Use of $SU(3)$ flavor projection operators to construct baryon-meson scattering amplitudes in the $1/N_c$ expansion*

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Abstract: An $SU(3)$ flavor projection operator technique is implemented to construct the baryon-meson scattering amplitude within the framework of the $1/N_c$ expansion of quantum chromodynamics (QCD), where N_c represents the number of color charges. The operator technique is implemented to evaluate not only the lowest-order scattering amplitude but also effects from the first-order perturbative $SU(3)$ flavor symmetry breaking and strong isospin breaking. The most general expression is obtained by explicitly accounting for the effects of the decuplet-octet baryon mass difference. At order $O(1/N_c^2)$, a large number of unknown operator coefficients appear, and therefore, there is little additional predictive power unless leading and subleading terms are retained. Although the resultant expression is sufficiently general that it can be applied to any incoming and outgoing baryons and pseudo scalar mesons, provided that the Gell-Mann--Nishijima scheme is respected, results for $N\pi \to N\pi$ scattering processes are explicitly considered.

Keywords: Baryon-meson scattering, $1/N_c$ expansion, projection operators

DOI: 10.1088/1674-1137/ae1197 **CSTR:** 32044.14.ChinesePhysicsC.50023106

I. INTRODUCTION

Quantum chromodynamics (QCD) is accepted as the theory of the strong interaction, with quarks and gluons as fundamental fields. QCD is a gauge theory with the local symmetry group $SU(N_c)$, which acts in the internal space of the color degrees of freedom with $N_c = 3$ color charges. However, the analytical computation of hadron properties from first principles is hampered because QCD is strongly coupled at low energies. Two major theories have shed light on the static properties of hadrons. One of them is the large-N_c limit, and the other one is the chiral perturbation theory (ChPT).

The generalization of QCD from $N_c = 3$ to $N_c \to \infty$, which is commonly referred to as large-N_c QCD, has become a remarkable tool for studying the structure and interactions of mesons [1, 2] and baryons [3] in more generality. Physical quantities evaluated in the large-N_c limit achieve corrections of relative orders $1/N_c$, $1/N_c^2$, and so on, which originates the $1/N_c$ expansion of QCD.

Baryon-meson scattering is a fundamental nuclear physics process that has been analyzed within the large-N_c limit (and of course ChPT and several other approaches). The earliest analysis of baryon-meson scattering amplitudes in the context of the $1/N_c$ expansion was introduced in the seminal paper by Witten [3]. Generally, it takes N_c quarks (in a totally antisymmetric color state) to make up a baryon, and therefore, Witten proposed splitting the problem into two parts to first use graphical methods to study n-quark forces in the large-N_c limit and then to use other methods for analyzing the effects of these forces on an N_c-body state. From the analysis of large-N_c counting rules for baryon-meson scattering, Witten concluded that the corresponding amplitude at a fixed energy must be of order one.

Subsequently, Gervais and Sakita [4] and Dashen and Manohar [5] independently proved that large-N_c QCD has a contracted $SU(4)$ symmetry (for two flavors of light quarks), and they derived a set of consistency conditions that must be satisfied. The equations obtained from these consistency conditions admit a unique (minimal) solution for baryon-meson coupling constants, which are identical to those of the Skyrme model or non-relativistic quark model. Dashen, Jenkins, and Manohar applied the approach to show that large-N_c power counting rules for multimeson–baryon-baryon scattering amplitudes lead to

Received 20 June 2025; Accepted 9 October 2025; Accepted manuscript online 10 October 2025
* The authors are grateful to Consejo Nacional de Humanidades, Ciencias y Tecnologías (Mexico) for support through the *Ciencia de Frontera* project CF-2023-I-162
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important constraints on baryon static properties [6, 7]. In the same context, Flores-Mendieta, Hofmann, and Jenkins [8] studied tree-level amplitudes for baryon-meson scattering and obtained generalized large-N_c consistency conditions valid to all orders in the baryon mass splitting $\Delta \equiv M_T - M_B$, where M_T and M_B represent the baryon decuplet and baryon octet masses, respectively. Cohen, Lebed, and collaborators implemented a systematic method to derive linear relationships among meson-baryon scattering amplitudes combining the $1/N_c$ expansion of QCD with the Wigner-Eckart theorem applied to both angular momentum and isospin [9−11]. In this framework, scattering amplitudes are expressed via partial wave expansions, where mesons carry fixed orbital angular momentum and baryons possess definite spin and isospin, while neglecting baryon recoil effects. The scattering matrix elements are further specified by the total spin and total isospin of the meson-baryon system. This approach, together with the $1/N_c$ corrections to the t-channel isospin and angular momentum exchange quantum numbers, $I_t = J_t$, enables deriving multiple linear relationships among partial-wave amplitudes for meson-baryon scattering.

In the context of baryon chiral perturbation theory (BChPT), important advancements have been made on baryon-meson scattering over the past three decades. A detailed account of phenomenological models and/or different approaches proposed prior 2016 is presented in Ref. [12]. Besides the heavy baryon approach (HBChPT) [13, 14], some fully relativistic methods are noteworthy, namely, the infrared regularization of covariant BChPT [15] and the extended-on-mass-shell scheme for BChPT [16, 17]. Further improvements in HBChPT to orders $O(p^3)$ and $O(p^4)$ have been performed recently [18, 19].

