D-1) \bmod 2$,

$$\sum_{i=1}^{k+\epsilon}\mu_i^2 = 1,$$

and

$$W \equiv \sum_{i=1}^{k+\epsilon}\frac{\mu_i^2}{\Xi_i}, \qquad U \equiv r^\epsilon \sum_{i=1}^{k+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{k}(r^2+a_j^2), \tag{4}$$

$$V \equiv r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{k}(r^2+a_i^2), \qquad \Xi_i \equiv 1+\lambda a_i^2. \tag{5}$$

The integer ϵ is known as the evenness number, which is 1 for even D and 0 for odd D. The metrics satisfy $R_{\mu\nu} = (D-1)\lambda g_{\mu\nu}$. The choices $\lambda > 0, \lambda = 0, \lambda < 0$ correspond to asymptotically de Sitter, flat (i.e. Myers-Perry), and anti-de Sitter cases, respectively. Moreover, for $\lambda \neq 0$, one has $|\lambda| = \ell^{-2}$, where ℓ is the (A)dS radius. The original solutions were presented in unit $G = 1$. However, since I will be discussing a formalism with the variable G, the explicit G-dependence is brought back carefully.

The outer horizon is located at $r = r_+$, where r_+ is the largest root of $V(r) - 2Gm = 0$. Therefore, one has

$$m = \frac{V(r_+)}{2G} = \frac{1}{2G}r_+^{\epsilon-2}(1-\lambda r_+^2)\prod_{i=1}^{k}(r_+^2+a_i^2). \tag{6}$$

The surface gravity κ and the area A of the event horizon are given by [24]

$$\kappa = r_+(1-\lambda r_+^2)\sum_i \frac{1}{r_+^2+a_i^2} - \frac{2-\epsilon+\epsilon\lambda r_+^2}{2r_+},$$

$$A = \mathcal{A}_{D-2}r_+^{\epsilon-1}\prod_i\frac{r_+^2+a_i^2}{\Xi_i},$$

where $\mathcal{A}_{D-2}$ is the volume of the unit $(D-2)$-sphere:

$$\mathcal{A}_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}.$$

The Hawking temperature and the entropy are then given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\left[r_+(1-\lambda r_+^2)\sum_i\frac{1}{r_+^2+a_i^2} - \frac{2-\epsilon+\epsilon\lambda r_+^2}{2r_+}\right], \tag{7}$$

$$S = \frac{A}{4G} = \frac{\mathcal{A}_{D-2}}{4G}r_+^{\epsilon-1}\prod_i\frac{r_+^2+a_i^2}{\Xi_i}. \tag{8}$$

The angular velocities, measured relative to a frame that is non-rotating at infinity, are given by

$$\Omega_i = \frac{(1-\lambda r_+^2)a_i}{r_+^2+a_i^2}, \tag{9}$$

and the angular momenta are given by

$$J_i = \frac{m a_i \mathcal{A}_{D-2}}{4\pi\Xi_i(\prod_j \Xi_j)}. \tag{10}$$

The physical mass E of the black holes are related to the mass parameter m via

$$E=\frac{m\mathcal{A}_{D-2}}{4\pi(\prod_j \Xi_j)}\left(\sum_{i=1}^{k}\frac{1}{\Xi_i}-\frac{1-\epsilon}{2}\right). \quad (11)$$

Finally, the first law of thermodynamics at fixed G reads

$$\tilde{\mathrm{d}}E=T\,\tilde{\mathrm{d}}S+\sum_i\Omega_i\,\tilde{\mathrm{d}}J_i, \quad (12)$$

where $\tilde{\mathrm{d}}$ denotes the total differential taken when G is considered to be a constant.

Before closing this section, note that, in [27], the black hole parameters such as the surface gravity κ, the area A of the event horizon, the physical mass E, and the Euclidean action I_D that will be used in the next section are presented separately for odd and even D. Here, I found it more convenient to rewrite these quantities for generic D in a unified form by using the evenness number ϵ. Inserting the corresponding values for ϵ will recover the original values of these quantities given in [27].

