Black hole tunneling in loop quantum gravity*

Hongwei Tan (谭鸿威)[1†] Rongzhen Guo (国荣祯)[2,3‡] Jingyi Zhang (张婧仪)[4§]

[1]School of Science, Hunan Institute of Technology, Hengyang 421002, China
[2]School of Fundamental Physics and Mathematical Sciences Hangzhou Institute for Advanced Study, University of Chinese Academy of Sciences, Hangzhou 310024, China
[3]School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[4]Center for Astrophysics, Guangzhou University, Guangzhou 510006, China

Abstract: In this study, we investigate the Hawking radiation of the quantum Oppenheimer-Snyde black hole using the tunneling scheme proposed by Parikh and Wilczek. We calculate the emission rate of massless scalar particles. Compared with the traditional results within the framework of General Relativity, our findings include quantum correction terms resulting from loop quantum gravity effects. Using the approach in [J. Zhang, Phys. Lett. B **668**(5), 353 (2008); J. Zhang, Phys. Lett. B **675**(1), 14 (2009)], we establish the entropy of the black hole. This entropy includes a logarithmic correction, which results from quantum gravity effects. Our result is consistent with the well-known result in the context of quantum gravity.

Keywords: black hole entropy, Hawking radiation, quantum tunneling, emission rate

DOI: 10.1088/1674-1137/adabd0 **CSTR:** 32044.14.ChinesePhysicsC.49055106

I. INTRODUCTION

As an important prediction of General Relativity (GR), black holes (BHs) have attracted considerable interest and have been studied widely (see *e.g.*, [1−5]). Both gravitational wave detections [6−10] and the observations of supermassive BHs by the Event Horizon Telescope [10−15] provide strong evidence for the existence of BHs. In the future, further gravitational and electromagnetic wave detection will provide stronger tests of the BH paradigm [16−21]. Despite the remarkable success in the classical paradigm of BHs, some problems remain, *e.g.*, BH singularity and the information paradox (see *e.g.*, [22−28]), where quantum gravity potentially plays a significant role.

Loop quantum gravity (LQG), a background-independent and non-perturbative quantum gravity, is a candidate resolution of these problems [29−36]. When LQG effects are considered, the BH singularity is replaced by a bounce, thus solving the singularity problem [37, 38]. Another advancement is in exploring BH entropy and the information paradox. LQG successfully reproduces BH entropy by counting the number of spacetime microstates [39, 40]. Recently, the concept of a black-to-white hole transition has been considered a promising candidate of a solution to the BH information paradox [41−49]. Addressing these problems will be accompanied by effects beyond the standard BH paradigm. Therefore, LQG is expected to introduce phenomenological corrections to BHs within the framework of GR.

A promising starting point to obtaining this modified model is studying the quantum corrections to the spherically symmetric self-gravitational collapse problem. The spherically symmetric gravitational collapse plays a crucial role in understanding the dynamical formation of BHs from both classical and quantum perspectives. In classical theory, the first collapse model was proposed by Oppenheimer and Snyder [50], known as the Oppenheimer-Snyder (OS) model. This model assumes that the matter field in the interior region is a pressure-less dust field. Thus, the interior dynamics can be described by standard Friedmann equations. The exterior region is characterized by the Schwarzschild solution according to Birkhoff's theorem. Owing to its simplicity, this model can be solved exactly, offering valuable insights into the nature of self-gravitational collapse.

In recent developments [51], LQG effects have been incorporated into the classical OS model, *i.e.*, the quantum OS model. It can be considered an effective model of LQG-modified BHs. It is formed as the intermediate product of the self-gravitational collapse process. Later, the collapsing phase transits to an expanding

Received 28 November 2024; Accepted 17 January 2025; Published online 18 January 2025
* Supported by the National Natural Science Foundation of China (11873025)

† E-mail: htan2018@fau.edu
‡ E-mail: guorongzhen@ucas.ac.cn(Corresponding author)
§ E-mail: zhangjy@gzhu.edu.cn

phase, resulting in a black-to-white hole transition. This BH has been extensively studied from various aspects, such as shadows and quasinormal modes [52−58]. In this framework, the interior region is treated using the Ashtekar-Pawlowski-Singh (APS) model, where the classical Friedmann equation is modified by quantum corrections resulting from loop quantum cosmology (LQC) effects. Appropriate junction conditions are then introduced to derive the spacetime metric in the exterior region. Thus, the spacetime metric is expressed as [1)]

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{1}$$

where M is the ADM energy of the spacetime, and $\alpha = 16\sqrt{3}\pi\gamma^3\ell_p^2$, with $\ell_p = \sqrt{\hbar}$ being the Planck length and γ being the Barbero-Immirzi parameter. Note that γ is dimensionless, $[\alpha]=[M]^2=[r]^2$.[2)] Compared with the classical Schwarzchild solution, a quantum correction results from LQG effects. We discuss this in detail later.

For simplicity, this paper focuses on scenarios in which the matter distribution in the interior spacetime region is assumed to be homogeneous and isotropic, according to the cosmological principle. However, to better reflect reality, we should also consider an anisotropic cosmological background for the interior spacetime region. In particular, the Kantowski-Sachs spacetime is critical in this context [59−62]. This model describes the anisotropic collapse of the matter field, where expansion occurs in one direction, whereas contraction occurs in the other directions. We will investigate these topics in our future research, which may potentially provide new insights into the study of quantum gravity.

For a given BH, we can calculate its Hawking radiation and entropy. Historically, various methods have been developed for this purpose (see *e.g.*, [63−71]). One of them is the tunneling approach, a semiclassical method introduced by Parikh and Wilczek (PW) in [65, 66] and further developed in Refs. [72, 73, 74]. In this approach, Hawking radiation is considered a tunneling effect across the BH horizon, and the back-reaction effects of emitted particles and energy conservation of the system are considered. This leads to a spectrum for emitted particles that is not purely thermal but includes a modification term. Additionally, the BH entropy can be derived from this spectrum.