Despite important progress achieved in the understanding of baryon-meson scattering processes in both the phenomenological and experimental bent [20], various challenges remain unsolved. In view of this, lattice QCD has become an essential non-perturbative tool for tackling some issues with first-principles QCD calculations that cannot be dealt with otherwise. A comprehensive description of the state-of-the-art computation of scattering amplitude for the baryon-meson system within lattice QCD can be found in Ref. [21].

The baryon-meson scattering problem is a mature area of research that has been tackled from a number of different perspectives. However, the aim of the present work is to analytically compute baryon-meson scattering amplitudes at leading and subleading orders in the framework of the $1/N_c$ expansion using the projection operator technique developed in Ref. [22]. This approach introduces new and unique elements into the theory of baryon-meson scattering, expanding existing concepts and insights. At the first stage in the analysis, the primary objective is to perform a calculation in the exact $SU(3)$ symmetry limit. At the second stage, the effects of the first-order perturbative $SU(3)$ flavor symmetry breaking (SB) and strong isospin symmetry breaking (IB) are expected to be separately incorporated. Thus, flavor projection operators can be useful to fully classify all flavor representations involved in the structure of the scattering amplitude. From this perspective, the present analysis is fundamentally different from previous works [9−11]. Loop graphs contributing to the scattering amplitude can be consistently analyzed in a combined formalism between chiral and $1/N_c$ corrections, which is the so-called large-N_c chiral perturbation theory based on the chiral Lagrangian introduced in Ref. [23]. However, this requires a non-negligible effort that will be deferred to subsequent work.

The remainder of this paper is organized as follows. Sec. II presents some elementary materials about scattering processes, along with a brief review of large-N_c QCD to introduce notation and conventions. The $1/N_c$ expansion of the baryon operator whose matrix elements between baryon states yields the scattering amplitude in the limit of the exact $SU(3)$ limit is constructed. The most complete form of this amplitude is obtained by accounting for the decuplet-octet baryon mass difference explicitly. In Sec. III, the results are particularized to the $N\pi$ system, and some isospin relationships are checked to be respected by the obtained expressions. In Sec. IV, the analysis is applied to two processes including strangeness only as case studies. In Sec. V, the effects of first-order SB are evaluated; for this purpose, flavor projection operators are constructed and extensively used to rigorously identify components from different $SU(3)$ flavor representations participating in the breaking. First-order IB effects to the scattering amplitude are also evaluated. Violations to the isospin relationships discussed in Sec. III are straightforward. A comparison of nucleon-pion scattering amplitudes within this formalism and HBChPT are outlined in Sec. V.B. Applications to scattering lengths are sketched in Sec. VI. Some concluding remarks are given in Sec. VII. In Appendix A, the baryon operator basis used in the scattering amplitude is listed. The paper is complemented by some supplementary material, loosely referred to as the Online Resource, which contains 1) the reduction of the different baryon structures in terms of an operator basis of linearly independent operators, 2) the full list of the pertinent coefficients that accompany the baryon operators of Appendix A, and 3) the operator basis used to evaluate SB effects along with their respective matrix elements listed in tables.

II. BARYON-MESON SCATTERING AMPLITUDE AT LEADING AND SUBLEADING ORDERS

In this section, the analytical computation of the amplitude of baryon-meson scattering presented in Ref. [8] is explicitly conducted, specialized to the process

$$B(p)+\pi^a(k)\to B'(p')+\pi^b(k'). \tag{1}$$

The amplitude for baryon-meson scattering at fixed meson energy is dominated in the large-N_c limit by the diagrams displayed in Fig. 1. In Eq. (1), π represents one of the nine pseudo scalar mesons π, K, η, and η' of momenta $k=(k^0,k^1,k^2,k^3)$ and $k'=(k'^0,k'^1,k'^2,k'^3)$ and flavors a and b for the incoming and outgoing mesons, respectively. B and B' represent the incoming and outgoing baryons of momenta p and p', respectively. Soft mesons with energies of order unity are considered in the process. The goal is to explicitly evaluate the corresponding scattering amplitude at leading and subleading orders, incorporating the effects of the baryon mass splitting Δ defined in the previous section. Before tackling the problem, it is convenient to introduce some key concepts on large-N_c QCD to set notation and conventions. Further details on the formalism can be found in Refs. [6, 7].

In the large-N_c limit, the baryon sector has a contracted $SU(2N_f)$ spin-flavor symmetry, where N_f represents the number of light quark flavors. For $N_f=3$, the lowest lying baryon states fall into a representation of the spin-flavor group $SU(6)$. When $N_c=3$, this corresponds to the **56** dimensional representation of $SU(6)$.

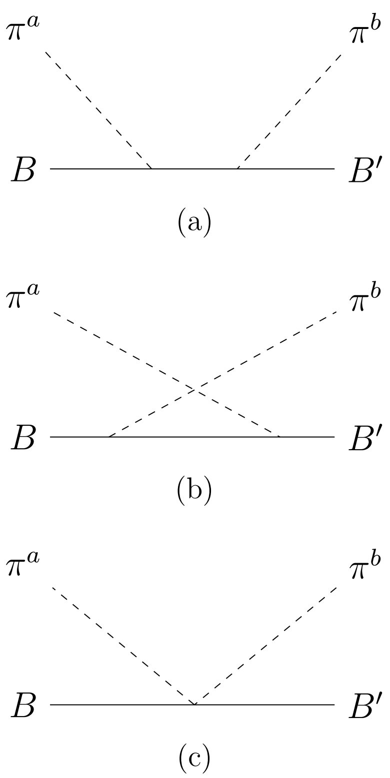

Fig. 1. Leading-order diagrams for the scattering $B+\pi\to B'+\pi$.

