Strong coupling from inclusive semileptonic decay of charmed mesons*

Jinfei Wu (吴金飞)[1,2†] Xinchou Lou (娄辛丑)[1,3,4] Yuzhi Che (车逾之)[1,5] Gang Li (李刚)[1]
Yanping Huang (黄燕萍)[1] Manqi Ruan (阮曼奇)[1] Jingbo Ye (叶竞波)[1]

[1]Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
[2]China Center of Advanced Science and Technology, Beijing 100190, China
[3]University of Texas at Dallas, Richardson 75083, Texas, USA
[4]Center for High Energy Physics, Henan Academy of Sciences, Zhengzhou 450046, China
[5]University of Chinese Academy of Sciences, Beijing 100049, China

Abstract: In this study, we employed the heavy quark expansion model with the kinetic scheme to evaluate $\alpha_S(m_c^2)$, the strong coupling constant at the charm quark mass m_c, using data on inclusive semileptonic decays of charmed mesons. Using the experimental values of the semileptonic decay widths of the D^0 and the D^+, the value of $\alpha_s(m_c^2)$ was determined to be $0.445\pm0.009\pm0.114$, where the first uncertainty is experimental and the second is systematic. This value of $\alpha_s(m_c^2)$ is in good agreement with the value of $\alpha_s(m_c^2)$ which calculated by running $\alpha_S(m_Z^2)$ at the Z^0 boson mass m_Z with the renormalization group evolution equation. In addition, the values of $\alpha_s(m_c^2)$ obtained individually from each of the D^0, D^+, and D_s^+ mesons were consistent, as they were of the same origin.

Keywords: strong coupling constant, charmed mesons, inclusive semileptonic decay, heavy quark expansion

DOI: 10.1088/1674-1137/ad8baf **CSTR:** 32044.14.ChinesePhysicsC.49023001

I. INTRODUCTION

In the Standard Model of elementary particle physics, Quantum Chromo-Dynamics (QCD) is the gauge field theory for the strong interaction. In QCD, gluons are force mediators, and α_S (the effective strong coupling constant) dictates many features of the strong interaction. Asymptotic freedom, in which the strength of α_S increases as the energy scale decreases, is one of the primary features of QCD. The value of α_S has been measured over the energy scale ranging from the τ lepton mass m_τ to several TeV, and it has been found to be consistent with the theoretical prediction. However, α_S has not been measured at energies below m_τ . In this regime, the QCD physics may enter the non-perturbative scheme and exhibit unknown behaviors. Therefore, measuring α_S at lower energies to further understand QCD and probe possible new physics is very desirable.

In the past five decades, significant progress has been achieved in the theoretical description of inclusive semileptonic decays of charmed and B mesons using the framework of the heavy quark expansion (HQE) model [1−7]. In the HQE framework, the features of the inclusive semileptonic decays of heavy quarks are expressed in terms of α_S, quark masses, Cabibbo–Kobayashi–Maskawa (CKM) matrix elements, and non-perturbative parameters. HQE calculations accurately describe experimental features of inclusive semileptonic decays of charmed and B mesons [5−10]. In addition, the HQE model has been employed as a reliable method for experimentally extracting the b quark mass and $|V_{cb}|$ with inclusive semileptonic decays of B mesons [8, 10−14]. In these studies, the b quark mass and $|V_{cb}|$ were determined from the fits to the observables of inclusive semileptonic decays of B mesons, where α_S was fixed to the value running from $\alpha_S(m_Z^2)$.

This procedure can also be applied to inclusive semileptonic decays of charmed mesons. Experimental measurements of m_c and $|V_{cs}|$ have become more precise [15−17], which will enable $\alpha_S(m_c^2)$ to be determined as a parameter from charmed mesons, either by fixing the values of m_c and $|V_{cs}|$ to those measured in processes other than semileptonic D decays, or through a fit that simultaneously extracts m_c, $|V_{cs}|$, and $\alpha_S(m_c^2)$ from inclusive semileptonic decays of charmed mesons. In this article, we present a determination of $\alpha_S(m_c^2)$ from inclusive semileptonic decays of charmed mesons.