In this study, we apply the PW approach to the quantum OS BH, focusing on massless scalar outgoing particles. We compute the emission rate Γ of the outgoing particles. Compared with the original result in classical GR [65], our results reveal additional quantum correction terms, which are interpreted as resulting from quantum gravity effects. Furthermore, by following the arguments in Refs. [72, 75], we extract the BH entropy from the emission rate. Compared with the classical results, quantum correction terms also appear in the entropy formula of the quantum OS BH. When the quantum gravity effects are considered, quantum corrections contribute to the BH entropy, where a logarithmic term appears (see *e.g.*, [76−88]):

$$\tilde{S} = S + a\log\left(\frac{\mathscr{A}}{l_p^2}\right)+O(1). \tag{2}$$

Here, a is a constant. $\mathscr{A}$ is the area of the BH horizon. S is the classical BH entropy, given by $S=\dfrac{\mathscr{A}}{4l_p^2}$. As the main result of our study, for the quantum OS BH, we extract the BH entropy from the emission spectrum, which also includes a logarithmic correction:

$$\tilde{S} = S + \frac{\sqrt{2}\pi\alpha}{l_p^2}\log\left(\frac{\mathscr{A}_{\rm Sch}}{l_p^2}\right)+O(\alpha^2). \tag{3}$$

Here, $\mathscr{A}_{\rm Sch}$ is the area of the Schwarzschild BH horizon. Our finding agrees with the traditional results (2). Moreover, we obtain this result using a semiclassical approach, providing evidence for the validity of the PW method.

The remainder of this paper is organized as follows. In Sec. II, we briefly review the quantum OS model. In Sec. III, we compute the emission rate of the massless scalar particles created near the BH horizon and analyze BH entropy. In Sec. IV, we provide conclusions and outlooks of this work.

II. BRIEF REVIEW OF THE QUANTUM OPPENHEIMER-SNYDER MODEL

Recently, the quantum OS model was investigated in Refs. [51−57]. The classical OS model is the first model to describe the collapse of a pressure-less dust field and the formation of a BH in a spherically symmetric background [2, 50]. In this model, space is divided into two regions: the interior region $\mathcal{M}^-$, which is filled with the matter field, and the exterior region $\mathcal{M}^+$, which is vacuum. The geometry of the interior region is described by the usual FRW metric:

1) Here we set $C=G=1$. In this article, the metric has the signature $(-,+,+,+)$.

2) $[A]$ denotes the dimension of the quantity $[A]$.

$$ds^2_{\rm APS} = -d\tau^2 + a(\tau)^2\left(d\tilde{r}^2 + \tilde{r}^2\, d\Omega^2\right), \tag{4}$$

where $(\tau, \tilde{r}, \theta, \varphi)$ denotes a coordinate system. $a(\tau)$ is the scalar factor, which satisfies the classical Friedmann equation

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho, \tag{5}$$

with H is the Hubble constant. The exterior region is vacuum, which is described by the Schwarzschild metric. Ref. [51] considers the quantum version of the OS model. The authors assume that the matter in the interior region satisfies an LQC-deformed Friedmann equation (see *e.g.*, [89–91]):

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right), \rho = \frac{M}{\frac{4}{3}\pi\tilde{r}_0^3 a^3}. \tag{6}$$

Here, M is the ADM mass of the matter field. The critical density is given by $\rho_c = \sqrt{3}/\left(32\pi^2\gamma^3\hbar\right)$. The corrected term $-\dfrac{8\pi\rho^2}{3\rho_c}$ results from LQC effects. In this work, we consider the mass of the matter to be on the scale of solar mass. The metric of the exterior region is given by applying appropriate junction conditions to both the reduced metric and the extrinsic curvature of the junction surface Σ:[1)]

$$h^+_{ab}|_\Sigma = h^-_{ab}|_\Sigma\,, \quad K^+_{ab}|_\Sigma = K^-_{ab}|_\Sigma\,. \tag{7}$$

Here, the "+" sign denotes the exterior region, whereas the "−" sign denotes the interior region. Σ is a timelike hypersurface, which connects the exterior and interior regions. h_{ab} is the reduced metric, and K_{ab} is the extrinsic curvature of Σ. In classical GR, the Israel junction conditions indicate that K_{ab} exhibits a jump at Σ only if the matter density is distributional [92], meaning that the matter has a surface density. However, in the quantum OS model, where the matter does not have a surface density, K_{ab} is assumed to be continuous, as discussed in [83]. The metric of the exterior region is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}\right)^{-1}dr^2 + r^2 d\Omega^2. \tag{8}$$

Compared with the Schwarzschild solution, Eq. (8) contains a quantum-corrected term $\dfrac{\alpha M^2}{r^4}$. This term is interpreted as a contribution of LQG. The position of the BH horizon is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4} = 0. \tag{9}$$

Eq. (9) has several roots. However, as demonstrated in [51], only two real roots exist when $M > M_{\min}$. Here, $M_{\min}$ is the lower bound of the mass required for BH formation, given by $M_{\min} := \dfrac{4}{3\sqrt{3}}\sqrt{\alpha}$. We focus only on the outer horizon, which is given by

$$r_h = \frac{M}{2} + \frac{1}{2}\sqrt{h} + \frac{1}{2}\sqrt{k}, \tag{10}$$

with

$$h = M^2 + \frac{1}{3}M\left(\frac{2\cdot 6^{2/3}\alpha}{\left(9M\alpha + \sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}} + \left(54M\alpha + 6\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}\right), \tag{11}$$

and

$$k = 2M^2 - \frac{M\left(18M\alpha + 2\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}}{3^{2/3}} - \frac{2\cdot 2^{2/3}M\alpha}{\left(27M\alpha + 3\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}} + \frac{2M^3}{\sqrt{M^2 + \frac{1}{3}M\left(\frac{2\cdot 6^{2/3}\alpha}{\left(9M\alpha + \sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}} + \left(54M\alpha + 6\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}\right)}}. \tag{12}$$

1) The Latin letters a, b, c are the abstract indices.

Note that $M^2 \gg \alpha > \frac{16}{24}\alpha$; thus, we find that the reality of the square roots above is preserved. Additionally, we can employ Taylor expansion to r_h, which yields

$$r_h = 2M - \frac{\alpha}{8M} + O(\alpha^2) < 2M. \tag{13}$$

The term $-\frac{\alpha}{8M}$ is the quantum correction of LQG.

III. HAWKING RADIATION AND BLACK HOLE ENTROPY

In this section, we investigate the Hawking radiation of the quantum OS BH using the PW approach. First, we rewrite the metric (8) in Painlevé-Gullstrand coordinates [93] and then compute the emission rate of the massless scalar particles near the horizon. Finally, BH entropy is discussed.