III. THERMODYNAMICS WITH VARIABLE NEWTON CONSTANT

The Euclidean actions for the black holes described in the preceding section are calculated explicitly in [27]. After properly restoring the Newton constant, the results read

$$I_D=\frac{\mathcal{A}_{D-2}}{8\pi T(\prod_j\Xi_j)}\left(m+\frac{\lambda r_+^{\epsilon}}{G}\prod_{i=1}^{k}(r_+^2+a_i^2)\right). \quad (13)$$

It was verified [27] that I_D obeys the identity

$$E-TS-\sum_i\Omega_i J_i=T\,I_D. \quad (14)$$

Equations (1) and (2) imply $\mu N=TI_D$; thus, Eq. (14) is recognized to be the Euler relation

$$E=TS+\sum_i\Omega_i J_i+\mu N. \quad (15)$$

Now, if G is considered to be a variable, one has a different total differential, e.g.,

$$\mathrm{d}m=\tilde{\mathrm{d}}m-\frac{m\mathrm{d}G}{G},\qquad \mathrm{d}S=\tilde{\mathrm{d}}S-\frac{S\,\mathrm{d}G}{G}.$$

Moreover, for any function of the form

$$f(m,a_i,r_+)=g(a_i,r_+)m, \quad (16)$$

one has

$$\begin{aligned}\mathrm{d}f&=m\,\mathrm{d}g+g\,\mathrm{d}m\\&=m\,\tilde{\mathrm{d}}g+g\left(\tilde{\mathrm{d}}m-\frac{m\mathrm{d}G}{G}\right)\\&=\tilde{\mathrm{d}}f-f\frac{\mathrm{d}G}{G}.\end{aligned} \quad (17)$$

Meanwhile, it follows from Eq. (1) that

$$\frac{\mathrm{d}G}{G}=-\frac{\mathrm{d}N}{N}.$$

Thus, Eq. (17) can also be written as

$$\tilde{\mathrm{d}}f=\mathrm{d}f-f\frac{\mathrm{d}N}{N}.$$

It is important to note that the quantities E and J_i are all proportional to m. For this reason,

$$\begin{aligned}\mathrm{d}E&=\tilde{\mathrm{d}}E+E\frac{\mathrm{d}N}{N}\\&=T\left(\mathrm{d}S-S\frac{\mathrm{d}N}{N}\right)+\sum_i\Omega_i\left(\mathrm{d}J_i-J_i\frac{\mathrm{d}N}{N}\right)+E\frac{\mathrm{d}N}{N}\\&=T\mathrm{d}S+\sum_i\Omega_i\,\mathrm{d}J_i+\left(E-TS-\sum_i\Omega_i J_i\right)\frac{\mathrm{d}N}{N}\\&=T\,\mathrm{d}S+\sum_i\Omega_i\,\mathrm{d}J_i+\mu\,\mathrm{d}N,\end{aligned} \quad (18)$$

where the Euler relation (15) has been used. The analysis does not rely on the choice of λ and the concrete value of D, provided that Eqs. (12) and (14) are valid, and λ and L are both kept as constants. Eqs. (15) and (18) lay down the fundamental relations in our formalism of black hole thermodynamics.

Please note that the inclusion of the (μ,N) variables implies that the first law (18) corresponds to an open thermodynamic system; the corresponding ensemble is grand canonical. One can, of course, consider the case with N fixed. Then, the first law (18) would fall back to (12), which corresponds to a closed thermodynamic system or a canonical ensemble. It should be stressed that even in the latter case, the variables (μ,N) are still meaningful, and the Euler relation (15) still holds. Therefore, our formalism is still different from the traditional formalism, which is governed only by the first law (12) and the generalized Smarr relation, without the Euler relation.

