

Holographic Einstein rings of AdS black holes in Horndeski theory*

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Abstract: Using the AdS/CFT correspondence and wave optics techniques, we conduct an extensive study of the imaging properties of holographic Einstein rings in the context of anti-de Sitter (AdS) black holes (BHs) in Horndeski theory. Our results indicate that the optical characteristics of these holographic Einstein rings are significantly influenced by the observer's position, physical parameters of the BH, nature of the wave source, and configuration of the optical system. Specifically, when the observer is positioned at the north pole of the AdS boundary, the holographic image prominently displays a ring structure aligning with the BH's photon sphere. We thoroughly analyze the effect of various physical parameters — including the observation position, event horizon radius, temperature, and the parameter γ in Horndeski theory — on the holographic Einstein rings. These parameters play a crucial role in determining the rings' radius and brightness, with variations potentially causing the ring structures to deform or even transform into bright spots. Furthermore, our comparative analysis between wave and geometric optics reveals a strong agreement in predicting the positions and brightness of both the photon and Einstein rings. This paper offers new insights into the spacetime geometry of BHs in Horndeski theory and proposes a promising framework for exploring the gravitational duals of strongly coupled systems.

Keywords: black hole, AdS/CFT correspondence, Horndeski theory, holographic Einstein rings

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I. INTRODUCTION

In a landmark discovery, the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors observed gravitational waves (GWs) emanating from the merger of two stellar-mass black holes (BHs) [1,2], thereby providing compelling direct evidence for the existence of BHs. This pivotal achievement was soon followed by the detection of additional GW signals from mergers of diverse compact objects, significantly expanding the realm of observational astronomy [3–7]. Complementing these findings, the Event Horizon Telescope (EHT) collaboration captured the first image of a supermassive BH at the center of the M87 galaxy [8–13], marking a historic milestone as the inaugural direct observation of a BH. More recently, the EHT collaboration unveiled an image of the supermassive BH at the core of our own Milky Way galaxy, SgrA* [14–19]. These images depict a dark region encircled by a luminous ring; the dark area corresponds to the BH shadow, whereas the bright ring is the photon sphere [20, 21]. The morphology and dimensions of these shadows vary depending on the BH type and the underlying gravitational theory. For

example, a Schwarzschild BH yields a perfectly circular shadow [22, 23], whereas a rotating Kerr BH manifests a slightly deformed D-shaped shadow owing to its spin [24–26]. In recent years, extensive investigations into BH shadows within various modified gravity theories have revealed that these shadows harbor rich information on spacetime geometry, offering novel perspectives for probing alternative gravitational models [27–66].

Currently, research on BH shadows predominantly employs ray-tracing techniques within the framework of geometric optics. However, Hashimoto *et al.* have recently introduced a wave optics-based approach grounded in the holographic principle and anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [67, 68]. The AdS/CFT correspondence, a concrete realization of the holographic principle, posits an equivalence between quantum gravity theories in AdS space and CFT defined on its boundary [69]. A prominent example of this duality is the correspondence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [70, 71]. This duality has been extensively applied to study strongly coupled systems, offering signi-

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ficant theoretical insights in fields such as quantum chromodynamics (QCD) and condensed matter physics, and it has stimulated further exploration into other holographic correspondences [72–93]. More specifically, Hashimoto *et al.* investigated a $(2+1)$ -dimensional boundary CFT at finite temperature, constructing the response function of the boundary quantum field theory (QFT) by introducing a temporally periodic localized Gaussian source as a scalar field with AdS boundary conditions [67, 68]. Subsequently, they transformed this response function into a holographic image of the dual BH using wave optics, resulting in an image structure analogous to an Einstein ring. This finding suggests that within a thermal QFT environment, a holographic "Einstein ring" image of the dual BH can be generated, providing a direct method for testing gravitational duality. Future experiments designed to observe holographic images of BHs in matter could significantly advance the study of quantum gravity. Motivated by this approach, Yang *et al.* explored the holographic Einstein rings of charged AdS BHs, revealing that while the radius of an Einstein ring remains unaffected by the chemical potential, it is notably influenced by temperature [94]. Subsequent investigations have extended these studies to holographic Einstein rings in various modified gravity backgrounds [95–98]. In this paper, we extend these explorations by examining the holographic Einstein rings of AdS BHs in Horndeski theory.

Horndeski theory, recognized as the most general scalar-tensor theory with second-order field equations, has attracted renewed interest since the introduction of the covariant Galileon model [99–102]. To address the Ostrogradsky instability inherent in such theories, Gleyzes *et al.* proposed the beyond Horndeski framework [103, 104], within which Babichev *et al.* discovered static BH solutions [105]. The investigation of BHs in Horndeski theory and its extensions has attracted considerable interest from the scientific community [106–114]. In our previous work [115], we explored photon dynamics in BH spacetimes within the Horndeski framework using ray-tracing techniques and constrained the theory's parameters based on EHT observations. Additionally, we examined the observational signatures of asymmetric thin-shell wormholes (ATWs) in Horndeski theory, identifying unique features such as "lens bands" and "photon ring sets" that distinguish BHs from wormholes [116]. Building on the methodology of Hashimoto *et al.*, this paper employs wave optics to investigate the holographic Einstein rings of AdS BHs in Horndeski theory, analyzing the correlation of holographic images with the parameters of the theory and the properties of the wave sources. Before delving into the details of our model, we note that the specific Horndeski BH solutions we study depend on parameters η and β , which together define $\gamma = \beta^2/\eta$. For physical solutions, $\eta \geq 0$ is required to ensure a healthy (ghost-free) scalar field, implying

$\gamma \geq 0$. Nevertheless, following our previous work [115, 116], we also consider $\eta < 0$ (and thus $\gamma < 0$), leading to negative-temperature solutions that are not physically meaningful. We include these negative- γ regimes purely as a theoretical extension to probe the dependence of imaging characteristics on the parameter space. Such mathematical continuations, while not realistic, can reveal the sensitivity of the image properties to changes in the underlying parameters.

The remainder of this paper is structured as follows: In Sec. II, we provide an overview of AdS BHs within the Horndeski theory framework, detailing the holographic setup and lensing response function in this spacetime. Section III discusses the utilization of an optical system comprising convex lenses and a spherical screen to study the holographic Einstein rings of AdS BHs in Horndeski theory, as well as a comparative analysis of wave and geometric optics results. Finally, Section IV concludes the paper. In this paper, we employ natural units (with $G = c = 1$).