Received 26 September 2024; Accepted 28 October 2024; Published online 29 October 2024

* Supported by the National Natural Science Foundation of China (12247119, 12042507)

† E-mail: wujf@ihep.ac.cn

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI. Article funded by SCOAP[3] and published under licence by Chinese Physical Society and the Institute of High Energy Physics of the Chinese Academy of Sciences and the Institute of Modern Physics of the Chinese Academy of Sciences and IOP Publishing Ltd

II. HEAVY QUARK EXPANSION MODEL IN THE KINETIC SCHEME

In this study, the theoretical calculation [5] of the inclusive semileptonic decay width (Γ_{SL}) for charmed mesons was employed to derive $\alpha_S(m_c^2)$. In [5], the authors considered $O(\alpha_S)$ and $O(\beta_0\alpha_S^2)$ corrections [18, 19], as well as $O(1/m_c^3)$ contributions [20], when calculating Γ_{SL}. As shown in Eq. (1) [5], Γ_{SL} is expressed in terms of $\alpha_S(m_c^2)$, quark masses, the CKM matrix element $|V_{cs}|$, and non-perturbative corrections. In Eq. (1), G_F is the Fermi coupling constant, r is the square of the ratio of the strange quark mass to the charm quark mass (m_s^2/m_c^2), $\alpha_S \equiv \alpha_S(m_c^2)$, μ_π^2 and μ_G^2 are the kinetic and chromomagnetic dimension five operators [21−23], respectively, and ρ_D^3 and ρ_{LS}^3 are the Darwin and the spin-orbital (LS) dimension six operators [23], respectively, in the HQE model. The weak annihilation (WA) contribution, B_{WA}, depends on the type of spectator quark within each charmed meson.

$$\begin{aligned}\Gamma_{SL} = \frac{G_F^2 m_c^5}{192\pi^3}|V_{cs}|^2\Big[f_0(r) + \frac{\alpha_S}{\pi}f_1(r) + \frac{\alpha_S^2}{\pi^2}f_2(r)\\ + \frac{\mu_\pi^2}{m_c^2}f_\pi(r) + \frac{\mu_G^2}{m_c^2}f_G(r) + \frac{\rho_{LS}^3}{m_c^3}f_{LS}(r)\\ + \frac{\rho_D^3}{m_c^3}f_D(r) + \frac{32\pi^2}{m_c^3}B_{WA}\Big]. \qquad (1)\end{aligned}$$

The coefficients of the perturbative and non-perturbative items, $f_{0,1,2}(r)$ and $f_{\pi,G,LS,D}(r)$, respectively, were calculated using Eq. (2) [5], where n_f is the number of active flavors and β_0 is the QCD beta function, $\beta_0 = 11 - 2n_f/3$.

$$\begin{aligned}f_0(r) &= 1 - 8r + 8r^3 - r^4 - 12r^2\cdot\log(r),\\ f_1(r) &= 2.86\sqrt{r} - 3.84r\cdot\log(r),\\ f_2(r) &= \beta_0[8.16\sqrt{r} - 1.21r\cdot\log(r) - 3.38],\\ f_\pi(r) &= -f_0(r)/2,\\ f_G(r) &= \frac{1}{2}f_0(r) - 2(1-r)^4,\\ f_{LS}(r) &= -f_G(r),\\ f_D(r) &= \frac{77}{6} + O(r) + 8\log(\frac{\mu_{WA}^2}{m_c^2}). \qquad (2)\end{aligned}$$

The infrared cutoff scale μ in the kinetic scheme was set to 0.5 GeV. In the theoretical expression for $f_D(r)$, 0.8 GeV was treated as the $\overline{\text{MS}}$ renormalization scale (μ_{WA}) associated with the mix of Darwin and WA operators [5, 24, 25]. In Eq. (1), only the process of $c\to sl\bar{\nu}$ (which was slightly different from experimental measurements [26, 27] because of missing Cabibbo-suppressed processes) was taken into account. A corresponding systematic uncertainty was assigned to cover the missing processes in the determination of $\alpha_S(m_c^2)$.

III. FITTING METHOD

The χ^2 minimization method was employed to determine $\alpha_S(m_c^2)$ from fits of $\hat{\Gamma}_{SL}$, which is the Γ_{SL} expression of Eq. (1) for different charmed mesons. The χ^2 function is expressed as

$$\chi^2(\alpha_S,\theta_j) = \sum_i \frac{[\Gamma_{SL,D_i} - \hat{\Gamma}_{SL}(\alpha_S,\theta_j)]^2}{\sigma^2_{\Gamma_{SL,D_i}}} + \sum_j \frac{(\theta_j - \theta_j')^2}{\sigma^2_{\theta_j'}}, \qquad (3)$$

where D_i denotes D^+, D^0, or D_s^+; Γ_{SL,D_i} and $\sigma_{\Gamma_{SL,D_i}}$ are the measured inclusive semileptonic decay width and the corresponding uncertainty of D_i, respectively; and $\theta_j = \{m_c$, m_s, $|V_{cs}|$, μ_G^2, μ_π^2, ρ_D^3, $\rho_{LS}^3\}$ represents the constrained parameters (the values and uncertainties of which are θ_j' and $\sigma_{\theta_j'}$, respectively).