The $1/N_c$ expansion of a QCD operator can be written in terms of $1/N_c$-suppressed operators with well-defined spin-flavor transformation properties. A complete set of operators can be constructed using the 0-body operator $\mathcal{I}$ and 1-body operators

$$J^k=q^\dagger\left[\frac{\sigma^k}{2}\otimes\mathcal{I}\right]q, \qquad (1,1) \tag{2a}$$

$$T^c=q^\dagger\left[\mathcal{I}\otimes\frac{\lambda^c}{2}\right]q, \qquad (0,8) \tag{2b}$$

$$G^{kc}=q^\dagger\left[\frac{\sigma}{2}\otimes\frac{\lambda^c}{2}\right]q, \qquad (1,8) \tag{2c}$$

where J^k, T^c, and G^{kc} represent the baryon spin, baryon flavor, and baryon spin-flavor generators, respectively, which transform under $SU(2)\times SU(3)$ as (j,dim), where j represent the spin and dim represents the dimension of the $SU(3)$ flavor representation. The $SU(2N_f)$ spin-flavor generators satisfy well-known commutation relationships [7].

The Feynman diagrams displayed in Fig. 1 will be analyzed separately as they contribute differently to the scattering process.

A. Scattering amplitude from Fig. 1(a,b)

The amplitude for the scattering process (1) represented in Fig. 1(a,b), *in the rest frame of the initial baryon*, can be represented by the baryon operator [8]

$$\mathcal{A}_{\rm LO}^{ab}=-\frac{1}{f^2}k^ik'^j\left[\frac{1}{k^0}\sum_{n=0}^{\infty}\frac{1}{k^{0n}}[A^{jb},\underbrace{[\mathcal{M},[\mathcal{M},\ldots[\mathcal{M}}_{n\text{ insertions}},A^{ia}\underbrace{]\ldots]]}]\right], \tag{3}$$

where $f\approx 93$ MeV represents the pion decay constant, A^{ia} represents the baryon axial vector current, and $\mathcal{M}$ represents the baryon mass operator. Explicitly, the $1/N_c$ expansion of A^{ia}, at $N_c=3$, is given by [7]

$$A^{ia}=a_1G^{ia}+\frac{1}{N_c}b_2\mathcal{D}_2^{ia}+\frac{1}{N_c^2}b_3\mathcal{D}_3^{ia}+\frac{1}{N_c^2}c_3\mathcal{O}_3^{ia}, \tag{4}$$

where a_1, b_2, b_3, and c_3 represent unknown coefficients of order one, and the two- and three-body operators $\mathcal{D}_2^{ia}$, $\mathcal{D}_3^{ia}$, and $\mathcal{O}_3^{ia}$ read

$$\mathcal{D}_2^{ia}=J^iT^a, \tag{5a}$$

$$\mathcal{D}_3^{ia}=\{J^i,\{J^r,G^{ra}\}\}, \tag{5b}$$

$$\mathcal{O}_3^{ia}=\{J^2,G^{ia}\}-\frac{1}{2}\{J^i,\{J^r,G^{ra}\}\}. \tag{5c}$$

The baryon mass operator is expressed as [7]

$$\mathcal{M} = m_0 N_c \mathcal{I} + \sum_{n=2,4}^{N_c-1} m_n \frac{1}{N_c^{n-1}} J^n, \tag{6}$$

where m_n represents unknown coefficients. Although the first term on the right-hand side is the overall spin-independent mass of the baryon multiplet, the remaining terms are spin-dependent and make up $\mathcal{M}_{\rm hyperfine}$. At $N_c = 3$, $\mathcal{M}_{\rm hyperfine}$ is simply

$$\mathcal{M}_{\rm hyperfine} = \frac{m_2}{N_c} J^2, \tag{7}$$

where m_2 can be set to Δ. Numerically, the average value is $\Delta = 0.237$ GeV [24].

The series (Eq. (3)) with the first three summands reads

$$\mathcal{A}_{\rm LO}^{ab} = -\frac{1}{f^2} k^i k'^j \left[\frac{1}{k^0}[A^{jb}, A^{ia}] + \frac{1}{k^{0^2}}[A^{jb}, [\mathcal{M}, A^{ia}]] + \frac{1}{k^{0^3}}[A^{jb}, [\mathcal{M}, [\mathcal{M}, A^{ia}]]] + \ldots \right]. \tag{8}$$

The constraint that $\mathcal{A}_{\rm LO}^{ab}$ should be at most $O(1)$ in the large-N_c limit sets the consistency conditions [5, 8]

$$[A^{jb}, A^{ia}] \leq O(N_c), \tag{9a}$$

$$[A^{jb}, [\mathcal{M}, A^{ia}]] \leq O(N_c), \tag{9b}$$

$$[A^{jb}, [\mathcal{M}, [\mathcal{M}, A^{ia}]]] \leq O(N_c), \tag{9c}$$

$$\vdots$$

where k^0, f, and Δ are of orders $O(1)$, $O(\sqrt{N_c})$, and $O(N_c^{-1})$ in that limit, respectively. This work focuses on the explicit analytical computations of the first three operator structures in Eq. (9); the results will be discussed in the following sections.