II. RESPONSE FUNCTION FOR ADS BHs IN HORNDESKI THEORY

To begin our analysis, we consider the action of Horndeski gravity with a negative cosmological constant Λ , expressed as [99]

$$S = \int d^4x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 - 2\Lambda), \quad (1)$$

with

$$\mathcal{L}_2 = G_2(X), \quad (2)$$

$$\mathcal{L}_3 = -G_3(X)\Box\phi, \quad (3)$$

$$\mathcal{L}_4 = G_4(X)\mathcal{R} + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], \quad (4)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ & - \frac{G_{5X}}{6}[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3]. \end{aligned} \quad (5)$$

In this context, $\mathcal{R}$ denotes the Ricci scalar, and $G_{\mu\nu}$ represents the Einstein tensor. The scalar field is ϕ , with its kinetic term defined as $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. The functions $G_i = G_i(\phi, X)$ depend on both ϕ and X , where the subscript X indicates differentiation with respect to X . Building on the work in Ref. [105], we consider a specific case

of the Horndeski action (1), characterized by

$$G_2 = \eta X, \quad G_4 = \zeta + \beta \sqrt{-X}, \quad G_3 = G_5 = 0. \quad (6)$$

By setting $\zeta = \frac{1}{2}$ and considering η and β as dimensionless parameters, the action in Eq. (1) reduces to

$$S = \int d^4x \sqrt{-g} \left\{ \left[\frac{1}{2} + \beta \sqrt{(\partial\phi)^2/2} \right] \mathcal{R} - \frac{\eta}{2} (\partial\phi)^2 - \frac{\beta}{\sqrt{2}(\partial\phi)^2} [(\Box\phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2] - 2\Lambda \right\}. \quad (7)$$

A spherically symmetric AdS BH solution in Horndeski theory can be derived from Eq. (7) and is given by [105]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (8)$$

in which

$$f(r) = 1 - \frac{2M}{r} - \frac{\gamma}{r^2} + \frac{r^2}{l^2}. \quad (9)$$

To simplify the expressions in Eq. (9), we introduce the parameter $\gamma = \beta^2/\eta$. Here, $l = \sqrt{-3\Lambda}$ denotes the radius of the AdS spacetime, and the constant M acts as an integration constant corresponding to the mass of the BH. By setting $f(r_h) = 0$, we determine the radius r_h of the BH's outer event horizon. Consequently, the BH mass can be re-expressed in terms of the event horizon radius r_h as

$$M = \frac{r_h^2 + r_h^4 - \gamma}{2r_h}, \quad (10)$$

where we have adopted the unit $l = 1$ in Eq. (10) for simplicity.

To develop a holographic model for AdS BHs in Horndeski theory, we reformulate the metric function by introducing a new variable $\xi = 1/r$. This substitution leads to the following expression:

$$f(r) = \frac{1}{\xi^2} f(\xi). \quad (11)$$

Furthermore, the Hawking temperature of the BH is

$$T = \frac{3 + \xi_h^2 + \xi_h^4 \gamma}{4\pi \xi_h}, \quad (12)$$

where ξ_h is the event horizon of the BH. Note that $\eta \geq 0$ ensures a canonical scalar field kinetic term. If $\eta < 0$, the field becomes ghost-like and $\gamma = \beta^2/\eta$ is negative, resulting in a negative BH temperature, which is not phys-

ically interpretable. Despite this, we present results for $\gamma < 0$ as a mathematical probe of parameter sensitivity, without implying that these are physically realizable BHs.

Moreover, we can express the metric as follows:

$$ds^2 = \frac{1}{\xi^2} \left[-f(\xi) dt^2 + \frac{1}{f(\xi)} d\xi^2 + d\theta^2 + \sin^2\theta d\varphi^2 \right], \quad (13)$$

where $f(\xi)$ is the metric function redefined in terms of ξ . To simplify calculations near the event horizon and analyze wave propagation more effectively, we adopt the ingoing Eddington–Finkelstein coordinates $(v, \xi, \theta, \varphi)$, defined by

$$v = t + \xi_* = t - \int \frac{d\xi}{f(\xi)}. \quad (14)$$

In these new coordinates, the metric (13) transforms to

$$ds^2 = \frac{1}{\xi^2} \left[-f(\xi) dv^2 - 2dv d\xi + d\theta^2 + \sin^2\theta d\varphi^2 \right]. \quad (15)$$

Now we consider an evolution of the scalar field ϕ , which is governed by the Klein-Gordon equation,

$$\Box\phi(v, \xi, \theta, \varphi) = 0. \quad (16)$$

Combining Eqs. (15) and (16), we arrive at

$$\xi^2 f(\xi) \partial_\xi^2 \phi + [\xi^2 f'(\xi) - 2\xi f(\xi)] \partial_\xi \phi - 2\xi^2 \partial_v \partial_\xi \phi + 2\xi \partial_v \phi + \xi^2 D_S^2 \phi = 0, \quad (17)$$

where $f'(\xi) = \partial_\xi f(\xi)$, and D_S^2 denotes the scalar Laplacian on the unit two-sphere S^2 . Following Ref. [94], the asymptotic solution of Eq. (17) near the AdS boundary is given by

$$\phi(v, \xi, \theta, \varphi) = \mathcal{J}_O(v, \theta, \varphi) + \xi \partial_v \mathcal{J}_O(v, \theta, \varphi) + \frac{1}{2} \xi^2 D_S^2 \mathcal{J}_O(v, \theta, \varphi) + \xi^3 \langle O \rangle + O(\xi^4), \quad (18)$$

where $\mathcal{J}_O(v, \theta, \varphi)$ is the scalar source term, and $\langle O \rangle$ is the corresponding response function. For a monochromatic axisymmetric Gaussian wave packet source located at the south pole of the AdS boundary, we have [95]

$$\mathcal{J}_O(v, \theta) = e^{i\omega v} \cdot \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(\pi - \theta)^2}{2\sigma^2}\right] = e^{i\omega v} \sum_{l=0}^{\infty} C_{l0} Y_{l0}(\theta), \quad (19)$$

where

$$C_{l0} = (-1)^l \sqrt{\frac{l+1/2}{2\pi}} \exp\left[-\frac{1}{2}(l+1/2)^2 \sigma^2\right]. \quad (20)$$