A. Quantum Oppenheimer-Snyder black hole in Painlevé-Gullstrand coordinates

To express the line element (8) in Painlevé-Gullstrand coordinates, we introduce the following coordinate transformation:

$$\tilde{t} = t + F(r), \tag{14}$$

where

$$F(r) = \int_0^r \sqrt{\frac{Mr'^4(2r'^3 - M\alpha)}{(-2Mr'^3 + r'^4 + M^2\alpha)^2}}\mathrm{d}r'. \tag{15}$$

The explicit expression is complicated, but it is not important in our discussions. Thus, we find

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}\right)\mathrm{d}\tilde{t}^2 + 2\sqrt{\frac{2M}{r} - \frac{\alpha M^2}{r^4}}\mathrm{d}\tilde{t}\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{16}$$

A brief review of Painlevé-Gullstrand coordinates and a detailed derivation of Eq. (16) is given in Appendix A. We observe that no coordinate singularity occurs at the horizon in (16), enabling us to investigate the tunneling process across the horizon. Furthermore, the timeslice of (16) is Euclidean, enabling the traditional Schrödinger equation to remain valid.

B. Parikh-Wilczek approach of the quantum Oppenheimer-Snyder black hole

In the PW approach [65], the Hawking radiation is interpreted as a quantum tunneling process. This approach considers the energy conservation condition and incorporates the backreaction of Hawking radiation.

We consider the emitted particles to be massless scalar particles. The radial null geodesic is

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{\alpha M^2}{r^4}}, \tag{17}$$

where $\dot{r}$ denotes the derivative of r with respective to $\tilde{t}$. The "+" sign corresponds to an outgoing particle, whereas the "−" sign corresponds to an ingoing particle. In the following discussion, we focus on the outgoing particle, which tunnels across the outer BH horizon. For simplicity, we assume the wave corresponding to the emitted particle is an s-wave. In classical GR, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. Nevertheless, the case of the quantum OS BH is different owing to the contributions of LQG effects. A recent study [94] addresses this problem by generalizing Birkhoff's theorem within the context of Polymerized Einstein Field Equations (PEFE), which incorporate quantum gravity corrections. The authors demonstrate that the exterior region of the quantum OS spacetime satisfies the generalized Birkhoff's theorem, with three distinct branches corresponding to $k = -1, 0,$ and $1,$ representing open, flat, and closed universes for the interior region, respectively. In this paper, we focus on the $k = 0$ case, as shown in (6) and (8). As demonstrated in Ref. [94], the exterior region of this type of quantum OS model is unique and characterized by a single parameter: the mass. Therefore, no graviton is emitted in this process. If the energy of this particle is ω, according to energy conservation, the energy of the BH changes as $M \to M - \omega$. Accordingly, the modified line element yields

$$\mathrm{d}s^2 = -\left(1 - \frac{2(M-\omega)}{r} + \frac{\alpha(M-\omega)^2}{r^4}\right)\mathrm{d}\tilde{t}^2 + 2\sqrt{\frac{2(M-\omega)}{r} - \frac{\alpha(M-\omega)^2}{r^4}}\mathrm{d}\tilde{t}\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{18}$$

and the modified radial null geodesic yields

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-\omega)}{r} - \frac{\alpha(M-\omega)^2}{r^4}}. \tag{19}$$

The tunneling process occurs very close to the horizon. The outgoing particle, as measured by a local observer near the horizon, experiences an ever-increasing blue shift. Consequently, the geometric optics approximation is suitable for describing the outgoing particles, and the WKB approximation is justified.

The outgoing particles are created inside the horizon. As they are emitted, they cross the horizon and travel to infinity. This process is classically forbidden but permitted through quantum tunneling. Next, we calculate the emission rate of these outgoing particles. With the WKB approximation, the emission rate is given by

$$\Gamma \sim \mathrm{e}^{-\frac{2}{\hbar}\mathrm{Im}A}, \tag{20}$$

where A is the action of the particle. The imaginary part of the action is given by

$$\mathrm{Im}A = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} p_r \mathrm{d}r = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} \int_0^{p_r} \mathrm{d}p_r' \mathrm{d}r. \tag{21}$$

Here, p_r is the conjugate momentum of r, and $r_{\rm in}$ and $r_{\rm out}$ are the locations of the particle before and after the tunneling across the event horizon, respectively. Hamiltonian equations imply

$$\dot{r} = +\left.\frac{\mathrm{d}H}{\mathrm{d}p_r}\right|_r, \tag{22}$$

with the Hamiltonian being $H = M - \omega$. Subsequently, with Eq. (19) and focusing on the outgoing particles, we find

$$\begin{aligned}\mathrm{Im}A &= \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} \int_0^{p_r} \mathrm{d}p_r' \mathrm{d}r \\ &= -\mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} \int_0^{\omega} \frac{\mathrm{d}r}{1-\sqrt{\dfrac{2(M-\omega')}{r}-\dfrac{\alpha(M-\omega')^2}{r^4}}}\mathrm{d}\omega'.\end{aligned} \tag{23}$$

As assumed previously, as $M^2 \gg \alpha$, we can apply Taylor expansion to the integrand of Eq. (23):

$$\begin{aligned}\frac{1}{1-\sqrt{\dfrac{2(M-\omega')}{r}-\dfrac{\alpha(M-\omega')^2}{r^4}}} &= \frac{1}{1-\sqrt{\dfrac{2(M-\omega')}{r}}} \\ &- \frac{(M-\omega')\sqrt{\dfrac{(M-\omega')}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega')}{r}}\right)^2}\alpha + O\left(\alpha^2\right).\end{aligned} \tag{24}$$

Note that $[\alpha] = [M]^2$. Hence, the left-hand side of (24) is dimensionless, which implies that any term on the right hand side of (24) is also dimensionless. In particular, $\left[\dfrac{(M-\omega')\sqrt{\dfrac{(M-\omega')}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega')}{r}}\right)^2}\alpha\right] = 1$. The integrand is singular at the location of the classical horizon. To evaluate Eq. (23), we apply Feynman's scheme. To ensure that the positive energy solutions decay over time, we deform the contour as $\omega \to \omega - \mathrm{i}\epsilon$ with $\epsilon \to 0$. By deforming the contour, the singularity in Eq. (24) is avoided, ensuring that the right-hand side of Eq. (24) converges uniformly. Consequently, the integration and summation can be interchanged. Thus, we find

$$\mathrm{Im}A = -\lim_{\epsilon\to 0}\mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{\omega}\left(\frac{1}{1-\sqrt{\dfrac{2(M-\omega'+\mathrm{i}\epsilon)}{r}}} - \frac{(M-\omega'+\mathrm{i}\epsilon)\sqrt{\dfrac{(M-\omega'+\mathrm{i}\epsilon)}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega'+\mathrm{i}\epsilon)}{r}}\right)^2}\alpha\right)\mathrm{d}r\mathrm{d}\omega' + O(\alpha^2). \tag{25}$$