The variable G_F was fixed at 1.1663788×10^{-5} [15]. According to [5], the values of B_{WA} for D^+, D^0, and D_s^+ are fixed at −0.001, −0.001, and −0.002 GeV^3, respectively. Except for G_F and B_{WA}, the parameters were allowed to float when determining $\alpha_S(m_c^2)$. The value of $|V_{cs}|$ has been measured to be 0.975 ± 0.006 [15]. In the kinetic scheme, the expected values of μ_G^2 and ρ_{LS}^3 do not run with respect to the energy scale, and they have been determined to be 0.288 ± 0.049 GeV^2 and -0.113 ± 0.090 GeV^3, respectively, from inclusive semileptonic B decays [10]. In [5, 8, 28], the values of μ_π^2(0.5 GeV) and ρ_D^3(0.5 GeV) were determined to be 0.26 ± 0.06 GeV^2 and 0.05 ± 0.04 GeV^3, respectively, which were evolved to $\mu = 0.5$ GeV using $O(\alpha_S^2)$ expressions from values of $\mu = 1$ GeV. The mass of the strange quark was set to 93.4 ± 8.6 MeV [15].

The convergence of the perturbative series in the Γ_{SL} expression is strongly affected by the mass definition of the charm quark [29−32]. In [16], the pole mass and the $\overline{\text{MS}}$ scheme exhibited bad convergence behaviors in the QCD corrections to Γ_{SL}. To avoid the divergence, the kinetic scheme [29, 31, 33] was introduced to calculate Γ_{SL}. The relationship between $\overline{\text{MS}}$ and the kinetic mass of the charm quark has been investigated to three-loop order (N^{3}LO) [16, 17]. For different choices of μ_s ($\overline{\text{MS}}$ scale), the value of m_c at a scale of 0.5 GeV in the kinetic scheme m_c^{kin}(0.5 GeV) has been obtained separately using the relationship in [16, 17]:

$$\begin{aligned}m_c^{\text{kin}}(0.5\text{ GeV}) &= 1336\text{ MeV for }\overline{m}_c(\mu_s = 3\text{ GeV}),\\ m_c^{\text{kin}}(0.5\text{ GeV}) &= 1372\text{ MeV for }\overline{m}_c(\mu_s = 2\text{ GeV}),\\ m_c^{\text{kin}}(0.5\text{ GeV}) &= 1404\text{ MeV for }\overline{m}_c(\mu_s = \overline{m}_c). \qquad (4)\end{aligned}$$

The average value of $m_c^{\rm kin}(0.5\ {\rm GeV})$ from different μ_s values was treated as the input value of $m_c(0.5\ {\rm GeV})$ in the χ^2 fit, which was determined to be 1370 MeV. For a conservative estimate, the largest difference between $m_c(0.5\ {\rm GeV})$ and $m_c^{\rm kin}(0.5\ {\rm GeV})$ was taken as the uncertainty in $m_c(0.5\ {\rm GeV})$. To evaluate the bias caused by the choice of m_c and $|V_{cs}|$, the first fit was performed with m_c as a free parameter and with $|V_{cs}|$ allowed to vary within one standard error; the second fit was performed with m_c and $|V_{cs}|$ both fixed at the world average. The results for $\alpha_S(m_c^2)$ from these fits were compared to check the consistency of the experiment.

IV. EXPERIMENTAL INPUTS

The experimental measurement of Γ_{SL} was derived from the inclusive semileptonic decay branch fraction, $\mathcal{B}_{SL}$ [26, 27], and the lifetime, τ [15], via Eq. (5), where D_i denotes D^+, D^0, or D_s^+:

$$\Gamma_{SL,\ D_i} = \frac{6.582\times10^{-25}\cdot\mathcal{B}_{SL}(D_i\to Xe\nu_e)}{\tau_{D_i}}\ {\rm GeV}. \quad (5)$$