Here, σ characterizes the width of the Gaussian wave packet, Y_{l0} represents the spherical harmonics, and ω is the angular frequency of the monochromatic wave source, determining the temporal oscillation of the wave. In the limit $\sigma \ll \pi$, the scalar field $\phi(v, \xi, \theta, \varphi)$ can be expanded as

$$\phi(v, \xi, \theta, \varphi) = e^{i\omega v} \sum_{l=0}^{\infty} c_{l0} U_l(\xi) Y_{l0}(\theta, \varphi), \quad (21)$$

and the response function can be expressed as

$$\langle O \rangle = e^{i\omega v} \sum_{l=0}^{\infty} \langle O \rangle_l Y_{l0}(\theta). \quad (22)$$

Substituting Eq. (21) into the field equation, we find

$$\begin{aligned} \xi^2 f(\xi) U_l'' + [\xi^2 f'(\xi) - 2\xi f(\xi) + 2i\omega \xi^2] U_l' \\ + [-2i\omega \xi - l(l+1)\xi^2] U_l = 0, \end{aligned} \quad (23)$$

with the asymptotic behavior of U_l near the boundary given by

$$\lim_{\xi \rightarrow 0} U_l = 1 - i\omega \xi + \frac{1}{2} [-l(l+1)] \xi^2 + \langle O \rangle_l \xi^3 + O(\xi^4). \quad (24)$$

The function U_l satisfies two boundary conditions. The first is at the horizon $\xi = \xi_h$:

$$[\xi_h^2 f'(\xi_h) + 2i\omega \xi_h^2] U_l' - [2i\omega \xi_h + l(l+1)\xi_h^2] U_l = 0, \quad (25)$$

and the second is the AdS boundary condition at $\xi = 0$:

$$U_l(0) = 1. \quad (26)$$

By imposing the boundary conditions (25) and (26), we can solve the differential Eq. (23) for U_l . Subsequently, the total response function $\langle O \rangle$ can be determined using Eqs. (22) and (24).

In Fig. 1, we display the amplitude of the response function as a function of source frequency ω for various values of the parameter γ . Note that while $\gamma < 0$ leads to a ghost field and negative-temperature solutions, which are not physically meaningful, we still include these cases in the present and subsequent discussions as a theoretical extension. This approach enables us to explore the dependence of the image properties on the parameter space. Figure 1(a) demonstrates that as the source frequency ω increases, both the amplitude and period of the response function decrease. In contrast, Fig. 1(b) shows that the amplitude of the response function diminishes with increasing values of γ . Furthermore, Fig. 2 explores the relationship between the amplitude of the response function and the temperature T for both positive and negative values of γ . The findings suggest that in both scenarios, the amplitude of the response function increases with increasing temperature.

III. HOLOGRAPHIC RINGS OF ADS BHs IN HORNDESKI THEORY

In this section, we introduce the optical system shown in Fig. 3(a) to simulate the imaging of the Einstein ring

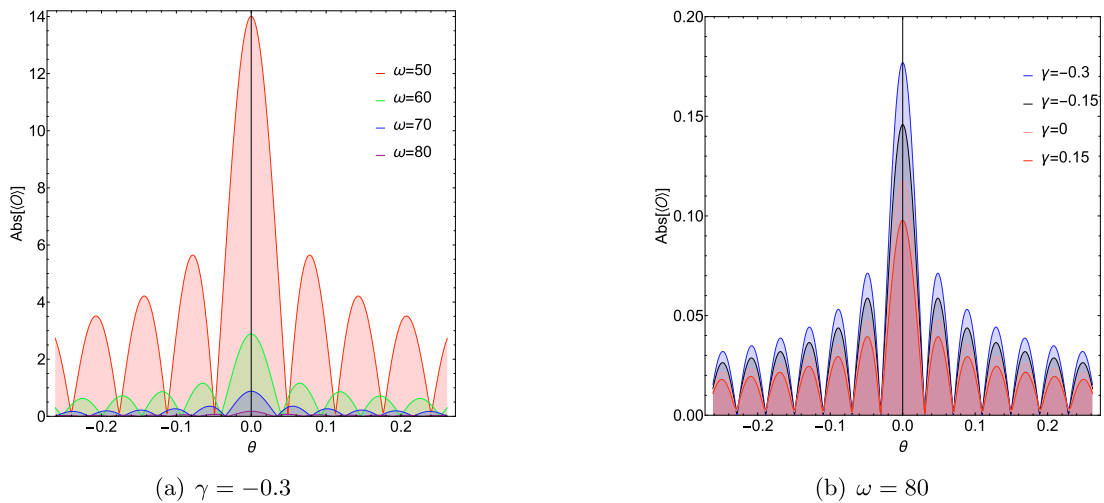


Fig. 1. (color online) Effects of different source frequencies $\omega = 50, 60, 70, 80$ and parameters $\gamma = -0.3, -0.15, 0, 0.15$ on the response function. In this analysis, we set $\xi_h = 1$.

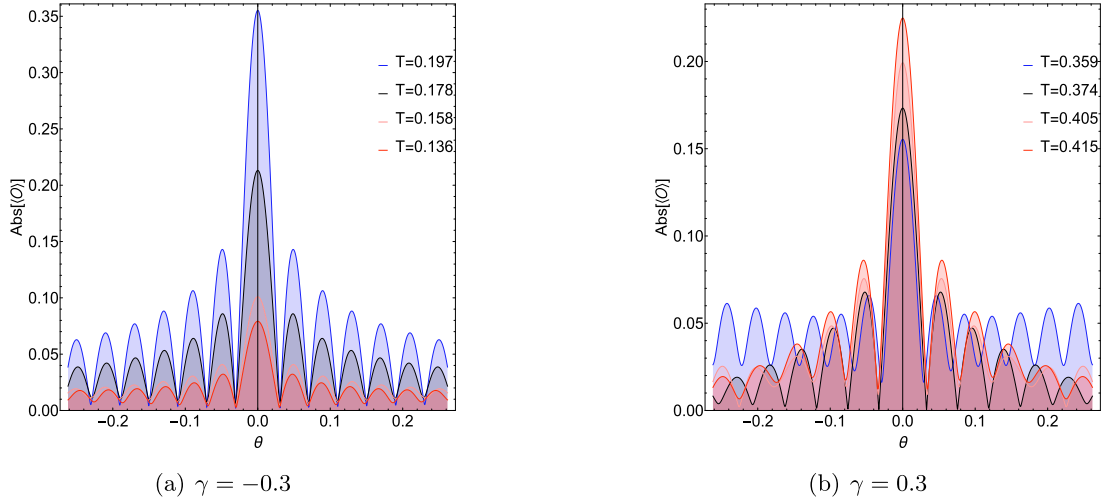


Fig. 2. (color online) Effects of different temperatures on the response function. In this analysis, we set $\omega = 80$.