Introducing the coordinate transformation $u = \sqrt{r}$, we find

$$\mathrm{Im}A = -\lim_{\epsilon\to 0}\mathrm{Im}\int_{u_{\rm in}}^{u_{\rm out}}\int_0^{\omega}\left(\frac{2u^2}{u-\sqrt{2(M-\omega'+\mathrm{i}\epsilon)}} - \frac{(M-\omega'+\mathrm{i}\epsilon)^{3/2}}{\sqrt{2}u^4\left(u-\sqrt{2(M-\omega'+\mathrm{i}\epsilon)}\right)^2}\alpha\right)\mathrm{d}u\mathrm{d}\omega' + O(\alpha^2). \tag{26}$$

We focus on the particles created near the outer horizon. A pole occurs at $u = \sqrt{2(M-\omega'+\mathrm{i}\epsilon)}$ in Eq. (26). Therefore, we find

$$\mathrm{Im}A = \lim_{\epsilon\to 0}\int_0^{\omega}\left(4\pi(M-\omega') + \mathrm{Im}\frac{\sqrt{2}\pi\mathrm{i}}{M-\omega'+\mathrm{i}\epsilon}\alpha\right)\mathrm{d}\omega' + O(\alpha^2). \tag{27}$$

For a solar mass BH, we have $M \gg \omega$. Therefore,

$$\begin{aligned}\operatorname{Im} A &= 4\pi M\omega\left(1-\frac{\omega}{2M}\right)+\sqrt{2}\pi\alpha\left(\log\left(\frac{M}{l_p}\right)-\log\left(\frac{M-\omega}{l_p}\right)\right)+O\left(\alpha^2\right)\\ &= 4\pi M\omega\left(1-\frac{\omega}{2M}\right)+\frac{\sqrt{2}}{2}\pi\alpha\left(\log\left(\frac{\mathscr{A}_{\rm Sch}(M)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\rm Sch}(M-\omega)}{l_p^2}\right)\right)+O\left(\alpha^2\right).\end{aligned} \tag{28}$$

Here, $\mathscr{A}_{\rm Sch}$ is the area of the Schwarzschild BH horizon with mass M. Compared with the original work [65], an additional term $\frac{\sqrt{2}}{2}\pi\alpha\left(\log\left(\frac{\mathscr{A}_{\rm Sch}(M)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\rm Sch}(M-\omega)}{l_p^2}\right)\right)$ appears. This term is interpreted as the quantum correction resulting from LQG. Thus, the emisson rate of the particles yields

$$\Gamma \sim \exp\left(-\frac{2}{\hbar}\operatorname{Im} A\right)=\exp\left(-\frac{8\pi M\omega}{l_p^2}\left(1-\frac{\omega}{2M}\right)+\frac{\sqrt{2}\pi\alpha}{l_p^2}\left(\log\left(\frac{\mathscr{A}_{\rm Sch}(M-\omega)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\rm Sch}(M)}{l_p^2}\right)\right)+O\left(\alpha^2\right)\right). \tag{29}$$

C. Entropy of the quantum Oppenheimer-Snyde black hole

In classical GR, the emission rated can be expressed as

$$\Gamma \sim \exp\Delta S, \tag{30}$$

where ΔS is the difference in BH entropy before and after the emission of the particle. This result implies the unitary of BH evaporation in GR [72, 75]. In quantum mechanics, the transition apmplitude between the initial and the final states is

$$\Gamma(i\to f)=|\mathcal{M}|^2\cdot(\text{ phase space factor}), \tag{31}$$

where $|\mathcal{M}|$ is the probabilty magnitude of this poccess. The phase space factor is given by

$$\text{phase space factor } =\frac{N_f}{N_i}=\frac{e^{S_f}}{e^{S_i}}=e^{\Delta S}. \tag{32}$$

Let us take the Schwarzschild BH as an example. For a Schwarzschild BH, its initial entropy is given by

$$S_i=\frac{4\pi M^2}{l_p^2}. \tag{33}$$

After the BH emits a particle with energy ω, the final entropy is

$$S_f=\frac{4\pi(M-\omega)^2}{l_p^2}. \tag{34}$$

Therefore,

$$\Delta S=S_f-S_i=-\frac{8\pi M\omega}{l_p^2}\left(1-\frac{\omega}{2M}\right)=-\frac{2}{\hbar}\operatorname{Im} A_{\rm Sch}, \tag{35}$$

where $A_{\rm Sch}$ is the action of the particle created near the Schwarzschild BH horizon. Hence, the emission rate of this particle is

$$\Gamma_{\rm Sch}\sim\exp\Delta S. \tag{36}$$

We now return to scenarios of the quantum OS BH. Quantum correction terms appear in Eq. (29). Hence, to express the emission rate Γ in the formula of Eq. (30), we introduce the quantum-corrected entropy as

$$\tilde{S}=S+\frac{\sqrt{2}\pi\alpha}{l_p^2}\log\left(\frac{\mathscr{A}_{\rm Sch}}{l_p^2}\right)+O(\alpha^2), \tag{37}$$

Here, S is the entropy of the Schwarzschild BH, given by $S=\frac{4\pi M^2}{l_p^2}$. Note that we are considering particles tunneling across the outer BH horizon. Therefore, $\tilde{S}$ is the entropy of the outer BH horizon, Our approach aligns with the scheme introduced in Refs. [65, 73, 74]. To compute the total BH entropy, we should consider a multi-horizon scenario [95, 96]. This approach is more intricate than it may initially appear, and we will explore this topic further in our future research. Finally, the emission rate becomes

$$\Gamma\sim\exp\left[\tilde{S}_f-\tilde{S}_i\right]=\exp\Delta\tilde{S}. \tag{38}$$