In Eq. (5), τ_{D_i} is the mean life of D_i, and $\mathcal{B}_{SL}(D_i \to Xe\nu_e)$ is the branch fraction of the inclusive semileptonic decay for D_i. The inclusive semileptonic branch fractions of D^+, D^0, and D_s^+ have been measured by the CLEO-c [26] detector using 818 pb^{-1} and 602 pb^{-1} open-charm data at $E_{CM} = 3.774$ GeV and 4.170 GeV. Because of limited statistics, the uncertainty in $\mathcal{B}_{SL,D_s^+}$ was much higher than that in $\mathcal{B}_{SL,D^+/D^0}$ in the CLEO-c measurements. Recently, $\mathcal{B}_{SL,D_s^+}$ has also been measured by the BESIII instrument using 3.19 fb^{-1}, 2.08 fb^{-1}, and 1.05 fb^{-1} e^+e^- collision data at $E_{CM} = 4.178$ GeV, 4.189–4.219 GeV, and 4.225 – 4.230 GeV [27]. The uncertainty in $\mathcal{B}_{SL,D_s^+}$ has been reduced by the additional data provided by the BESIII measurements. The $\mathcal{B}_{SL,D^+/D^0}$ value from CLEO-c and the $\mathcal{B}_{SL,D_s^+}$ value from BESIII were adopted to calculate the Γ_{SL} values of D^+, D^0, and D_s^+. In Table 1, the input values of $\mathcal{B}_{SL}(D_i \to Xe\nu_e)$, τ_{D_i}, and Γ_{SL,D_i} are displayed. The consistent Γ_{SL} values of D^0 and D^+ indicate the reliability of the HQE model for inclusive semileptonic decays of D^0 and D^+.

Except for $\mathcal{B}_{SL}$, the distributions of electron momentum ($|p_{e^+}|$) in the laboratory frame have been measured for inclusive semileptonic decays of D^+, D^0, and D_s^+ by CLEO-c and BESIII [26, 27], as shown in Fig. 1. The average $|p_{e^+}|$ values of D^0, D^+, and D_s^+ are also plotted in Fig. 1. Kolmogorov-Smirnov (KS) tests [34] between the distributions of $|p_{e^+}|$ and $\overline{|p_{e^+}|}$ were performed to further validate the reliability of the HQE model for inclusive semileptonic decays of charmed mesons. The results of the KS tests are shown in Table 2.

The $|p_{e^+}|$ distributions for D^+, D^0, and D_s^+ were consistent. This was a strong indication that the HQE model was reliable for inclusive semileptonic decays of charmed mesons. Because experimental measurements of $|p_{e^+}|$ were not available in the center-of-mass frame of the charmed mesons, only Γ_{SL} was used to extract $\alpha_S(m_c^2)$ in this study.

Table 1. Input values of $\mathcal{B}_{SL}(D_i \to Xe\nu_e)$, τ_{D_i}, and Γ_{SL,D_i}.

D_i	$\mathcal{B}_{SL}$ (%)	$\tau\ (10^{-13}{\rm s})$	$\Gamma_{SL}\ (10^{-15}{\rm GeV})$
D^0	$6.46\pm0.09\pm0.11$	4.10 ± 0.01	104 ± 2
D^+	$16.13\pm0.10\pm0.29$	10.33 ± 0.05	103 ± 2
D_s^+	$6.30\pm0.13\pm0.10$	5.04 ± 0.04	82 ± 2

V. RESULTS

The value of $\alpha_S(m_c^2)$ was extracted from D^+, D^0, and D_s^+, including

- D^+, D^0, and D_s^+, respectively.
- D^+ and D^0 combined.

In the χ^2 fit, high-order perturbative corrections needed to be taken into account for the inclusive

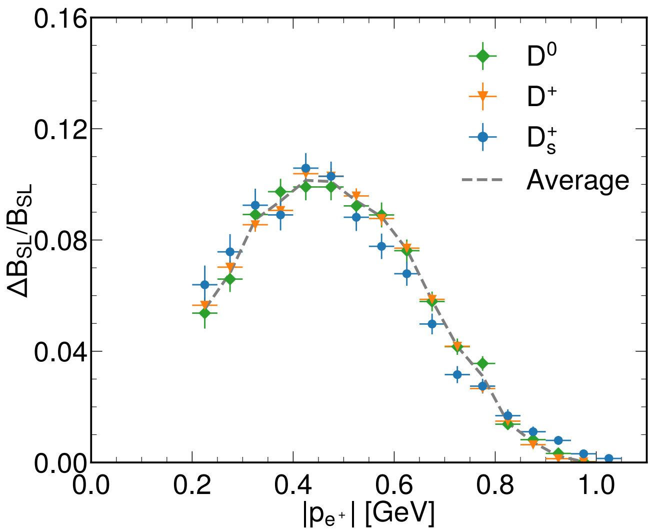

Fig. 1. (color online) Distributions of $|p_{e^+}|$ with $|p_{e^+}| > 200$ MeV from inclusive semileptonic decays of D^+, D^0, and D_s^+ in the laboratory frame. The green diamonds and orange triangles are the results of D^0 and D^+, respectively, measured by CLEO-c [26]. The blue dots are the results of D_s^+ measured by BESIII [27]. The dashed gray line is the average of D^0 and D^+.

Table 2. Results of the KS tests, in which the null hypothesis was that the two tested distributions are identical.