1. *Spin-flavor transformation properties of* $\mathcal{A}_{\rm LO}^{ab}$

The baryon operator $\mathcal{A}_{\rm LO}^{ab}$ is a spin-zero object and contains two adjoint (octet) indices. The tensor product of two adjoint representations $8 \otimes 8$ can be split into the symmetric product $(8\otimes 8)_S$ and the antisymmetric product $(8\otimes 8)_A$ [7], which in turn can be decomposed in terms of $SU(3)$ multiplets as

$$(8\otimes 8)_S = 1 \oplus 8 \oplus 27, \tag{10a}$$

$$(8\otimes 8)_A = 8 \oplus 10 \oplus \overline{10}. \tag{10b}$$

To exploit the transformation properties of $\mathcal{A}_{\rm LO}^{ab}$ under the $SU(2)\times SU(3)$ spin-flavor symmetry, the spin and flavor projectors introduced in Ref. [22] become handy. In a few words, this technique exploits the decomposition of the tensor space formed by the product of the adjoint space with itself n times, $\prod_{i=1}^n adj\otimes$, into subspaces that can be labeled by a specific eigenvalue of the quadratic Casimir operator C of the Lie algebra of $SU(N)$. For the product of two $SU(3)$ adjoints, the flavor projectors $[\mathcal{P}^{(\rm dim)}]^{abcd}$ for the irreducible representation of dimension dim contained in Eq. (10) are given by [22]

$$[\mathcal{P}^{(1)}]^{abcd} = \frac{1}{N_f^2-1}\delta^{ab}\delta^{cd}, \tag{11}$$

$$[\mathcal{P}^{(8)}]^{abcd} = \frac{N_f}{N_f^2-4} d^{abe} d^{cde}, \tag{12}$$

$$[\mathcal{P}^{(27)}]^{abcd} = \frac{1}{2}(\delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}) - \frac{1}{N_f^2-1}\delta^{ab}\delta^{cd} - \frac{N_f}{N_f^2-4} d^{abe} d^{cde}, \tag{13}$$

$$[\mathcal{P}^{(8_A)}]^{abcd} = \frac{1}{N_f} f^{abe} f^{cde}, \tag{14}$$

and

$$[\mathcal{P}^{(10+\overline{10})}]^{abcd} = \frac{1}{2}(\delta^{ac}\delta^{bd} - \delta^{bc}\delta^{ad}) - \frac{1}{N_f} f^{abe} f^{cde}, \tag{15}$$

which fulfill the completeness relationship

$$[\mathcal{P}^{(1)} + \mathcal{P}^{(8)} + \mathcal{P}^{(27)} + \mathcal{P}^{(8_A)} + \mathcal{P}^{(10+\overline{10})}]^{abcd} = \delta^{ac}\delta^{bd}. \tag{16}$$

Therefore, $[\mathcal{P}^{(\rm dim)}\mathcal{A}_{\rm LO}]^{ab}$ effectively projects out the piece of $\mathcal{A}_{\rm LO}^{ab}$ that transforms under the flavor representation of dimension dim according to the decomposition (Eq. (10)). However, for computational purposes, it is more convenient to group the operators $[\mathcal{P}^{(\rm dim)}\mathcal{A}_{\rm LO}]^{ab}$ based on their symmetry transformation properties under the interchange of a and b. Accordingly, $[\mathcal{P}^{(1)}+\mathcal{P}^{(8)}+\mathcal{P}^{(27)}]^{abcd}$ and $[\mathcal{P}^{(8_A)}+\mathcal{P}^{(10+\overline{10})}]^{abcd}$, acting on the symmetric and antisymmetric [antisymmetric and symmetric] pieces of $\mathcal{A}_{\rm LO}^{cd}$, respectively, will provide the symmetric [antisymmetric] piece of $\mathcal{A}_{\rm LO}^{ab}$ under the interchange of a and b.

2. *Explicit form of the scattering amplitude*

A more specialized and detailed calculation beyond the qualitative analyses of baryon-meson scattering presented in previous works [5, 8] can be performed by explicitly evaluating the first summands expressed in Eq. (8); succinctly, all baryon operators allowed at $N_c=3$ (*i.e.*, up to relative order $1/N_c^2$ not discussed so far in the literature) are accounted for in the terms kept in the series (Eq. 8).

As a starting point, it should be recalled that the commutator of an n-body operator and m-body operator is an $(n+m-1)$-body operator, *i.e.*,

$$[O_m, O_n] = O_{m+n-1}. \tag{17}$$

A close inspection of Eq. (8) reveals that $[A^{jb}, A^{ia}]$ for $N_c=3$ yields at most the operator structure $[O_3^{jb}, O_3^{ia}]$, and according to Eq. (17), it retains up to five-body operators. Sequential insertions of one and two J^2 operators add up six- and seven-body operators, respectively. Therefore, the first three summands displayed in Eq. (8) will be explicitly evaluated and sufficient to draw some conclusions. Clearly, it would be desirable to perform a calculation including eight-body operators and higher; however, this is beyond the scope of this work because of the considerable amount of group theory involved.

The task now is to rewrite the baryon operators involved in $[\mathcal{P}^{(\mathrm{dim})}\mathcal{A}_{\mathrm{LO}}]^{ab}$ in terms of a set of linearly independent operators up to seven-body operators. A convenient operator basis $S_m^{(ij)(ab)}$ for $m=1,\ldots,139$ is listed in Appendix A. This is straightforward albeit the long and tedious exercise to compute those reductions. However, owing to the length and unilluminating nature of the full expressions, only symmetric and antisymmetric pieces rather than individual results for each representation are listed in the Online Resource. In passing, it is straightforward to verify that the consistency conditions (Eq. (9)) are fulfilled by all these reduced structures.