The pre-factor of the logarithmic correction of the BH entropy relates to the micro-states of BH horizon. In our findings, the sign of this pre-factor is positive. Although this differs from the results obtained within the framework of ordinary LQG [76–78], it is consistent with the entropy of the effective loop quantum BH computed using other approaches [82, 83]. Ref. [81] provides profound insights into this discrepancy. On the one hand, it states that the quantum correction to the BH entropy, resulting from quantum gravity effects, reduces the entropy. On the other hand, it states that the logarithmic correction, which originates from thermal fluctuations, increases the entropy owing to the enhanced uncertainty introduced by these fluctuations. Based on these insights, our finding (37) might suggest the logarithmic correction for quantum OS BH entropy incorporates the contributions from both quantum gravity effect and (effective) thermal fluctuations. For precision, let us denote a_q as the pre-factor for the logarithmic correction contributed by quantum gravity effect and a_f as the pre-factor for the logarithmic correction contributed by (effective) thermal fluctuation; thus, $a_q+a_f=\sqrt{2}\pi\alpha/l_p^2$. We will continue this interesting topic in our future research. Moreover, the BH entropy is related to the dimension of the spacetime [97]. Our finding is consistent with the scenarios of four-dimensional spacetime discussed in Ref. [97]. Unlike the arguments based on quantum gravity, we obtain this correction using the semiclassical method. Therefore, our result supports the validity of the semiclassical method in the context of quantum gravity.

D. Comparison with other scenarios for computing black hole entropy

In addition to the previously introduced PW approach, several other methods for computing BH entropy incorporate correction terms that modify the standard entropy formula. In this subsection, we compare these methods with our results.

• Noncommutative spacetime geometry approach

This approach is based on the assumption that the coordinates for the spacetime manifold do not commute to each other but instead satisfy the following relationship:[1)]

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{39}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix that encodes the noncommutativity of spacetime [98−100]. The spherical and static solution within this framework is [101, 102]

$$\mathrm{d}s^2 = -\left(1-\frac{2M_\theta}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M_\theta}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2. \tag{40}$$

Here, M_θ is the mass of the spacetime satisfying the Gaussian distribution of minimal width $\sqrt{\theta}$

$$M_\theta = \int_0^r \rho_\theta(r)4\pi r^2\mathrm{d}r = \frac{2M}{\sqrt{\pi}}\int_0^{\frac{r^2}{4\theta}} x^{\frac{1}{2}}\mathrm{e}^{-x}\mathrm{d}x, \tag{41}$$

where M is a constant, and the distribution function $\rho(r)$ is given by

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}}\exp\left(-\frac{r^2}{4\theta}\right). \tag{42}$$

We can show that

$$\lim_{\theta\to 0} M_\theta = M. \tag{43}$$

Here, Eq. (42) reduces to the ordinary Schwarzschild solution. Within this approach, the BH entropy entropy becomes

$$S_{\rm NSG} = 4\pi\frac{M^2}{l_p^2}\mathcal{E}\left(\frac{M}{l_p\sqrt{\theta}}\right)-6\pi\theta\mathcal{E}\left(\frac{M}{l_p\sqrt{\theta}}\right) + 12\sqrt{\pi\theta}\frac{M}{l_p}\exp\left(-\frac{M^2}{l_p^2\theta}\right), \tag{44}$$

where $\epsilon(x)$ is the Gauss error function:

$$\mathcal{E}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\lambda^2}\mathrm{d}\lambda. \tag{45}$$

In particular, in the large mass limit $\dfrac{M}{l_p\sqrt{\theta}} \gg 1$, we find

$$S_{\rm NSG} = \frac{4\pi M^2}{l_p^2}+12\sqrt{\pi\theta}\frac{M}{l_p}\exp\left(-\frac{M^2}{l_p^2\theta}\right). \tag{46}$$

The correction in Eq. (46) is exponential, differing from the logarithmic correction in our result (37). Ref. [102] demonstrates that BHs cannot evaporate completely in the noncommutative spacetime geometry framework. Instead, a lower bound exists for the mass, M_0, during the later stages of BH evaporation, and the information can be stored in the remnant, offering a possible resolution to the BH information paradox. Our findings may lead to a similar conclusion because the pre-factor of the logarithmic correction in Eq. (37) is positive. However, a subtlety occurs because our analysis assumes $M^2 \gg \alpha$, which may not hold during the later stages of evaporation. This suggests that these two frameworks might describe different phases of BH evaporation. Investigating the relationship between these two approaches would be an interesting avenue for future research. Moreover, as discussed in Ref. [51], a bounce occurs at the end of the collapse. Ex-

1) The Greek letters μ, ν are the abstract indices, which take the value $\mu, \nu = 0, 1, 2, 3$.

ploring how to incorporate both the bounce and BH evaporation into a unified framework remains a challenging but intriguing task.

• **Modified dispersion relations approach**

Inspired by quantum gravity, the modified dispersion relations (MDR) approach results from modifying the energy-momentum dispersion relation [103, 104]

$$\boldsymbol{p}^2 = E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + \alpha_3 l_p^3 E^5 + \alpha_4 l_p^4 E^6 + O\left(l_p^5 E^7\right), \tag{47}$$

where $E \ll 1/l_p$. The value of the coefficients α_i depend on the specific quantum gravity theory. Applying the modified dispersion relation, the BH entropy is given by

$$\begin{aligned} S_{\text{MDR}} \simeq & \frac{\mathscr{A}}{4l_p^2} + \frac{\alpha_1 \pi^{1/2}}{l_p} \mathscr{A}^{1/2} + \pi\left(\frac{3}{2}\alpha_2 - \frac{3}{8}\alpha_1^2\right) \log \frac{\mathscr{A}}{l_p^2} \\ & - 4\pi^{3/2} l_p \left(-\alpha_1\alpha_2 + \frac{1}{4}\alpha_1^3 + 2\alpha_3\right) \mathscr{A}^{-1/2} \\ & - 4\pi^2 l_p^2 \left(-\frac{5}{4}\alpha_1\alpha_3 - \frac{5}{8}\alpha_2^2 + \frac{15}{16}\alpha_1^2\alpha_2 - \frac{25}{128}\alpha_1^4\right) \mathscr{A}^{-1} \\ & - \frac{16}{3}\pi^{5/2} l_p^3 \left(\frac{9}{8}\alpha_1^2\alpha_3 - \frac{45}{48}\alpha_1^3\alpha_2 + \frac{9}{8}\alpha_1\alpha_2^2 \right. \\ & \left. + \frac{21}{128}\alpha_1^5 - \frac{3}{2}\alpha_2\alpha_3\right) \mathscr{A}^{-3/2}. \end{aligned} \tag{48}$$

As observed in Ref. [104], if the odd powers of energy in Eq. (47) are set to 0 ($\alpha_1 = \alpha_3 = 0$), then the BH entropy reduces to

$$S_{\text{MDR}} \simeq \frac{\mathscr{A}}{4l_p^2} + \frac{3}{2}\pi\alpha_2 \log \frac{\mathscr{A}}{l_p^2} + \frac{5}{2}\pi^2\alpha_2^2 \frac{l_p^2}{\mathscr{A}}. \tag{49}$$

This result is consistent with our finding (37). BH entropy formula imposes constraints on the form of MDR.