Test Distributions	Test Statistic	P Value
$\lvert p_{e^+,D^0}\rvert$ and $\overline{\lvert p_{e^+}\rvert}$	0.125	1.000
$\lvert p_{e^+,D^+}\rvert$ and $\overline{\lvert p_{e^+}\rvert}$	0.125	1.000
$\lvert p_{e^+,D_s^+}\rvert$ and $\overline{\lvert p_{e^+}\rvert}$	0.132	0.992

semileptonic decays of the charmed mesons. The α_S^3 order correction to $b \to c l \bar{\nu}$ has been determined to be less than 1% in the kinetic scheme [7]. For a conservative estimate, 5% of Γ_{SL} was taken as the high-order perturbative corrections for the inclusive semileptonic decays of the charmed mesons. Furthermore, the theoretical calculation of Γ_{SL} in Eq. (1) was the contribution of $c \to s l \bar{\nu}$, in which Cabibbo-suppressed processes were missed. To cover missed Cabibbo-suppressed processes, $|V_{cd}|^2/(|V_{cd}|^2+|V_{cs}|^2) \approx 5\%$ was treated as the uncertainty in the Γ_{SL} expression. In total, 10% is taken as the theoretical uncertainty in the calculation of Γ_{SL} for a more conservative estimate. The input values of the dimension six HQE matrix elements were evolved from the results obtained in the inclusive semileptonic B decays at $\mu = 1$ GeV. The treatment of the inputs of the dimension six HQE matrix elements may have impacted the systematic uncertainties, which can be improved by obtaining more precise measurements of the inclusive semileptonic decays in the charm sector. Despite the fact that the kinetic scheme was adopted to improve the convergence of the perturbative series, the contribution of higher-order corrections was larger owing to the slow convergence behavior in the charm sector, which may have caused the systematic uncertainties to be underestimated. To reduce the corresponding systematic uncertainty, more measurements in the charm sector, such as spectral moments, can benefit the determination of higher-order corrections. High-order perturbative corrections played an important role in this study, and advanced theoretical calculations of high-order perturbative corrections are highly desirable.

In Fig. 2 and Table 3, the fitted $\alpha_S(m_c^2)$ value of each sample is shown and compared to $\alpha_S(m_c^2)$ running from $\alpha_S(m_Z^2)$ using **RunDec** [35] with a renormalization group evolution equation. Because of relatively heavy spectator quarks in D_s^+, the combined result of D^+ and D^0 was chosen to measure $\alpha_S(m_c^2)$ in this study. Using the combined sample of D^0 and D^+, $\alpha_S(m_c^2)$ was determined to be $0.445 \pm 0.009_{\rm exp.} \pm 0.081_{m_c} \pm 0.056_{\rm trun.} \pm 0.057_{\rm others}$ at $m_c =$ 1.3701 GeV, where the first uncertainty is experimental, the second is the uncertainty in m_c, the third is associated with high-order perturbative corrections in the Γ_{SL} expression, and the fourth is related to other sources. As shown in Fig. 3, the measured value of $\alpha_S(m_c^2)$ was consistent within 1σ of the value running from $\alpha_S(m_Z^2)$. The consistent values of $\alpha_S(m_c^2)$ among different charmed mesons indicated the robustness of this method. In the fit for the combined D^0 and D^+ sample, the value of χ^2/dof of the fit was 0.1/6, indicating good agreement between the data and the model. In Fig. 4, the profile contours of different samples confirmed the consistency among these charmed mesons and the robustness of this method.

To check the stability of the results of this study, the value of m_c was fixed at 1.370 ± 0.034 GeV, and the corresponding uncertainty was estimated by varying the value of m_c within $\pm 1\sigma$. Usually, the value of $|V_{cs}|$ is obtained from exclusive semileptonic or leptonic charmed meson decays; however, this technique could have introduced bias in this study. Hence, obtaining a value for $|V_{cs}|$ without involving semileptonic charmed meson decays was necessary to validate the results of this study. Using $|V_{cd}| = 0.2181 \pm 0.0049 \pm 0.0007$ from the leptonic decays of D^+ [47] and $|V_{cb}| = (41.1 \pm 1.2) \times 10^{-3}$ [15], the value of $|V_{cs}|$ without involving semileptonic charmed meson decays was calculated to be 0.975 ± 0.001 via Eq. (6), which has a negligible bias in the determination of $\alpha_S(m_c^2)$.