The matrix elements of $\mathcal{A}_{\mathrm{LO}}^{ab}$ given in Eq. (3) between $SU(6)$ baryon states, where mesons are labeled with flavors a and b, yield the corresponding scattering amplitude, namely,

$$\mathcal{A}_{\mathrm{LO}}(B+\pi^a\to B'+\pi^b) \equiv \langle \pi^b B'|\mathcal{A}_{\mathrm{LO}}^{ab}|\pi^a B\rangle. \tag{18}$$

The flavors associated to mesons are conventionally given by $\left\{\frac{1\mp i2}{\sqrt{2}}, 3, \frac{4\mp i5}{\sqrt{2}}, \frac{6-i7}{\sqrt{2}}, \frac{6+i7}{\sqrt{2}}, 8\right\}$ for $\{\pi^\pm, \pi^0, K^\pm, K^0, \overline{K}^0, \eta\}$, respectively.[1)] For instance, an expressions such as $\mathcal{A}_{\mathrm{LO}}(p+\pi^-\to n+\pi^0)$ should be understood as $\langle \pi^0 n|\mathcal{A}_{\mathrm{LO}}^{13}+\mathrm{i}\mathcal{A}_{\mathrm{LO}}^{23}|\pi^- p\rangle/\sqrt{2}$.

Thus, with the operator reductions listed in the Online Resource, the scattering amplitude for process (1) arising from Fig. 1(a,b) can be organized as

$$\begin{aligned}&\mathcal{A}_{\mathrm{LO}}(B+\pi^a\to B'+\pi^b)\\ &= -\frac{1}{f^2k^0}\sum_{m=1}^{139}(c_m^{(\mathrm{s})}+c_m^{(\mathrm{a})})k^ik'^j\langle \pi^b B'|S_m^{(ij)(ab)}|\pi^a B\rangle,\end{aligned} \tag{19}$$

where $S_m^{(ij)(ab)}$ constitute a basis of linearly independent spin-2 baryon operators with two adjoint indices, and $c_m^{(\mathrm{s})}$ and $c_m^{(\mathrm{a})}$ are well-defined coefficients that come along with the symmetric and antisymmetric pieces of $\mathcal{A}_{\mathrm{LO}}^{ab}$; these coefficients are listed in the Online Resource. In Eq. (19), the sum over spin indices is implicit.

B. Scattering amplitude from Fig. 1(c)

The two-meson-baryon-baryon contact interaction represented in Fig. 1(c) contributes to the baryon-meson scattering amplitude with a term [8]

$$\mathcal{A}_{\mathrm{vertex}}^{ab} = -\frac{1}{2f^2}(2k^0+M-M')\mathrm{i}f^{abc}T^c, \tag{20}$$

where M and M' represents the masses of the initial and final baryons, respectively. Since $\mathcal{A}_{\mathrm{vertex}}^{ab}$ is already antisymmetric under the interchange of a and b, the only term that remains once the projection operators are applied is $[\mathcal{P}^{(8_A)}A_{\mathrm{vertex}}]^{ab}$, and therefore, this term only contributes to the octet piece. Further, both $\mathcal{A}_{\mathrm{vertex}}^{ab}$, of order $O(1)$, and $\mathcal{A}_{\mathrm{LO}}^{ab}$ (Eq. (3)) yield the leading order $O(1)$ scattering amplitude for baryons with spin $J\sim O(1)$.

III. APPLICATION: $N\pi\to N\pi$ SCATTERING AMPLITUDE

The formalism presented so far can be implemented to study the scattering processes of the form $B+\pi^a\to B'+\pi^b$ provided that reactions in which these particles are produced have equal total strangeness on each side, according to the Gell-Mann–Nishijima scheme. Since B and B' can be either octet or decuplet baryons from the theoretical point of view, the possibilities are numerous. The examples include $\Lambda+K^+\to p+\pi^0$, $\Xi^{*-}+K^+\to\Sigma^{*0}+\pi^0$, $\Xi^{*-}+K^0\to\Sigma^-+\pi^0$, and so on. For definiteness, the $N\pi\to N\pi$ scattering processes will be analyzed to exemplify the approach.

A pion $I=1$ and a nucleon $I=1/2$ can be combined in a $I=3/2$ or a $I=1/2$ state following the usual addition rules of angular momenta [27]. The allowed states

1) For simplicity only the octet of mesons is considered. Extending the analysis to include the η' is straightforward by using the baryon axial vector current $A^i\equiv A^{i9}$, which is written in terms of the 1-body operators $G^{i9}=\frac{1}{\sqrt{6}}J^i$ and $T^9=\frac{1}{\sqrt{6}}N_c\mathcal{I}$ [23].

for the $N\pi$ system are listed in Table 1.