• **Generalized uncertainty principle approach**

The general uncertainty principle (GUP) approach is based on the modification of the ordinary Heisenberg uncertainty principle (HUP) [103, 105]:

$$\delta x \geq \frac{1}{\delta p} + \beta l_p^2 \delta p + O\left(l_p^3 \delta p^2\right), \tag{50}$$

where β is a small parameter that depends on the specific quantum gravity theory. Based on this assumption, the BH entropy is computed as

$$S_{\text{GUP}} \simeq \frac{\mathscr{A}}{4l_p^2} - \beta\pi \ln \frac{\mathscr{A}}{l_p^2}. \tag{51}$$

This result is consistent with our finding (37) up to a difference in the sign. Furthermore, by considering higher order effects, Ref. [105] derives the BH entropy with additional corrections and predicts that the existence of a remnant as the BH mass M approaches to the order of Planck mass M_p during evaporation. This prediction corresponds with the results of the noncommutative spacetime geometry approach. Incorporating the GUP framework into the quantum OS BH model would be an intriguing direction for future research.

• **Polymeric quantization approach**

The polymeric quantization approach is a method for quantizing gravity, inspired by techniques from LQG and quantum mechanics in the polymer representation [106, 107]. It is based on the premise that, in a quantum theory of gravity, spacetime is discrete and a minimum measurable length scale exists in the order of the Planck length. Within this framework, the classical phase space is modified to incorporate discrete structures, which is different from the standard continuous treatment. Ref. [108] computes the BH entropy in this framework. We begin by introducing the polymeric area of the BH in terms of the horizon area, expressed as

$$\mathscr{A}^{(\text{poly})} = \mathscr{A}\left(1 - \frac{\left(1+\mu^2 M_P/8\right)}{8\pi}\left(\frac{M_P}{M}\right)^2\right)^2. \tag{52}$$

Here, $\mu = \mu_0/\hbar$, where μ_0 is the discreteness parameter. The continuum limit is given by $\mu \to 0$. In the high-temperature limit, the BH entropy is given by

$$\begin{aligned} S_{\text{poly}} = & \frac{\mathscr{A}^{(\text{poly})}}{4l_P^2} - \frac{1}{2}\ln\left[\frac{\mathscr{A}^{(\text{poly})}}{4l_P^2}\right] + M\left(1 - \frac{\mathscr{A}^{(\text{poly})}}{\mathscr{A}}\right) \\ & + O\left[\mathscr{A}^{(\text{poly})^{-1}}\right]. \end{aligned} \tag{53}$$

In the limit $\dfrac{M_p}{M} \to 0$, the polymeric area $\mathscr{A}^{(\text{poly})}$ reduces to the original horizon area $\mathscr{A}$. The entropy reduces to the well-known form

$$S_{\text{poly}} = \frac{\mathscr{A}}{4l_P^2} - \frac{1}{2}\ln\left[\frac{\mathscr{A}}{4l_P^2}\right] + O\left[\mathscr{A}^{-1}\right]. \tag{54}$$

This result is also consistent with our finding (37), up to a minus sign.

IV. CONCLUSION AND OUTLOOK

In this study, we apply the PW approach to the quantum OS BH and evaluate the emission rate of the outgoing massless scalar particles. Compared with the original results in Refs. [65, 66], our findings include quantum correction terms resulting from LQG effects.

Using the scheme in Refs. [72, 75], we establish the entropy of the OS BH. This entropy formula includes a logarithmic correction, which is consistent with well-known result in the context of quantum gravity [76–78].

Thus far, we have only considered the massless scalar particles as the emitted particles. It would be interesting to extend the study to the tunneling process of massive particles. This topic has been explored within the framework of classical GR in Refs. [72, 75, 74], where BH thermodynamics plays a crucial role. Investigating the tunneling process of massive particles in the OS model potentially provides us deeper insights into the BH thermodynamics in the context of LQG.

Recently, the island scheme has been proposed as a resolution to the BH information paradox [109–113]. This approach utilizes the concepts of the minimal quantum extremal surface [114–116] to evaluate the BH entropy, successfully recovering the Page curve [117, 118] and resolving the BH information paradox. In our future work, we plan to extend the island scheme to the OS BH and compare the results with those presented in this paper.

APPENDIX A: A BRIEF REVIEW OF PAINLEVÉ-GULLSTRAND COORDINATES

In this appendix, we briefly review the scheme of obtaining the Painlevé-Gullstrand coordinate transformation of a general static spacetime. Firstly, we introduce the line element as

$$\mathrm{d}s^2 = -(1-g(r))\,\mathrm{d}t^2 + \frac{1}{1-g(r)}\mathrm{d}r^2 + \mathrm{d}\Omega^2, \tag{A1}$$

with $g(r)$ is a function of r. We then introduce the coordinate transformation

$$t = \tilde{t} + F(r). \tag{A2}$$

Thus, we have

$$\mathrm{d}t = \mathrm{d}\tilde{t} + F'(r)\mathrm{d}r. \tag{A3}$$

In Painlevé-Gullstrand coordinates, the time slice is required to be Euclidean, which necessitates the coefficient of $\mathrm{d}r^2$ to be 1. Consequently, we obtain the following equations for $F(r)$

$$\frac{1}{1-g(r)} - [1-g(r)]\left[F'(r)\right]^2 = 1. \tag{A4}$$

Thus, the new line element becomes

$$\mathrm{d}s^2 = -[1-g(r)]\mathrm{d}t^2 \pm 2\sqrt{g(r)}\mathrm{d}t\,\mathrm{d}r + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{A5}$$

In our case,

$$g(r) = \frac{2M}{r} - \frac{\alpha M^2}{r^4}. \tag{A6}$$

Hence, $F(r)$ is given by

$$F(r) = \int_0^r \sqrt{\frac{Mr'^4(2r'^3 - Ma)}{(-2Mr'^3 + r'^4 + M^2\alpha)^2}}\mathrm{d}r'. \tag{A7}$$