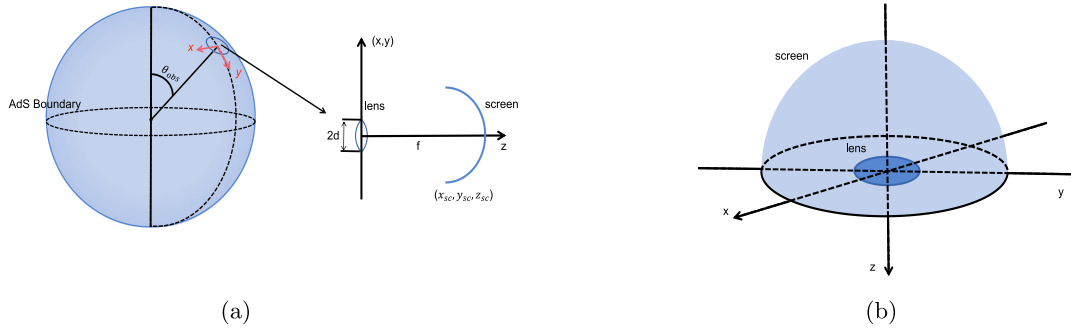


Fig. 3. (color online) Optical system.

and subsequently explore its wave characteristics. The system consists of a spherical screen and an infinitely thin convex lens, with the focal length f of the lens being much greater than its size d . The observation center is set at $(\theta_{\text{obs}}, 0)$, marked in blue in Fig. 3(a). This optical system is positioned at the observation center, with the lens at $\vec{x} = (x, y, 0)$ and the screen at $\vec{x}_S = (x_S, y_S, z_S)$. The response function is first transformed into a transmitted wave $\Psi_s(\vec{x})$ by the convex lens, which is then projected onto the screen and converted into an observable wave $\Psi_{sc}(\vec{x}_S)$, as depicted in Fig. 3(b). Therefore, we can obtain [95]

$$\Psi_{sc}(\vec{x}_S) = \int_{|\vec{x}| < d} d^2x \Psi_s(\vec{x}) e^{-i\omega L} = \int_{|\vec{x}| < d} d^2x e^{-i\omega \frac{|\vec{x}|^2}{2f}} \Psi_p(\vec{x}) e^{-i\omega L}, \quad (27)$$

where L represents the distance between $\vec{x}$ and $\vec{x}_S$, whereas f is defined by

$$f^2 = x_S^2 + y_S^2 + z_S^2. \quad (28)$$

By further combining the expression

$$L = \sqrt{(x_S - x)^2 + (y_S - y)^2 + z_S^2} \simeq f - \frac{\vec{x}_S \cdot \vec{x}}{f} + \frac{|\vec{x}|^2}{2f}, \quad (29)$$

where the Fresnel approximation $f \gg |\vec{x}|$, Eq. (27) can be expressed as

$$\Psi_{sc}(\vec{x}_S) \propto \int_{|\vec{x}| < d} d^2x \Psi_p(\vec{x}) \varpi(\vec{x}) e^{-\frac{i\omega}{f} \vec{x} \cdot \vec{x}_S}, \quad (30)$$

where the window function is

$$\varpi(\vec{x}) = \begin{cases} 1, & 0 \leq |\vec{x}| \leq d, \\ 0, & |\vec{x}| > d. \end{cases} \quad (31)$$

Equation (30) demonstrates that the incident wave can be transformed into the observed wave on the screen via a Fourier transformation, thereby delineating the profiles of the double BHs on the observer's screen. In Fig. 4, we present holographic Einstein images at various observation angles θ and different values of γ . The results reveal that the holographic Einstein images exhibit significant variations depending on the observation angle and value of γ . When $\theta = 0^\circ$, with the observer positioned at the