The elastic scattering amplitude for the process (Eq. (1)) can be decomposed using the usual Clebsch-Gordan technique into two non-interfering amplitudes $\mathcal{A}^{(T)}$ with $I=3/2$ and $I=1/2$. Thus, starting from the s-channel isospin eigenstates

$$|\pi^+ p\rangle = \left|\frac{3}{2},+\frac{3}{2}\right\rangle, \tag{21}$$

$$|\pi^+ n\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle, \tag{22}$$

$$|\pi^0 p\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle, \tag{23}$$

$$|\pi^0 n\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle, \tag{24}$$

$$|\pi^- p\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle, \tag{25}$$

$$|\pi^- n\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle, \tag{26}$$

it is straightforward to obtain [28]

$$\begin{aligned}
&\mathcal{A}_{\rm LO}(p+\pi^+\to p+\pi^+) = \mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-) = \mathcal{A}^{(3/2)},\\
&\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-) = \mathcal{A}_{\rm LO}(n+\pi^+\to n+\pi^+)\\
&= \frac{1}{3}\mathcal{A}^{(3/2)} + \frac{2}{3}\mathcal{A}^{(1/2)},\\
&\mathcal{A}_{\rm LO}(p+\pi^0\to p+\pi^0) = \mathcal{A}_{\rm LO}(n+\pi^0\to n+\pi^0)\\
&= \frac{2}{3}\mathcal{A}^{(3/2)} + \frac{1}{3}\mathcal{A}^{(1/2)},\\
&\sqrt{2}\mathcal{A}_{\rm LO}(p+\pi^-\to n+\pi^0) = \sqrt{2}\mathcal{A}_{\rm LO}(n+\pi^+\to p+\pi^0)\\
&= \frac{2}{3}\mathcal{A}^{(3/2)} - \frac{2}{3}\mathcal{A}^{(1/2)}.
\end{aligned} \tag{27}$$

In addition, an alternative set of invariant amplitudes $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ can be introduced for the $N\pi$ system, which are defined as [29]

$$\mathcal{A}^{(+)} = \frac{2}{3}\mathcal{A}^{(3/2)} + \frac{1}{3}\mathcal{A}^{(1/2)},$$

Table 1. Allowed states for the $N\pi$ system.

	$I=\frac{3}{2}$	$I=\frac{1}{2}$
$I_3=+\frac{3}{2}$	$\lvert\pi^+p\rangle$	
$I_3=+\frac{1}{2}$	$\sqrt{\frac{1}{3}}\lvert\pi^+n\rangle+\sqrt{\frac{2}{3}}\lvert\pi^0p\rangle$	$\sqrt{\frac{2}{3}}\lvert\pi^+n\rangle-\sqrt{\frac{1}{3}}\lvert\pi^0p\rangle$
$I_3=-\frac{1}{2}$	$\sqrt{\frac{2}{3}}\lvert\pi^0n\rangle+\sqrt{\frac{1}{3}}\lvert\pi^-p\rangle$	$\sqrt{\frac{1}{3}}\lvert\pi^0n\rangle-\sqrt{\frac{2}{3}}\lvert\pi^-p\rangle$
$I_3=-\frac{3}{2}$	$\lvert\pi^-p\rangle$	

$$\mathcal{A}^{(-)} = -\frac{1}{3}\mathcal{A}^{(3/2)} + \frac{1}{3}\mathcal{A}^{(1/2)}, \tag{28}$$

and therefore,

$$\begin{aligned}
\mathcal{A}^{(3/2)} &= \mathcal{A}^{(+)} - \mathcal{A}^{(-)},\\
\mathcal{A}^{(1/2)} &= \mathcal{A}^{(+)} + 2\mathcal{A}^{(-)}.
\end{aligned} \tag{29}$$

The non-trivial matrix elements $k^ik'^j\langle\pi^bB'|S_r^{(ij)(ab)}|\pi^aB\rangle$ are displayed in Tables 2 and 3 for proton-pion and neutron-pion processes ($N\pi\to N\pi$ processes for short), respectively.[1)] It can be easily verified that the symmetric and antisymmetric pieces of $\mathcal{A}_{\rm LO}(B+\pi^a\to B'+\pi^b)$ are respectively proportional to $\mathbf{k}\cdot\mathbf{k}'$ and the third component of $\mathrm{i}(\mathbf{k}\times\mathbf{k}')$, which will be denoted hereafter by $\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$. The latter can also be rewritten as $\mathrm{i}\epsilon^{ij3}k^ik'^j = \mathrm{i}(k^1k'^2 - k^2k'^1)$.

B. Scattering amplitude from Fig. 1(a,b)

Collecting partial results from Eq. (19), the scattering amplitude for the $N\pi$ system can be cast into

$$\begin{aligned}
&f^2k^0\mathcal{A}_{\rm LO}(p+\pi^+\to p+\pi^+)\\
&= \left[-\frac{25}{72}a_1^2 - \frac{5}{36}a_1b_2 - \frac{25}{108}a_1b_3 - \frac{1}{72}b_2^2 - \frac{5}{108}b_2b_3\right.\\
&\quad\left. - \frac{25}{648}b_3^2 + \frac{2}{9}\left[1 - \frac{2\Delta}{k^0} + \frac{\Delta^2}{k^{0\,2}}\right]\left[a_1^2 + a_1c_3 + \frac{1}{4}c_3^2\right]\right]\mathbf{k}\cdot\mathbf{k}'\\
&\quad + \left[\frac{25}{72}a_1^2 + \frac{5}{36}a_1b_2 + \frac{25}{108}a_1b_3\right.\\
&\quad + \frac{1}{72}b_2^2 + \frac{5}{108}b_2b_3 + \frac{25}{648}b_3^2\\
&\quad\left. - \frac{2}{9}\left[1 - \frac{1}{2}\frac{\Delta}{k^0} + \frac{\Delta^2}{k^{0\,2}}\right]\left[a_1^2 + a_1c_3 + \frac{1}{4}c_3^2\right]\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3\\
&\quad + O\left[\frac{\Delta^3}{k^{0\,3}}\right]\\
&= f^2k^0\mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-),
\end{aligned} \tag{30}$$

1) Here, non-trivial matrix elements are those which are either zero or obtained as anticommutators with J^2. For instance, for the $N\pi$ system, $k^ik'^j\langle\pi^bB'|S_{17}^{(ij)(ab)}|\pi^aB\rangle$ vanishes whereas $k^ik'^j\langle\pi^bB'|S_{37}^{(ij)(ab)}|\pi^aB\rangle = \frac{3}{2}k^ik'^j\langle\pi^bB'|S_{15}^{(ij)(ab)}|\pi^aB\rangle$, so they are not listed.