References

[1] R. M. Wald, *General relativity* (Chicago: University of Chicago Press, 2010)
[2] E. Poisson, *A relativist's toolkit: the mathematics of black-hole mechanics* (Cambridge: Cambridge university Press, 2004)
[3] V. P. Frolov and A. Zelnikov, *Introduction to black hole physics* (Oxford: OUP, 2011)
[4] N. Afshordi, A. Ashtekar, E. Barausse *et al.*, arXiv: 2410.14414
[5] S. Chandrasekhar and K. S. Thorne, *The mathematical theory of black holes* (Oxford: Oxford University Press, 1985)
[6] LIGO Scientific, Virgo Collaboration, B. P. Abbott *et al.*, Phys. Rev. Lett. **116**(6), 061102 (2016), arXiv: 1602.03837
[7] LIGO Scientific, Virgo Collaboration, B. P. Abbott *et al.*, Phys. Rev. Lett. **116**(24), 241103 (2016), arXiv: 1606.04855
[8] B. P. Abbott *et al.* (LIGO Scientific, Virgo Collaboration), Phys. Rev. X **6**(4), 041015 (2016) [Erratum: Phys. Rev. X **8**, 039903 (2018)], arXiv: 1606.04856
[9] B. P. Abbott *et al.* (LIGO Scientific, VIRGO Collaboration), Phys. Rev. Lett. **118**, 221101 (2017) [Erratum: Phys. Rev. Lett. **121**, 129901 (2018)], arXiv: 1706.01812
[10] E. H. T. Collaboration *et al.*, arXiv: 1906.11238
[11] K. Akiyama, A. Alberdi, W. Alef *et al.*, ApJL **875**(1), L2 (2019)
[12] K. Akiyama, A. Alberdi, W. Alef *et al.*, ApJL **875**(1), L3 (2019)
[13] K. Akiyama, A. Alberdi, W. Alef *et al.*, ApJL **875**(1), L4 (2019)
[14] K. Akiyama, A. Alberdi, W. Alef *et al.*, ApJL **875**(1), L5 (2019)
[15] K. Akiyama, A. Alberdi, W. Alef *et al.*, ApJL **875**(1), L6 (2019)
[16] M. Punturo, *et al.*, Class. Quant. Grav. **27**, 194002 (2010)
[17] M. Evans *et al.*, Cosmic Explorer: *A Submission to the NSF MPSAC ngGW Subcommittee*, arXiv: 2306.13745

[18] M. Evans *et al*., arXiv: 2109.09882
[19] M. Colpi *et al*., arXiv: 2402.07571
[20] TianQin Collaboration, J. Luo, *et al*., Class. Quant. Grav. **33**(3), 035010 (2016), arXiv: 1512.02076
[21] W.-R. Hu and Y.-L. Wu, Natl. Sci. Rev. **4**(5), 685 (2017)
[22] R. Penrose, Phys. Rev. Lett. **14**(3), 57 (1965)
[23] S. W. Hawking and R. Penrose, Proc. R. Soc. Lond. A **314**(1519), 529 (1970)
[24] P. R. Brady and J. D. Smith, Phys. Rev. Lett. **75**(7), 1256 (1995)
[25] S. W. Hawking, Nature **248**(5443), 30 (1974)
[26] S. Hawking, *Black holes and the information paradox*, in *General Relativity and Gravitation* (Singapore: World Scientific, 2005), p. 56
[27] J. D. Bekenstein, Contemporary Physics **45**(1), 31 (2004)
[28] F. Landgren, *The information paradox, a modern review*, .
[29] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press, 2008)
[30] C. Rovelli, *Quantum Gravity* (Cambridge: Cambridge University Press, 2004)
[31] M. Han, Y. Ma, and W. Huang, Int. J. Mod. Phys. D **16**(09), 1397 (2007)
[32] T. Thiemann, *Lectures on loop quantum gravity*, in *Quantum gravity: From Theory to Experimental Search* (New York: Springer, 2003), p. 41
[33] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **21**(15), R53 (2004)
[34] K. Giesel and H. Sahlmann, arXiv: 1203.2733
[35] C. Rovelli, arXiv: 1102.3660
[36] A. Perez, Living Reviews in Relativity **16**, 1 (2013)
[37] V. Husain, J. G. Kelly, R. Santacruz *et al*., Phys. Rev. Lett. **128**(12), 121301 (2022)
[38] V. Husain, J. G. Kelly, R. Santacruz *et al*., Phys. Rev. D **106**(2), 024014 (2022)
[39] C. Rovelli, Phys. Rev. Lett. **77**(16), 3288 (1996)
[40] A. Perez, Rep. Prog. Phys. **80**(12), 126901 (2017)
[41] C. Rovelli and F. Vidotto, Int. J. Mod. Phys. D **23**(12), 1442026 (2014)
[42] H. M. Haggard and C. Rovelli, Phys. Rev. D **92**(10), 104020 (2015)
[43] T. De Lorenzo and A. Perez, Phys. Rev. D **93**(12), 124018 (2016)
[44] M. Christodoulou, C. Rovelli, S. Speziale *et al*., Phys. Rev. D **94**(8), 084035 (2016)
[45] E. Bianchi, M. Christodoulou, F. d'Ambrosio *et al*., Class. Quant. Grav. **35**(22), 225003 (2018)
[46] F. D'Ambrosio, M. Christodoulou, P. Martin-Dussaud *et al*., Phys. Rev. D **103**, 106014 (2021)
[47] F. Soltani, C. Rovelli, and P. Martin-Dussaud, Phys. Rev. D **104**(6), 066015 (2021)
[48] A. Rignon-Bret and C. Rovelli, Phys. Rev. D **105**(8), 086003 (2022)
[49] M. Han, C. Rovelli, and F. Soltani, Phys. Rev. D **107**(6), 064011 (2023)
[50] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**(5), 455 (1939)
[51] J. Lewandowski, Y. Ma, J. Yang *et al*., Phys. Rev. Lett. **130**(10), 101501 (2023)
[52] J. Yang, C. Zhang, and Y. Ma, Eur. Phys. J. C **83**(7), 619 (2023)
[53] O. Stashko, Phys. Rev. D **110**(8), 084016 (2024)
[54] C. Zhang, Y. Ma, and J. Yang, Phys. Rev. D **108**(10), 104004 (2023)
[55] H. Gong, S. Li, D. Zhang *et al*., Phys. Rev. D **110**(4), 044040 (2024)
[56] S. Yang, Y.-P. Zhang, T. Zhu *et al*., arXiv: 2407.00283
[57] H. Liu, M.-Y. Lai, X.-Y. Pan *et al*., Phys. Rev. D **110**(10), 104039 (2024)
[58] T. Zi and S. Kumar, arXiv: 2409.17765
[59] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**(3), 443 (1966)
[60] A. Joe, *Anisotropic Spacetimes and Black Hole Interiors in Loop Quantum Gravity*. Ph.D. thesis (Thibodaux: Louisiana State Uni., 2015)
[61] L. Modesto, Int. J. Theor. Phys. **45**(12), 2235 (2006)
[62] R. Casadio, A. Kamenshchik, and J. Ovalle, Phys. Rev. D **110**(4), 044001 (2024)
[63] S. W. Hawking, Commun.Math. Phys. **43**(3), 199 (1975)
[64] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**(10), 2752 (1977)
[65] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**(24), 5042 (2000)
[66] M. Parikh, Int. J. Mod. Phys. D **13**(10), 2351 (2004)
[67] M. A. Rahman and M. I. Hossain, Phys. Lett. B **712**(1-2), 1 (2012)
[68] S. P. Kim, Int. J. Mod. Phys. D **25**(13), 1645005 (2016)
[69] G. Hooft, Nucl. Phys. B **256**, 727 (1985)
[70] R. Emparan, JHEP **2006**(06), 012 (2006)
[71] A. Almheiri, T. Hartman, J. Maldacena *et al*., Rev. Mod. Phys. **93**(3), 035002 (2021)
[72] J. Zhang, Phys. Lett. B **668**(5), 353 (2008)
[73] J.-Y. Zhang and Z. Zhao, Mod. Phys. Lett. A **20**, 1673 (2005)
[74] J.-Y. Zhang and Z. Zhao, Nucl. Phys. B **725**, 173 (2005)
[75] J. Zhang, Phys. Lett. B **675**(1), 14 (2009)
[76] A. Ghosh and P. Mitra, Phys. Rev. D **71**(2), 027502 (2005)
[77] R. K. Kaul and P. Majumdar, Phys. Rev. Lett. **84**(23), 5255 (2000)
[78] M. Domagala and J. Lewandowski, Class. Quant. Grav. **21**(22), 5233 (2004)
[79] K. A. Meissner, Class. Quant. Grav. **21**(22), 5245 (2004)
[80] A. Chatterjee and P. Majumdar, Phys. Rev. Lett. **92**(14), 141301 (2004)
[81] A. Medved, Class. Quant. Grav. **22**(1), 133 (2004)
[82] J. Lin and X. Zhang, Phys. Rev. D **110**(2), 026002 (2024)
[83] Z. Shi, X. Zhang, and Y. Ma, Phys. Rev. D **110**(10), 104074 (2024)
[84] R. Banerjee and B. R. Majhi, Phys. Lett. B **662**(1), 62 (2008)
[85] R. Banerjee and B. R. Majhi, JHEP **2008**(06), 095 (2008)
[86] R. Banerjee and B. R. Majhi, Phys. Lett. B **674**(3), 218 (2009)
[87] B. R. Majhi, Phys. Rev. D **79**(4), 044005 (2009)
[88] B. R. Majhi and S. Samanta, Ann. Phys. **325**(11), 2410 (2010)
[89] A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. Lett. **96**(14), 141301 (2006)
[90] J. Yang, Y. Ding, and Y. Ma, Phys. Lett. B **682**(1), 1 (2009)
[91] M. Assanioussi, A. Dapor, and K. Liegener, Phys. Rev. Lett. **121**, 081303 (2018)
[92] W. Israel, Il Nuovo Cimento B (1965-1970) **44**(1), 1 (1966)
[93] P. Painlevé, Comptes Rendus Academie des Sciences (serie non specifiee) **173**, 677 (1921)
[94] L. Cafaro and J. Lewandowski, Phys. Rev. D **110**(2),