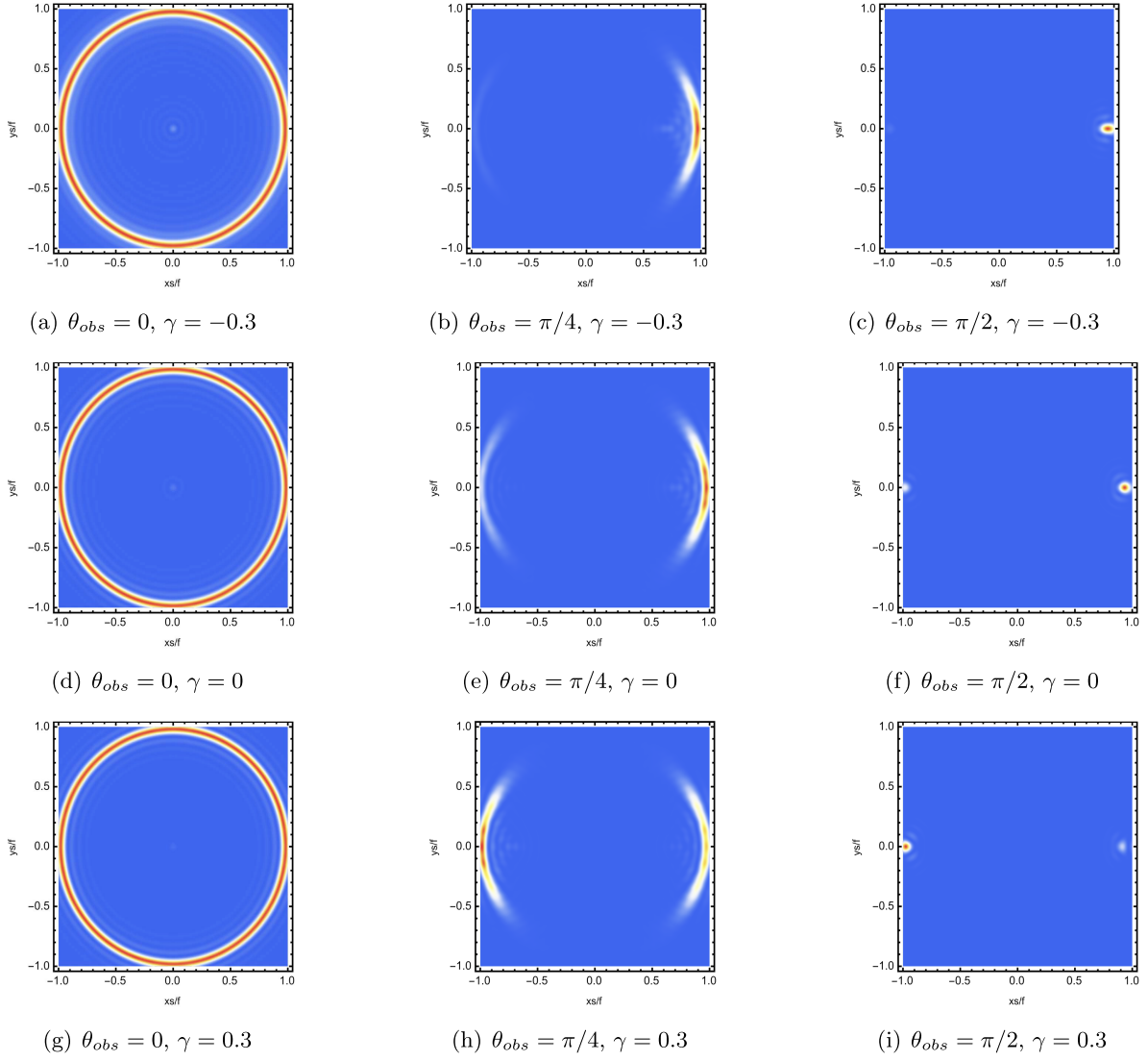


Fig. 4. (color online) Holographic images at different observation angles θ_{obs} and parameter values γ for the AdS BH in Horndeski theory. In this analysis, we set $\xi_h = 1$ and $\omega = 80$.

north pole of the AdS boundary, Figs. 4(a), 4(d), and 4(g) depict the holographic Einstein images as a series of axially symmetric concentric rings, with the brightest ring located at the center, forming a Poisson-like spot. Notably, the brightness of these rings remains relatively unchanged as γ increases. As the observation angle increases to $\theta = 45^\circ$, Figs. 4(b), 4(e), and 4(h) show that the holographic Einstein images transition into two arcs with varying brightness, where the left arc appears dimmer than the right one. As γ increases, the brightness of the left arc gradually intensifies. Finally, when the observation angle reaches $\theta = 90^\circ$, Figs. 4(c), 4(f), and 4(i) indicate that the two arcs further evolve into two small bright spots. As γ increases, the brightness of the left spot also gradually increases. These findings underscore that both θ and γ have a significant influence on the morphology and brightness of the holographic Einstein images. This

suggests that the spacetime characteristics of the BH are effectively encoded in these holographic images, offering a potential tool for probing the underlying geometry of BHs through observational data.

In Fig. 5, we explore the effect of varying frequencies on the holographic Einstein images generated by the wave source. Figs. 5(a), 5(b), and 5(c) demonstrate that the rings become progressively sharper as the frequency increases. This sharpening is expected, as the geometric optics approximation becomes increasingly accurate at higher frequencies, enabling finer image details to be captured. Additionally, Figs. 5(d), 5(e), and 5(f) depict the corresponding lensing response function for the holographic Einstein images. The results suggest that as the frequency increases, the brightness of the lensing response decreases significantly. This decrease in brightness may be attributed to the diminishing diffraction ef-

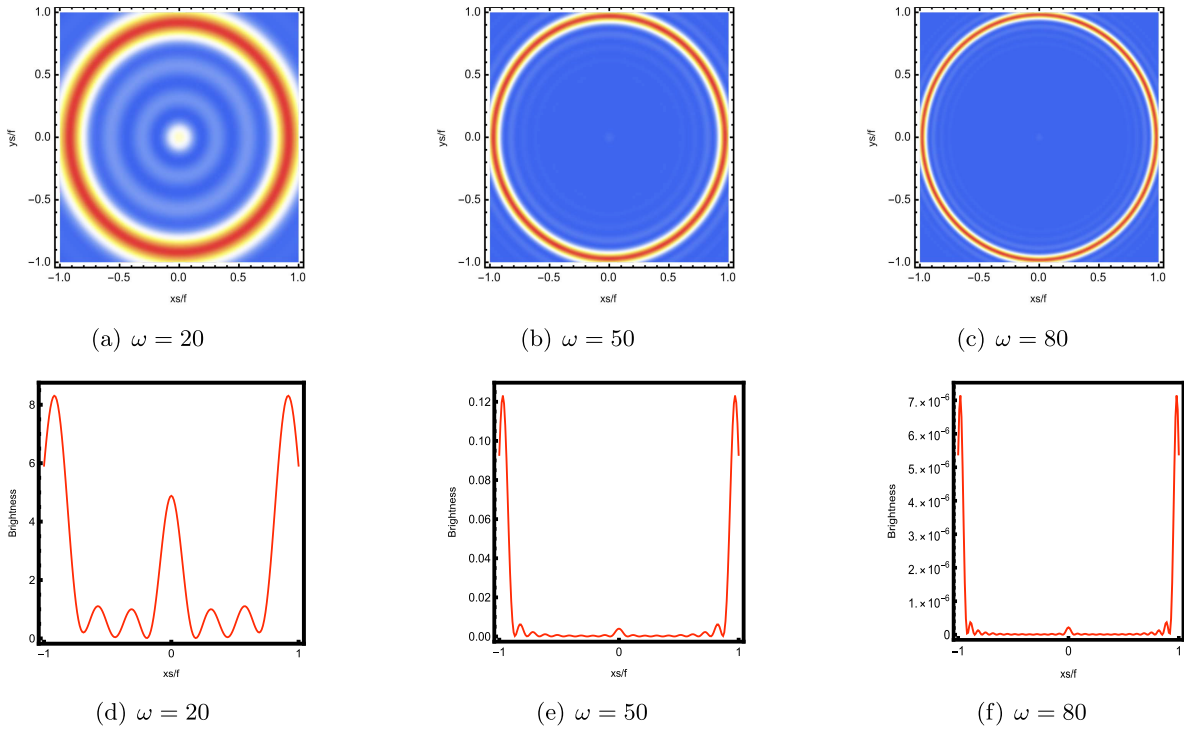


Fig. 5. (color online) Holographic images and the corresponding lensed response brightness at different source frequencies $\omega = 20, 50, 80$. In this analysis, we set $\gamma = -0.3$, $\xi_h = 1$, and $\theta_{\text{obs}} = 0$.