Table 2. Non-trivial matrix elements of operators involved in proton-pion scattering processes.

Operator	$p+\pi^+\to p+\pi^+$	$p+\pi^-\to p+\pi^-$	$p+\pi^0\to p+\pi^0$	$p+\pi^-\to n+\pi^0$
$k^ik'^j\langle S_1^{(ij)(ab)}\rangle$	$-\mathbf{k}\cdot\mathbf{k}'$	$\mathbf{k}\cdot\mathbf{k}'$	0	$-\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_2^{(ij)(ab)}\rangle$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_3^{(ij)(ab)}\rangle$	$\frac{1}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_4^{(ij)(ab)}\rangle$	$\mathbf{k}\cdot\mathbf{k}'$	$\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_5^{(ij)(ab)}\rangle$	$\frac{3}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{3}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{3}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_6^{(ij)(ab)}\rangle$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'+\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{24}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_7^{(ij)(ab)}\rangle$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'+\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{24}\mathbf{k}\cdot\mathbf{k}'$	$\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_8^{(ij)(ab)}\rangle$	$\frac{19}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{19}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{19}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_9^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{10}^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{11}^{(ij)(ab)}\rangle$	$\frac{1}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{12}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{12}^{(ij)(ab)}\rangle$	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{13}^{(ij)(ab)}\rangle$	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{14}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{15}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{16}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{18}^{(ij)(ab)}\rangle$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{19}^{(ij)(ab)}\rangle$	$\frac{19}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{20}^{(ij)(ab)}\rangle$	$-\mathbf{k}\cdot\mathbf{k}'$	$\mathbf{k}\cdot\mathbf{k}'$	0	$-\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{23}^{(ij)(ab)}\rangle$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{24}^{(ij)(ab)}\rangle$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{25}^{(ij)(ab)}\rangle$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{26}^{(ij)(ab)}\rangle$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{27}^{(ij)(ab)}\rangle$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{24}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{28}^{(ij)(ab)}\rangle$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{24}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{29}^{(ij)(ab)}\rangle$	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{30}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{38}^{(ij)(ab)}\rangle$	$-\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{39}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{40}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{47}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{49}^{(ij)(ab)}\rangle$	$4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{54}^{(ij)(ab)}\rangle$	$4\mathbf{k}\cdot\mathbf{k}'-\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$4\mathbf{k}\cdot\mathbf{k}'+\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$2\mathbf{k}\cdot\mathbf{k}'$	$-\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{56}^{(ij)(ab)}\rangle$	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{60}^{(ij)(ab)}\rangle$	$-8\mathbf{k}\cdot\mathbf{k}'+2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-8\mathbf{k}\cdot\mathbf{k}'-2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-4\mathbf{k}\cdot\mathbf{k}'$	$2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{67}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{68}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{83}^{(ij)(ab)}\rangle$	$\frac{25}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{84}^{(ij)(ab)}\rangle$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{85}^{(ij)(ab)}\rangle$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{86}^{(ij)(ab)}\rangle$	$4\mathbf{k}\cdot\mathbf{k}'$	$-4\mathbf{k}\cdot\mathbf{k}'$	0	$4\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{91}^{(ij)(ab)}\rangle$	$6\mathbf{k}\cdot\mathbf{k}'-6\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-6\mathbf{k}\cdot\mathbf{k}'-6\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-3\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$6\mathbf{k}\cdot\mathbf{k}'$

Table 3. Non-trivial matrix elements of operators involved in neutron-pion scattering processes.