IV. HAWKING-PAGE TRANSITIONS WITH NEGATIVE λ

Before delving into the detailed analysis of the thermodynamic behaviors, let me first make a brief discussion about the possible HP transitions in either the canonical or grand canonical ensembles, i.e., regardless of whether G is variable or not.

The HP transition [26] is a particular kind of transition between the AdS black hole state and a thermal gas state which is characterized by a vanishing Gibbs free energy or, equivalently, a vanishing chemical potential.

Using definition (2) and Eq. (13), one has

$$\mu = \frac{\mathcal{A}_{D-2}}{8\pi N(\prod_j \Xi_j)}\left(m + \frac{\lambda r_+^\epsilon}{G}\prod_i (r_+^2 + a_i^2)\right). \tag{19}$$

It is evident that μ can become zero only when $\lambda < 0$. The zero appears when

$$m + \frac{\lambda r_+^\epsilon}{G}\prod_i (r_+^2 + a_i^2) = 0. \tag{20}$$

Substituting Eq. (6) into Eq. (20), one obtains

$$\frac{1}{2r_+^2}(1-\lambda r_+^2) + \lambda = 0. \tag{21}$$

Writing $\lambda = -\ell^{-2}$, the solution to Eq. (21) is found to be

$$(r_+)_{\rm HP} = \ell.$$

Therefore, the HP transition occurs precisely when the radius of the event horizon reaches the AdS radius.

The temperature at which the HP transition occurs is known as the HP temperature. In the present case, the HP temperature can be expressed analytically using the parameters ℓ and a_i. The result reads

$$T_{\rm HP} = \frac{\ell}{\pi}\sum_i \frac{1}{\ell^2 + a_i^2} - \frac{1-\epsilon}{\ell}.$$

V. PHYSICAL MASS AS A MACRO STATE FUNCTION

The first law (18) and the Euler relation (15) imply that the variables S, J_i, N are extensive, and their conjugates, T, Ω_i, μ, are intensive. This formalism conforms with the standard extensive thermodynamics; therefore, one naturally expects that the usual practice for analyzing the thermodynamic properties of macroscopic systems should also be applicable here. In particular, the physical mass and the intensive variables should all be expressible as homogeneous macro state functions in the extensive variables S, J_i, and N.

A. Generic λ

In the cases with generic λ, one can obtain from Eqs. (6) and (8) that

$$\prod_i \Xi_i = \frac{m\mathcal{A}_{D-2} r_+}{2S(1-\lambda r_+^2)}. \tag{22}$$

Inserting Eq. (22) into (10), one has

$$J_i = \frac{a_i S(1-\lambda r_+^2)}{2\pi r_+ (1+\lambda a_i^2)}. \tag{23}$$

Equation (23) can be viewed as an algebraic equation for a_i, whose solution gives a_i as functions in S, J_i, r_+,

$$a_i = a_i(S, J_i, r_+).$$

By inserting the functions $a_i(S, J_i, r_+)$ into Eq. (8), a very complicated equation for r_+ will arise, the solution of which gives a function

$$r_+ = r_+(S, \mathcal{J}), \tag{24}$$

where $\mathcal{J}$ denotes the sequence of all J_i. This in turn implies that a_i are actually functions in S and $\mathcal{J}$ because r_+ is no longer an independent variable:

$$a_i = a_i(S, J_i, r_+(S, \mathcal{J})). \tag{25}$$

By scaling arguments, it can be seen that the functions $r_+(S,\mathcal{J})$ and $a_i(S, J_i, r_+(S,\mathcal{J}))$ are all zeroth order homogeneous functions in $S, \mathcal{J}$. Finally, inserting Eqs. (1), (24), and (25) into (11), (7), (9), and (19), the macro state parameters E, T, Ω_i, μ can all be expressed as functions in $S, \mathcal{J}$, and N.