$$|V_{cs}| = \sqrt{1-|V_{cd}|^2-|V_{cb}|^2} = 0.975 \pm 0.001 \tag{6}$$

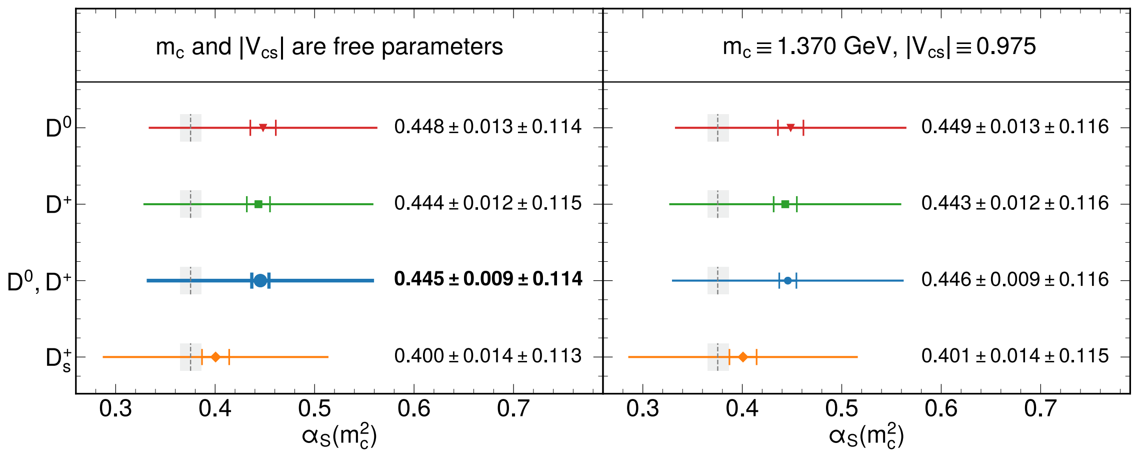

Fig. 2. (color online) In the left panel, m_c and $|V_{cs}|$ were allowed to float in the fit; in the right panel, m_c and $|V_{cs}|$ were fixed in the fit. Points with error bars are the determined central values of $\alpha_S(m_c^2)$, and the inner and the outer error bars are the experimental and total uncertainties, respectively. The dashed gray line and shaded box indicate the value of and uncertainty in $\alpha_S(m_c^2)$ running to m_c from $\alpha_S(m_Z^2)$, respectively.

Table 3. Values of $\alpha_S(m_c^2)$ obtained for each sample, where the values of m_c and $|V_{cs}|$ were allowed to change in the fit. The first and second uncertainties of $\alpha_S(m_c^2)$ are the experimental and theoretical uncertainties, respectively. The result, which was jointly obtained from D^0 and D^+ (bold), was similar to 0.375 ± 0.011, the value of α_S running from m_Z down to m_c.

Sample	D^0	D^+	D_s^+, D^0	D_s^+
m_c[GeV]	1.3701 ± 0.0339	1.3699 ± 0.0340	$\mathbf{1.3701\pm0.0338}$	1.3699 ± 0.0340
$\alpha_S(m_c^2)[10^{-3}]$	$448\pm13\pm114$	$444\pm12\pm115$	$\mathbf{445\pm9\pm114}$	$400\pm14\pm113$

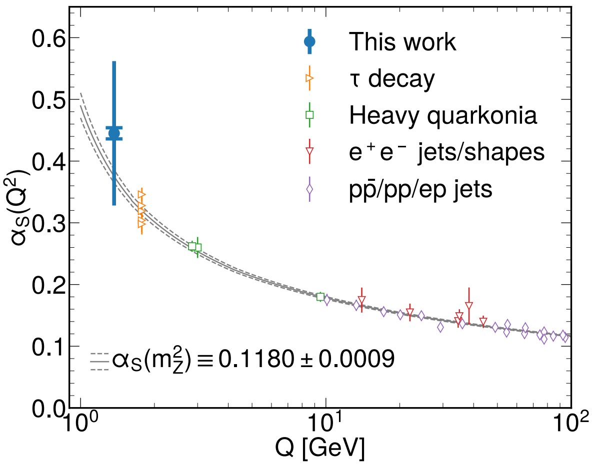

Fig. 3. (color online) Values of α_S at different energy scales. The blue dot is the measured $\alpha_S(m_c^2)$ value obtained in this study, where the inner and the outer error bars are the experimental and total uncertainties, respectively. The other points are measurements of α_S at different energy scales [36–46]. The solid and dashed gray lines are the values and uncertainties of α_S running from $\alpha_S(m_Z^2)$, respectively.

Figure 2 presents the fitted $\alpha_S(m_c^2)$ values for different D meson samples for a fixed m_c. The robustness of this study was confirmed by the consistent values of $\alpha_S(m_c^2)$ obtained via fits with fixed and floating values of m_c.