024072 (2024)
[95] C. Singha, Gen. Relativ. Gravit. **54**(4), 38 (2022)
[96] S. Saghafi and K. Nozari, Gen. Relativ. Gravit. **55**(1), 20 (2023)
[97] K. Nozari and A. Sefidgar, Gen. Relativ. Gravit. **39**, 501 (2007)
[98] H. S. Snyder, Phys. Rev. **71**(1), 38 (1947)
[99] N. Seiberg and E. Witten, JHEP **1999**(09), 032 (1999)
[100] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**(4), 977 (2001)
[101] P. Aschieri, C. Blohmann, M. Dimitrijević *et al*., Class. Quant. Grav. **22**(17), 3511 (2005)
[102] K. Nozari and S. H. Mehdipour, Class. Quant. Grav. **25**(17), 175015 (2008)
[103] G. Amelino-Camelia, M. Arzano, Y. Ling *et al*., Class. Quant. Grav. **23**(7), 2585 (2006)
[104] K. Nozari and A. Sefidgar, Phys. Lett. B **635**(2-3), 156 (2006)
[105] K. Nozari and S. Saghafi, JHEP **2012**(11), 1 (2012)
[106] A. Ashtekar, S. Fairhurst, and J. L. Willis, Class. Quant. Grav. **20**(6), 1031 (2003)
[107] A. Corichi, T. Vukašinac, and J. A. Zapata, Phys. Rev. D **76**(4), 044016 (2007)
[108] M. Gorji, K. Nozari, and B. Vakili, Phys. Lett. B **735**, 62 (2014)
[109] G. Penington, JHEP **2020**(9), 1 (2020)
[110] A. Almheiri, N. Engelhardt, D. Marolf *et al*., JHEP **2019**(12), 1 (2019)
[111] A. Almheiri, R. Mahajan, J. Maldacena *et al*., JHEP **2020**(3), 1 (2020)
[112] A. Almheiri, R. Mahajan, and J. Maldacena, arXiv: 1910.11077
[113] K. Hashimoto, N. Iizuka, and Y. Matsuo, JHEP **2020**(6), 1 (2020)
[114] S. Ryu and T. Takayanagi, JHEP **2006**(08), 045 (2006)
[115] V. E. Hubeny, M. Rangamani, and T. Takayanagi, JHEP **2007**(07), 062 (2007)
[116] N. Engelhardt and A. C. Wall, JHEP **2015**(1), 1 (2015)
[117] D. N. Page, Phys. Rev. Lett. **71**(23), 3743 (1993)
[118] D. N. Page, JCAP **2013**(09), 028 (2013)