Although the corresponding functions are very complicated and are not worth explicitly presenting here, some key features can be recognized without much difficulty. In particular, $E(S,\mathcal{J},N)$ is a homogeneous function of the first order, and $T(S,\mathcal{J},N)$, $\Omega_i(S,\mathcal{J},N)$, and $\mu(S,\mathcal{J},N)$ are homogeneous functions of the zeroth order. These homogeneity behaviors are desired for the thermodynamic potential and intensive variables in any extensive thermodynamic system.

B. λ=0: Myers-Perry cases

The overwhelmingly complicated form for the macro state functions $E(S,\mathcal{J},N)$ and $T(S,\mathcal{J},N)$, $\Omega_i(S,\mathcal{J},N)$, $\mu(S,\mathcal{J},N)$ can be avoided if one considers only the cases with $\lambda = 0$, i.e., the Myers-Perry cases. In such cases, one has

$$\Xi_i = 1. \tag{26}$$

Hence, Eqs. (22) and (23) become

$$\frac{m\mathcal{A}_{D-2}}{2} = \frac{S}{r_+}, \tag{27}$$

$$\frac{a_i}{r_+} = \frac{2\pi J_i}{S}. \tag{28}$$

Substituting Eqs. (26) and (28) into Eq. (8), one obtains

$$S = \frac{N\mathcal{A}_{D-2}}{4L^{D-2}} r_+^{D-2} \prod_i \left[1 + \left(\frac{2\pi J_i}{S}\right)^2\right]. \tag{29}$$

This is an algebraic equation for r_+, whose solution reads

$$r_+ = L\left(\frac{4S}{\mathcal{A}_{D-2}N\prod_i\left[1+\left(\frac{2\pi J_i}{S}\right)^2\right]}\right)^{1/(D-2)}. \tag{30}$$

Inserting the above result into Eq. (28) one obtains an expression for a_i as a macro state function:

$$a_i = 2\pi L\left(\frac{J_i}{S}\right)\left(\frac{4S}{\mathcal{A}_{D-2}N\prod_j\left[1+\left(\frac{2\pi J_j}{S}\right)^2\right]}\right)^{1/(D-2)}. \tag{31}$$

Notice that if all J_i are equal to each other, a_i are also equal to each other. Finally, substituting Eqs. (26) and (30) into Eq. (11), one obtains an explicit and very compact expression for the physical mass E as a macro state function $E(S,\mathcal{J},N)$,

$$E(S,\mathcal{J},N) = (D-2)KNA\prod_i B_i, \tag{32}$$

where

$$K = \frac{(\mathcal{A}_{D-2})^{1/(D-2)}}{4^{(D-1)/(D-2)}\pi L}$$

is a constant factor, and

$$A = \left(\frac{S}{N}\right)^{(D-3)/(D-2)}, \quad B_i = \left[1+\left(\frac{2\pi J_i}{S}\right)^2\right]^{1/(D-2)}.$$

It is evident from Eq. (32) that the physical mass is proportional to N, with the coefficient of proportionality being a zeroth order homogeneous function in the extensive variables.

In principle, one can also obtain the macro state functions $T(S,\mathcal{J},N)$, $\Omega(S,\mathcal{J},N)$, and $\mu(S,\mathcal{J},N)$ explicitly by substituting Eqs. (26), (30), and (31) into the appropriate equations presented in Sec. II. However, the resulting expressions will be somewhat complicated and require some effort to simplify. For the sake of simplicity, I will proceed in an alternative way, i.e., by using the first law (18) and treating the intensive variables as partial derivatives of E. The results will be presented in the next section.

VI. EQUATIONS OF STATES FOR MYERS-PERRY BLACK HOLES

In this section, I will present the explicit form for the macro state functions $T(S,\mathcal{J},N)$, $\Omega(S,\mathcal{J},N)$, and $\mu(S,\mathcal{J},N)$ as the equation of states (EOS) for Myers-Perry black holes.