VI. SUMMARY

In summary, the value of $\alpha_S(m_c^2)$ at $m_c = 1.37$ GeV was determined to be $0.445\pm0.009\pm0.114$ using the

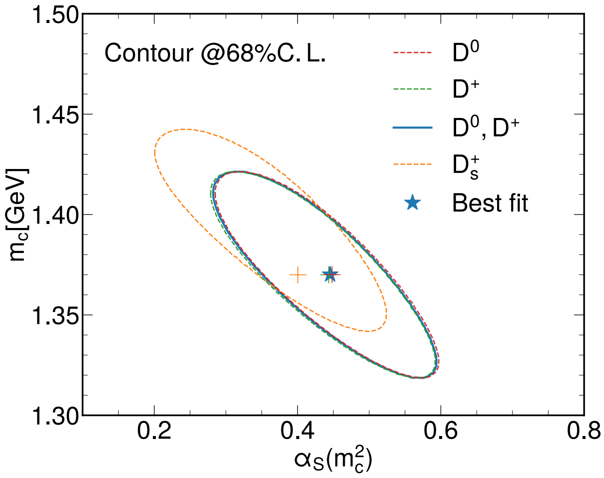

Fig. 4. (color online) Profile contours of the different samples at the 68% confidence level. The solid blue curve and star are the contour and best-fit value for D^0 and D^+ combined, respectively. The dashed orange curve and cross are the contour and best-fit value for D_s^+, respectively. The dashed red and the green curves and crosses are the contours and best-fit values for D^0 and D^+, respectively.

semileptonic decay widths of the D^0 and D^+ measured by CLEO-c, and it was cross-checked using the Γ_{SL} of the D_s^+ meson reported by BESIII. This result for $\alpha_S(m_c^2)$ was in good agreement with the value obtained by running $\alpha_S(m_Z^2)$ to m_c. The values of $\alpha_S(m_c^2)$ were derived for each of the D^0, D^+, and D_s^+ mesons, and were found to be within $\pm1\sigma$ of each other, illustrating the robustness of the analysis method. The leading uncertainty in $\alpha_S(m_c^2)$ was from the theoretical calculation of Γ_{SL}, which can be reduced by detailed experimental studies on the semileptonic decays of the D mesons as well as superior HQE calculations. This study represents the first measurement of $\alpha_S(m_c^2)$ obtained using a new approach. With additional statistical data and enhanced modeling of the HQE, the systematic uncertainty in the value of $\alpha_S(m_c^2)$ may be significantly reduced in the future.

VII. ACKNOWLEDGMENTS

The authors thank H. B. Li, X. T. Huang, X. Chen, G. Y. Zhang, J. L. Pei, H. Q. Zhang, Y. Q. Fang, and L. G. Shao for fruitful discussions.

References

[1] J. R. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B **100**, 313 (1975)
[2] N. Isgur, D. Scora, B. Grinstein *et al.*, Phys. Rev. D **39**, 799 (1989)
[3] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev *et al.*, Phys. Rev. Lett. **71**, 496 (1993), arXiv: hep-ph/9304225
[4] B. Blok, L. Koyrakh, M. Shifman *et al.*, Phys. Rev. D **49**, 3356 (1994)
[5] P. Gambino and J. F. Kamenik, Nucl. Phys. B **840**, 424 (2010), arXiv: 1004.0114 [hep-ph]
[6] M. Fael, T. Mannel, and K. K. Vos, JHEP **12**, 067 (2019)
[7] M. Fael, K. Schönwald, and M. Steinhauser, Phys. Rev. D **104**, 016003 (2021), arXiv: 2011.13654 [hep-ph]
[8] O. L. Buchmüller and H. U. Flächer, Phys. Rev. D **73**,