fects at higher frequencies, which leads to a more concentrated but less intense image. Overall, these findings highlight the critical role that frequency plays in determining the resolution and brightness of holographic Einstein images, with higher frequencies enhancing the precision of the observed features while concurrently reducing the overall intensity of the lensing response.

Additionally, we examine the impact of horizon temperature on the holographic Einstein images for the parameter $\gamma = -0.3$, as depicted in Fig. 6. As shown in Figs. 6(a) and 6(d), when the horizon radius is $\xi_h = 4$ and the horizon temperature is $T = -1.14989$, the holographic Einstein image is characterized by a series of axially symmetric bright rings and a central bright spot. The brightest ring is more luminous and positioned farther from the central spot. As shown in Figs. 6(b) and 6(e), when the horizon radius is increased to $\xi_h = 6$ and the temperature decreases to $T = -4.63937$, the radii of these bright rings decrease, causing the brightest ring to move closer to the central spot. Upon further increasing the horizon radius to $\xi_h = 8$, as depicted in Figs. 6(c) and 6(f), with the horizon temperature now at $T = -11.5566$, the radii of the bright rings continue to shrink, and the brightness of the brightest ring diminishes, becoming less intense than that of the central spot. These findings suggest that as the horizon radius ξ increases, leading to a decrease in the horizon temperature, the radii of the bright rings progressively diminish, whereas the central spot becomes relatively brighter than the rings. This behavior underscores the in-

tricate relationship between the horizon temperature and morphology of holographic Einstein images, indicating that changes in thermodynamic properties of the BH can be directly reflected in the observed image structure. We also examine the effect of horizon temperature on the holographic Einstein images for the parameter $\gamma = 0.3$, as shown in Fig. 7. As the radius of the event horizon increases from $\xi_h = 0.2$ ($T = 1.20977$) to $\xi_h = 1.2$ ($T = 0.33569$) and then to $\xi_h = 2.2$ ($T = 0.537787$), the ring gradually contracts. This result is consistent with the conclusions drawn from Fig. 6. However, we observe that the correlation between the ring radius and temperature is not very pronounced. This is because, when γ in Horndeski theory is positive, the BH temperature is no longer a monotonic function of the event horizon, as shown in Fig. 8.

Analyzing the holographic Einstein ring solely through the lens of wave optics provides an incomplete perspective on the system's behavior. To achieve a more comprehensive understanding, we now approach the relationship between the photon and holographic Einstein rings from the perspective of geometric optics. In describing photon motion, two conserved quantities play a crucial role: the energy of the photon, $E = f(r)(\partial t/\partial \lambda)$, and its angular momentum, $\tilde{L} = r^2(\partial \varphi/\partial \lambda)$, where λ is the affine parameter. Considering photon motion in the equatorial plane ($\theta = \pi/2$), its four-velocity vector $v^\alpha = (d/d\lambda)^\alpha$ satisfies the following condition:

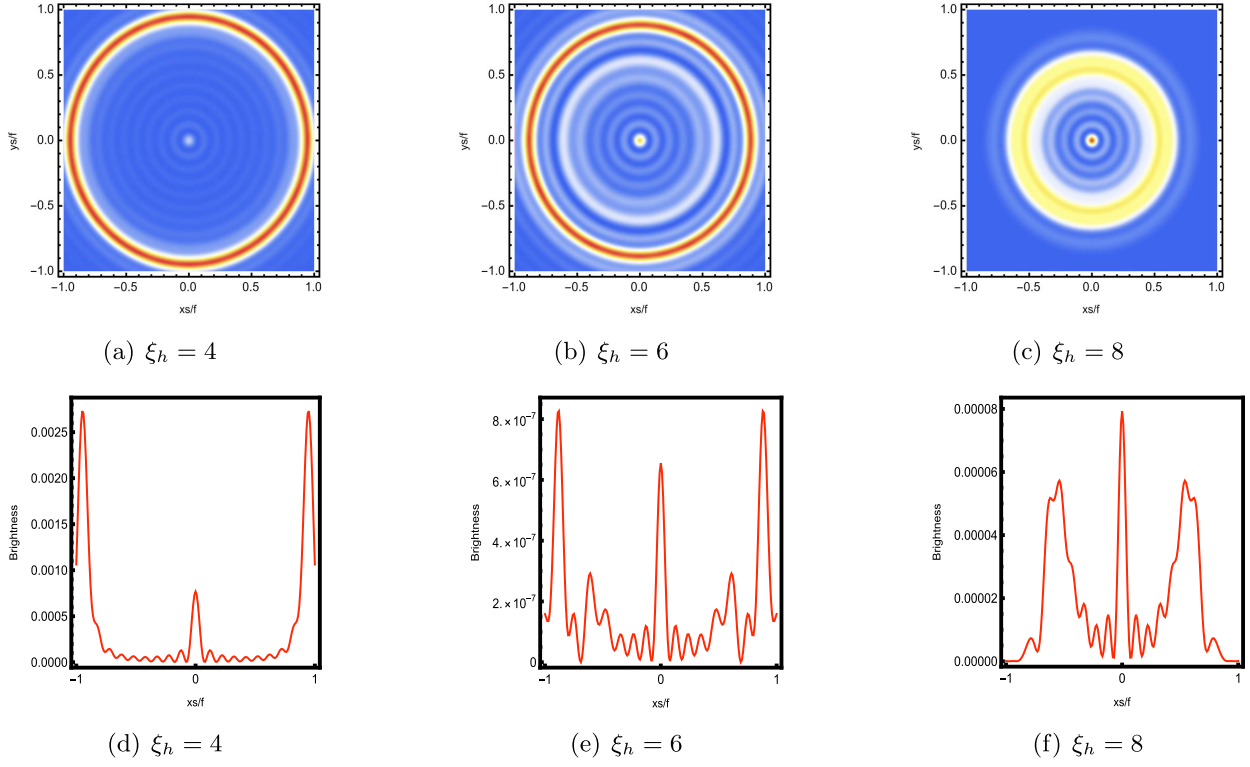


Fig. 6. (color online) Holographic images and the corresponding lensed response brightness at different temperatures. In this analysis, we set $\gamma = -0.3$, $\omega = 50$, and $\theta_{\text{obs}} = 0$.