To begin with, it is necessary to write down the partial derivatives of the intermediate functions $A(S,N)$ and $B_i(S,J_i)$. These are given as follows:

$$\frac{\partial A}{\partial S} = \left(\frac{D-3}{D-2}\right)\frac{A}{S},$$
$$\frac{\partial A}{\partial N} = -\left(\frac{D-3}{D-2}\right)\frac{A}{N},$$
$$\frac{\partial B_i}{\partial S} = -\frac{2\pi J_i B_i}{(D-2)(S^2+2\pi J_i S)},$$
$$\frac{\partial B_i}{\partial J_i} = \frac{2\pi B_i}{(D-2)(S+2\pi J_i)}.$$

Using these relations, one finds

$$\frac{\partial}{\partial S}\left(A\prod_j B_j\right) = \frac{\chi(S,\mathcal{J})}{(D-2)S}A\prod_j B_j,$$
$$\frac{\partial}{\partial J_i}\left(A\prod_j B_j\right) = \frac{2\pi}{(D-2)(S+2\pi J_i)}A\prod_j B_j,$$
$$\frac{\partial}{\partial N}\left(A\prod_j B_j\right) = \frac{1}{(D-2)N}A\prod_j B_j,$$

where

$$\chi(S,\mathcal{J}) \equiv D-3-\sum_{i=1}^{k}\frac{2\pi J_i}{S+2\pi J_i}. \tag{33}$$

Therefore,

$$T(S,\mathcal{J},N) = \left(\frac{\partial E}{\partial S}\right)_{\mathcal{J},N} = K\left(\frac{N}{S}\right)\chi(S,\mathcal{J})A\prod_j B_j, \quad (34)$$

$$\Omega_i(S,\mathcal{J},N) = \left(\frac{\partial E}{\partial J_i}\right)_{S,\mathcal{J}\setminus J_i,N} = K\left(\frac{2\pi N}{S+2\pi J_i}\right)A\prod_j B_j, \quad (35)$$

$$\mu(S,\mathcal{J},N) = \left(\frac{\partial E}{\partial N}\right)_{S,\mathcal{J}} = KA\prod_j B_j. \quad (36)$$

Several remarks need to be noted as follows.

1) The explicit EOS allows for a straightforward re-verification of the Euler relation (15). Moreover, one can also find other mass formulae using the EOS, e.g.,

$$E = \frac{D-2}{D-3}\left(TS + \sum_i \Omega_i J_i\right), \quad (37)$$

$$E = (D-2)\mu N. \quad (38)$$

Equation (37) is already known as the Smarr relation.

2) The chemical potential $\mu(S,\mathcal{J},N)$ is strictly positive, which indicates that there is no HP transition in the asymptotically flat cases, and that the microscopic degrees of freedom are repulsive. This latter feature may be a signature for thermodynamic instability. More confirmative evidence for the thermodynamic instabilities will be given in the next section by analysis of the heat capacity.

3) The condition $T(S,\mathcal{J},N) \geq 0$ requires

$$\chi(S,\mathcal{J}) = D-3-\sum_{i=1}^{k}\frac{2\pi J_i}{S+2\pi J_i} \geq 0. \quad (39)$$

Since the expression $\dfrac{2\pi J_i}{S+2\pi J_i}$ increases monotonically with J_i and approaches the value 1 as $J_i \to \infty$ with finite S (recall here that $k = (D-1-\epsilon)/2$), one has

$$\min\chi(S,\mathcal{J}) = D-3-\frac{D-1-\epsilon}{2} = \frac{1}{2}(D+\epsilon-5).$$

For $D<4$, the bound (39) can be violated, signifying that the angular momentum cannot be too large. $D=4,5$ are critical in the sense that the bound (39) can be at most saturated but not violated. Therefore, the existence of extremal black holes of the Myers-Perry class cannot be excluded by use of the bound (39) alone in these dimensions. For $D>5$, T is always strictly positive, which excludes the existence of extremal Myers-Perry black holes in higher dimensions.