073008 (2006)
[9] D. King, A. Lenz, M. L. Piscopo *et al*., JHEP **08**, 241 (2022)
[10] G. Finauri and P. Gambino, arXiv: 2310.20324 [hep-ph]
[11] P. Gambino, JHEP **09**, 055 (2011)
[12] P. Gambino and C. Schwanda, Phys. Rev. D **89**, 014022 (2014), arXiv: 1307.4551 [hep-ph]
[13] A. Alberti, P. Gambino, K. J. Healey *et al*., Phys. Rev. Lett. **114**, 061802 (2015), arXiv: 1411.6560 [hep-ph]
[14] F. Bernlochner, M. Fael, K. Olschewsky *et al*., arXiv: 2205.10274 [hep-ph]
[15] R. L. Workman *et al*. (Particle Data Group), PTEP **2022**, 083C01 (2022)
[16] M. Fael, K. Schönwald, and M. Steinhauser, Phys. Rev. Lett. **125**, 052003 (2020), arXiv: 2005.06487 [hep-ph]
[17] M. Fael, K. Schönwald, and M. Steinhauser, Phys. Rev. D **103**, 014005 (2021), arXiv: 2011.11655 [hep-ph]
[18] A. Czarnecki and M. Jezabek, Nucl. Phys. B **427**, 3 (1994), arXiv: hep-ph/9402326
[19] V. Aquila, P. Gambino, G. Ridolfi *et al*., Nucl. Phys. B **719**, 77 (2005), arXiv: hep-ph/0503083
[20] M. Gremm and A. Kapustin, Phys. Rev. D **55**, 6924 (1997), arXiv: hep-ph/9603448
[21] I. I. Y. Bigi, N. G. Uraltsev, and A. I. Vainshtein, Phys. Lett. B **293**, 430 (1992), arXiv: hep-ph/9207214
[22] B. Blok and M. A. Shifman, Nucl. Phys. B **399**, 459 (1993), arXiv: hep-ph/9209289
[23] I. I. Y. Bigi, M. A. Shifman, and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. **47**, 591 (1997), arXiv: hep-ph/9703290
[24] P. Gambino, G. Ossola, and N. Uraltsev, JHEP **09**, 010 (2005), arXiv: hep-ph/0505091
[25] P. Gambino, P. Giordano, G. Ossola *et al*., JHEP **10**, 058 (2007)
[26] D. M. Asner *et al*. (CLEO collaboration), Phys. Rev. D **81**, 052007 (2010), arXiv: 0912.4232 [hep-ex]
[27] M. Ablikim *et al*. (BESIII collaboration), Phys. Rev. D **104**, 012003 (2021), arXiv: 2104.07311 [hep-ex]
[28] P. Gambino and N. Uraltsev, Eur. Phys. J. C **34**, 181 (2004), arXiv: hep-ph/0401063
[29] M. Beneke and V. M. Braun, Nucl. Phys. B **426**, 301 (1994), arXiv: hep-ph/9402364
[30] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev *et al*., Phys. Rev. D **50**, 2234 (1994), arXiv: hep-ph/9402360
[31] I. I. Y. Bigi, M. A. Shifman, N. Uraltsev *et al*., Phys. Rev. D **56**, 4017 (1997), arXiv: hep-ph/9704245
[32] K. Melnikov and T. v. Ritbergen, Phys. Lett. B **482**, 99 (2000), arXiv: hep-ph/9912391
[33] M. Antonelli *et al*., Phys. Rept. **494**, 197 (2010), arXiv: 0907.5386 [hep-ph]
[34] P. Virtanen *et al*., Nature Meth. **17**, 261 (2020), arXiv: 1907.10121 [cs.MS]
[35] F. Herren and M. Steinhauser, Comput. Phys. Commun. **224**, 333 (2018), arXiv: 1703.03751 [hep-ph]
[36] A. Pich and A. Rodríguez-Sánchez, Phys. Rev. D **94**, 034027 (2016), arXiv: 1605.06830 [hep-ph]
[37] M. Davier, A. Höcker, B. Malaescu *et al*., Eur. Phys. J. C **74**, 2803 (2014), arXiv: 1312.1501 [hep-ex]
[38] C. Peset, A. Pineda, and J. Segovia, JHEP **09**, 167 (2018)
[39] D. Boito, M. Golterman, K. Maltman *et al*., Phys. Rev. D **91**, 034003 (2015), arXiv: 1410.3528 [hep-ph]
[40] D. Boito, M. Golterman, A. Keshavarzi *et al*., Phys. Rev. D **98**, 074030 (2018), arXiv: 1805.08176 [hep-ph]
[41] S. Narison, Int. J. Mod. Phys. A **33**, 1850045 (2018), arXiv: 1801.00592 [hep-ph]
[42] J. Schieck, S. Bethke, S. Kluth *et al*. (JADE collaboration), Eur. Phys. J. C **73**, 2332 (2013), arXiv: 1205.3714 [hep-ex]
[43] V. M. Abazov *et al*., Phys. Rev. D **80**, 111107 (2009), arXiv: 0911.2710 [hep-ex]
[44] H. Abramowicz *et al*. (ZEUS collaboration), Nucl. Phys. B **864**, 1 (2012), arXiv: 1205.6153 [hep-ex]
[45] V. M. Abazov *et al*. (D0 collaboration), Phys. Lett. B **718**, 56 (2012), arXiv: 1207.4957 [hep-ex]
[46] V. Andreev *et al*. (H1 collaboration), Eur. Phys. J. C **77**, 791 (2017), arXiv: 1709.07251 [hep-ex]
[47] Y. S. Amhis *et al*. (HFLAV collaboration), Phys. Rev. D **107**, 052008 (2023), arXiv: 2206.07501 [hep-ex]