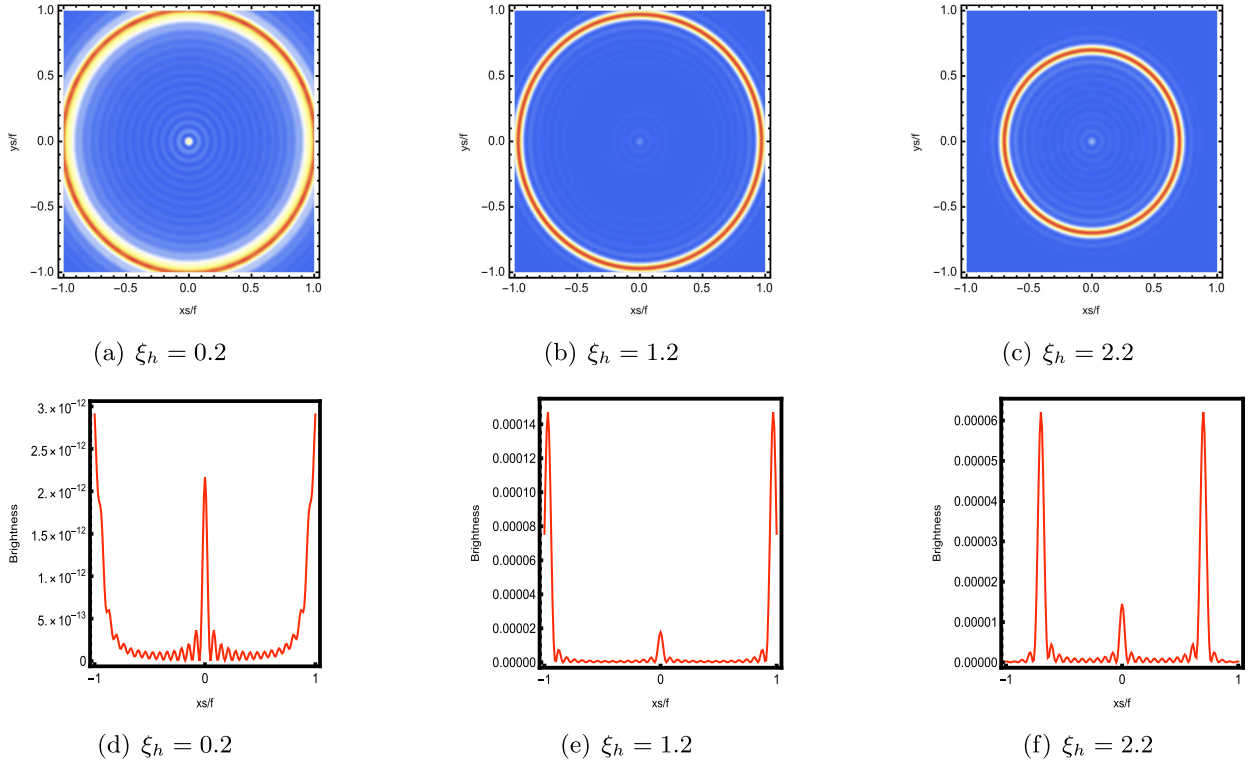


Fig. 7. (color online) Holographic images and the corresponding lensed response brightness at different temperature. In this analysis, we set $\gamma = 0.3$, $\omega = 80$, and $\theta_{\text{obs}} = 0$.

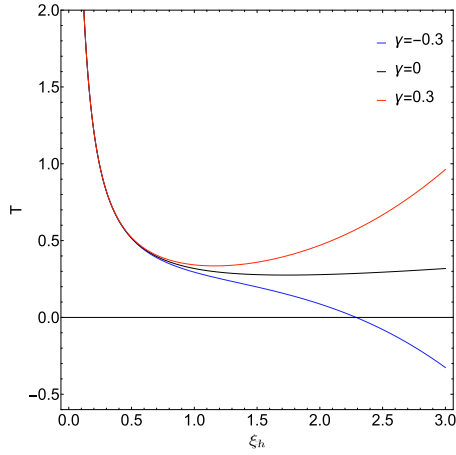


Fig. 8. (color online) Temperature of the AdS BH in Horndeski theory represented by three lines: the blue line corresponds to $\gamma = -0.3$, the black line to $\gamma = 0$, and the red line to $\gamma = 0.3$.

$$-f(r) \left(\frac{dt}{d\lambda} \right)^2 + f(r)^{-1} \left(\frac{dr}{d\lambda} \right)^2 + r^2 \sin^2 \theta \left(\frac{d\varphi}{d\lambda} \right)^2 = 0, \quad (32)$$

which simplifies to the equation

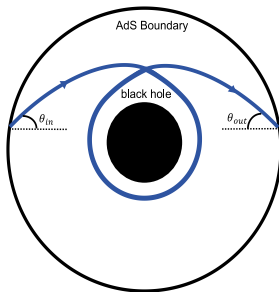
$$\dot{r}^2 = E - z(r)\tilde{L}, \quad (33)$$

where

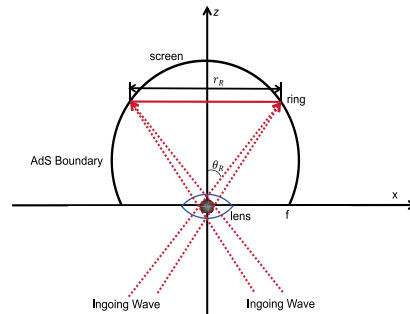
$$z(r) = f(r)/r^2. \quad (34)$$

Building on Refs. [94–96], we consider an incident angle defined by the boundary's normal vector θ_{in} and the direction $n^\zeta = \partial/\partial r^\zeta$. Under these conditions, we obtain

$$\cos \theta_{in} = \frac{g_{\alpha\beta} v^\alpha n^\beta}{|v||n|} \Big|_{r=\infty} = \sqrt{\frac{\dot{r}^2/f}{\dot{r}^2/f + \tilde{L}/r^2}} \Big|_{r=\infty}. \quad (35)$$



(a)



(b)

Fig. 9. (color online) (a) The trajectory of an incident photon completes a full rotation around the BH; (b) relationship between the ring radius and ring angular momentum.