VII. HEAT CAPACITY OF MYERS-PERRY BLACK HOLES

The explicit form of the EOS allows for an analytical calculation for the heat capacity of Myers-Perry black holes. I particularly concentrate on the heat capacity associated with the macro processes with fixed $\mathcal{J}$ and N, i.e.,

$$C_{\mathcal{J},N} = T\left(\frac{\partial S}{\partial T}\right)_{\mathcal{J},N}.$$

The calculation of the heat capacity $C_{\mathcal{J},N}$ is essentially the calculation of the partial derivative $\left(\dfrac{\partial S}{\partial T}\right)_{\mathcal{J},N}$. This partial derivative cannot be calculated directly because S has not been written as an explicit function in $T,\mathcal{J}$, and N. However, using the EOS (34), one can calculate its inverse, i.e., $\left(\dfrac{\partial T}{\partial S}\right)_{\mathcal{J},N}$. To make the calculation more concise, it is better to start with the partial derivative of $\chi(S,\mathcal{J})$, which is defined in (33) with respect to S as follows:

$$\frac{\partial}{\partial S}\chi(S,\mathcal{J}) = \sum_i \frac{2\pi J_i}{(S+2\pi J_i)^2}.$$

Using the above result, one has

$$\begin{aligned}
\left(\frac{\partial T}{\partial S}\right)_{\mathcal{J},N} =& K\left[\frac{\partial}{\partial S}\left(\frac{N}{S}\right)\right]\chi(S,\mathcal{J})A\prod_j B_j \\
&+ K\left(\frac{N}{S}\right)\left(\frac{\partial}{\partial S}\chi(S,\mathcal{J})\right)A\prod_j B_j \\
&+ K\left(\frac{N}{S}\right)\chi(S,\mathcal{J})\frac{\partial}{\partial S}\left(A\prod_j B_j\right) \\
=& -\frac{T}{S} + K\left(\frac{N}{S}\right)\sum_i \frac{2\pi J_i}{(S+2\pi J_i)^2}A\prod_j B_j \\
&+ \frac{\chi(S,\mathcal{J})}{(D-2)S}T = \chi^{-1}(S,\mathcal{J})\left[\sum_i \frac{2\pi J_i}{(S+2\pi J_i)^2}\right. \\
&\left.+\frac{\chi^2(S,\mathcal{J})}{(D-2)S} - \frac{\chi(S,\mathcal{J})}{S}\right]T.
\end{aligned}$$

Consequently, the heat capacity can be written as

$$C_{\mathcal{J},N} = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{\mathcal{J},N}} = \chi(S,\mathcal{J}) \times \left[\sum_i \frac{2\pi J_i}{(S+2\pi J_i)^2} + \frac{\chi^2(S,\mathcal{J})}{(D-2)S} - \frac{\chi(S,\mathcal{J})}{S}\right]^{-1}. \quad (40)$$

The analytical result (40) for the heat capacity makes it possible to analyze the thermodynamic (in)stability for Myers-Perry black holes in generic dimensions. For arbitrary choices of J_i, the detailed analysis can still be quite complicated; therefore, I will proceed only with some simplified cases.

1) $J_i = 0$ for all *i*, i.e. the Schwarzschild-Tangherlini cases

In such cases, one has

$$\chi(S,\mathcal{J}) = D-3,$$

and consequently,

$$C_{\mathcal{J},N} = -(D-2)S < 0,$$

which shows that the higher dimensional Schwarzschild-Tangherlini black holes are thermodynamically unstable.

2) $J_1 = J, J_i = 0$ for all $i \geq 2$, i.e., the cases with a single rotation parameter

In these cases, one has

$$\chi(S,\mathcal{J}) = D-3-\frac{2\pi J}{S+2\pi J} > 0 \quad \text{for} \quad D \geq 5 \text{ and } J < \infty,$$
$$C_{\mathcal{J},N} = \chi(S,\mathcal{J})\mathcal{D}^{-1}(S,\mathcal{J}),$$

where

$$\mathcal{D}(S,\mathcal{J}) \equiv \frac{2\pi J}{(S+2\pi J)^2} + \frac{\chi^2(S,\mathcal{J})}{(D-2)S} - \frac{\chi(S,\mathcal{J})}{S} = -\frac{(D-4)(4\pi J+S)^2+(D-2)S^2}{2(D-2)S(2\pi J+S)^2} < 0 \quad \text{for} \quad D \geq 5.$$

Therefore, the higher dimensional Kerr black holes with a single rotation parameter always have a negative heat capacity, indicating that such black holes are thermodynamically unstable.

3) $J_i = J \neq 0$ for all *i*, i.e., the cases with *k* equal rotation parameters

In these cases, one has

$$\chi(S,\mathcal{J}) = D-3-\frac{D-1-\epsilon}{2}\left(\frac{2\pi J}{S+2\pi J}\right) > 0 \quad \text{for} \quad D \geq 5 \text{ and } J < \infty, \quad (41)$$

$$C_{\mathcal{J},N} = \chi(S,\mathcal{J})\tilde{\mathcal{D}}^{-1}(S,\mathcal{J}), \quad (42)$$

where

$$\tilde{\mathcal{D}}(S,\mathcal{J}) \equiv \frac{D-1-\epsilon}{2}\frac{2\pi J}{(S+2\pi J)^2} + \frac{\chi^2(S,\mathcal{J})}{(D-2)S} - \frac{\chi(S,\mathcal{J})}{S} = -\frac{(D-4)(D\pi J+S)^2+(D-2)^2S^2}{D(D-2)S(2\pi J+S)^2} < 0 \text{ for } D \geq 5.$$

One thus concludes that for all $D \geq 5$, the heat capacity (42) is always negative, indicating that the higher dimensional Myers-Perry black holes with equal rotation parameters are all thermodynamically unstable.

Before concluding this section, note that the negativeness of the heat capacity of Myers-Perry black holes has already been studied in previous works using different methods in various limiting cases; see [28, 29]. However, to the best of my knowledge, the representation of the heat capacity purely in terms of the extensive variables has not been previously presented.

VIII. SUMMARY AND CONCLUSIONS

The major achievements and conclusions of the present paper are summarized as follows.

1) The variable Newton constant formalism for black hole thermodynamics holds for general rotating black hole solutions in higher dimensional Einstein gravity with or without a cosmological constant. In this formalism, the first law and the Euler relation hold simultaneously, and the physical mass is fully extensive.

2) It can be inferred from the zero of the chemical potential $\mu = GTI_D/L^{D-2}$ that the HP transitions appear only in asymptotically AdS cases and only when the radius of the event horizon approaches the AdS radius. The HP temperature can be expressed analytically in terms of the AdS radius ℓ and the rotation parameters a_i.

3) For Myers-Perry black holes, the physical mass and the intensive variables can be written as explicit functions in the extensive variables, and the results have a remarkable simple and compact form. The homogeneity behaviors of these macro state functions are transparent.

4) The calculation for the heat capacity of Myers-Perry black holes can be carried out analytically and can

be shown to be always negative in the example cases with zero, one, and k equal rotation parameters. The results indicate that the corresponding black holes are thermodynamically unstable. The thermodynamic instability might also be inferred from the strict positivity of the chemical potential in the asymptotically flat cases.

The above results provide more evidence for the applicability and strength of the new formalism for black hole thermodynamics proposed in [1-3]. It is expected that this formalism should also be applicable to black holes in extended theories of gravity. Further studies in this direction are currently underway, and progresses will be reported soon.

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