Therefore, we derive

$$\sin^2 \theta_{in} = 1 - \cos^2 \theta_{in} = \frac{\tilde{L}^2 z(r)}{\dot{r}^2 + \tilde{L}^2 z(r)} \Big|_{r \rightarrow \infty} = \frac{\tilde{L}^2}{E^2}. \quad (36)$$

This relationship indicates that the incident angle θ_{in} is connected to the photon's energy E and angular momentum $\tilde{L}$ as

$$\sin \theta_{in} = \frac{\tilde{L}}{E}. \quad (37)$$

As shown in Fig. 9(a), when the photon's entry and exit points align with the BH's center along a straight line, its trajectory forms a ring-shaped image. Upon reaching the photon ring, the photon enters a critical orbit around the BH. We denote the angular momentum of the photon in this critical state as L , and its orbital equation satisfies

$$\dot{r} = 0, \quad \frac{dz}{dr} = 0. \quad (38)$$

We define the radius of the circle in Fig. 9(b) as r_R , leading to the relation

$$\sin \theta_R = \frac{r_R}{f}. \quad (39)$$

For sufficiently large angular momentum, we have $\sin \theta_R = \sin \theta_{in}$; thus, we derive the relationship

$$\frac{r_R}{f} = \frac{L}{E}. \quad (40)$$

To validate Eq. (40), we present in Fig. 10 the Einstein ring radius under different event horizons. The results demonstrate that the incident angle obtained via geometric optics agrees closely with the observational angle de-

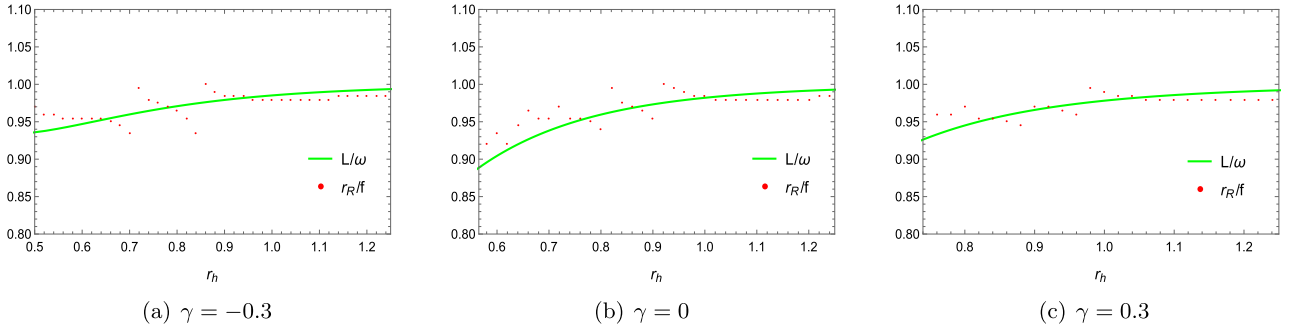


Fig. 10. (color online) Einstein ring radius shown as a function of the event horizon under different values of γ .

rived from wave optics across various γ values. Moreover, this conclusion holds regardless of the γ value in Horndeski theory.

IV. CONCLUSIONS

In this paper, we have investigated holographic Einstein rings resulting from AdS BHs in Horndeski theory, employing both wave and geometric optics. Our analysis reveals that the characteristics of the holographic Einstein ring, such as its radius and brightness, are influenced by multiple factors, including the observer's position, properties of the wave source, BH's event horizon, temperature, and parameter γ in Horndeski theory. Additionally, while physically realizable solutions require $\eta \geq 0$ and thus $\gamma \geq 0$, we have also explored the regime $\gamma < 0$ as a theoretical extension. Although negative γ values (and the corresponding negative-temperature solutions) are not physically meaningful, examining this parameter space enables us to probe the sensitivity of the holographic Einstein ring images to changes in the underlying parameters. Such an exploration provides further insights into the rich structure of the Horndeski theory's solution space, even in regions that do not correspond to physically realizable BHs.

We have demonstrated that when the observer is located at the north pole of the AdS boundary and the wave source is placed at the south pole, the resultant image forms a circular Einstein ring with a central bright spot, coinciding with the location of the BH's photon sphere. The size and brightness of this ring are affected by the wave source's frequency and BH's temperature. As the source frequency increases, the ring sharpens, whereas the overall brightness decreases owing to reduced diffraction effects. In contrast, as the BH's temperature in-

creases, the radius of the event horizon and the Einstein ring both shrink, causing the central bright spot to become more prominent. Furthermore, we examine the relationship between the Einstein ring and BH parameters, including the parameter γ in Horndeski theory. Our findings indicate that the ring's morphology and brightness are significantly influenced by γ . As γ increases, the Einstein ring evolves from symmetric concentric rings into arcs or even distinct bright spots depending on the observer's angle.

In addition, we explore the effect of horizon temperature on the Einstein ring structure. As the event horizon radius increases, resulting in lower temperatures, the ring's radius shrinks, and the central bright spot becomes more prominent. This behavior underscores the sensitive relationship between the thermodynamic properties of the BH and the observed holographic image. We find that the influence of the BH's temperature varies with the value of γ . For a positive γ , the BH temperature is no longer a monotonic function of the event horizon, leading to more complex ring behaviors. The comparison between wave and geometric optics further verifies that the observed angle in wave optics corresponds well with the incident angle in geometric optics, validating the consistency between these two approaches for analyzing the Einstein ring.

Overall, this study provides a deeper understanding of the holographic images of BHs in Horndeski theory and highlights the potential of using holographic Einstein rings as tools to probe the underlying geometry and thermodynamic properties of BHs. Future research can extend this approach to other gravitational theories and BH spacetimes, further enriching our understanding of holographic dualities and BH physics.

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