Operator	$n+\pi^+\to n+\pi^+$	$n+\pi^-\to n+\pi^-$	$n+\pi^0\to n+\pi^0$	$n+\pi^+\to p+\pi^0$
$k^ik'^j\langle S_1^{(ij)(ab)}\rangle$	$\mathbf{k}\cdot\mathbf{k}'$	$-\mathbf{k}\cdot\mathbf{k}'$	0	$-\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_2^{(ij)(ab)}\rangle$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_3^{(ij)(ab)}\rangle$	$\frac{1}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{1}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_4^{(ij)(ab)}\rangle$	$\mathbf{k}\cdot\mathbf{k}'$	$\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_5^{(ij)(ab)}\rangle$	$\frac{3}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{3}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{3}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_6^{(ij)(ab)}\rangle$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'+\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{24}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_7^{(ij)(ab)}\rangle$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'-\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{12}\mathbf{k}\cdot\mathbf{k}'+\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{24}\mathbf{k}\cdot\mathbf{k}'$	$\frac{11}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_8^{(ij)(ab)}\rangle$	$\frac{19}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{19}{4}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{19}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_9^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{10}^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{6}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{11}^{(ij)(ab)}\rangle$	$\frac{1}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{6}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{12}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{12}^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{13}^{(ij)(ab)}\rangle$	$\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{6}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{14}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{15}^{(ij)(ab)}\rangle$	$-\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{16}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{18}^{(ij)(ab)}\rangle$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{1}{2}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{19}^{(ij)(ab)}\rangle$	$\frac{19}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{19}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{20}^{(ij)(ab)}\rangle$	$\mathbf{k}\cdot\mathbf{k}'$	$-\mathbf{k}\cdot\mathbf{k}'$	0	$-\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{23}^{(ij)(ab)}\rangle$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{24}^{(ij)(ab)}\rangle$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{25}^{(ij)(ab)}\rangle$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'-\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{26}^{(ij)(ab)}\rangle$	$-\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'+\frac{4}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{2}{3}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{4}{3}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{27}^{(ij)(ab)}\rangle$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{24}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{28}^{(ij)(ab)}\rangle$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{12}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{24}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{29}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{30}^{(ij)(ab)}\rangle$	$-\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{38}^{(ij)(ab)}\rangle$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$-\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	0	$-\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{39}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{40}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{5}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{47}^{(ij)(ab)}\rangle$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{1}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{49}^{(ij)(ab)}\rangle$	$-4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$4\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{54}^{(ij)(ab)}\rangle$	$4\mathbf{k}\cdot\mathbf{k}'+\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$4\mathbf{k}\cdot\mathbf{k}'-\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$2\mathbf{k}\cdot\mathbf{k}'$	$-\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{56}^{(ij)(ab)}\rangle$	$-\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{60}^{(ij)(ab)}\rangle$	$-8\mathbf{k}\cdot\mathbf{k}'-2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-8\mathbf{k}\cdot\mathbf{k}'+2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-4\mathbf{k}\cdot\mathbf{k}'$	$2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$
$k^ik'^j\langle S_{67}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{68}^{(ij)(ab)}\rangle$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{8}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{83}^{(ij)(ab)}\rangle$	$\frac{25}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{4}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$\frac{25}{8}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	0
$k^ik'^j\langle S_{84}^{(ij)(ab)}\rangle$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{85}^{(ij)(ab)}\rangle$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{2}\mathbf{k}\cdot\mathbf{k}'$	$\frac{5}{4}\mathbf{k}\cdot\mathbf{k}'$	0
$k^ik'^j\langle S_{86}^{(ij)(ab)}\rangle$	$-4\mathbf{k}\cdot\mathbf{k}'$	$4\mathbf{k}\cdot\mathbf{k}'$	0	$4\mathbf{k}\cdot\mathbf{k}'$
$k^ik'^j\langle S_{91}^{(ij)(ab)}\rangle$	$-6\mathbf{k}\cdot\mathbf{k}'-6\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$6\mathbf{k}\cdot\mathbf{k}'-6\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$-3\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3$	$6\mathbf{k}\cdot\mathbf{k}'$

$$\begin{aligned}
&f^2k^0\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-)\\
&=\left[\frac{25}{72}a_1^2+\frac{5}{36}a_1b_2+\frac{25}{108}a_1b_3+\frac{1}{72}b_2^2+\frac{5}{108}b_2b_3+\frac{25}{648}b_3^2-\frac{2}{9}\left[1+\frac{2\Delta}{k^0}+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathbf{k}\cdot\mathbf{k}'\\
&+\left[\frac{25}{72}a_1^2+\frac{5}{36}a_1b_2+\frac{25}{108}a_1b_3+\frac{1}{72}b_2^2+\frac{5}{108}b_2b_3+\frac{25}{648}b_3^2-\frac{2}{9}\left[1+\frac{1}{2}\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta^3}{k^{03}}\right]\\
&=f^2k^0\mathcal{A}_{\rm LO}(n+\pi^+\to n+\pi^+),
\end{aligned}\tag{31}$$

$$\begin{aligned}
&f^2k^0\mathcal{A}_{\rm LO}(p+\pi^0\to p+\pi^0)\\
&=-\frac{4}{9}\frac{\Delta}{k^0}\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\mathbf{k}\cdot\mathbf{k}'+\left[\frac{25}{72}a_1^2+\frac{5}{36}a_1b_2+\frac{25}{108}a_1b_3+\frac{1}{72}b_2^2+\frac{5}{108}b_2b_3+\frac{25}{648}b_3^2\right.\\
&\left.-\frac{2}{9}\left[1+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta^3}{k^{03}}\right]=f^2k^0\mathcal{A}_{\rm LO}(n+\pi^0\to n+\pi^0),
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\sqrt{2}f^2k^0\mathcal{A}_{\rm LO}(p+\pi^-\to n+\pi^0)\\
&=\left[-\frac{25}{36}a_1^2-\frac{5}{18}a_1b_2-\frac{25}{54}a_1b_3-\frac{1}{36}b_2^2-\frac{5}{54}b_2b_3-\frac{25}{324}b_3^2+\frac{4}{9}\left[1+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathbf{k}\cdot\mathbf{k}'\\
&+\frac{2}{9}\frac{\Delta}{k^0}\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta^3}{k^{03}}\right]=\sqrt{2}f^2k^0\mathcal{A}_{\rm LO}(n+\pi^+\to p+\pi^0).
\end{aligned}\tag{33}$$

Scattering amplitudes including all operator structures enabled for $N_c=3$ in Eq. (8) can be evaluated because the above results can be rewritten in terms of the $SU(3)$ invariant couplings D, F, C, and $\mathcal{H}$ introduced in HBChPT [25, 26]. These couplings are related to the $1/N_c$ coefficients a_1, b_2, b_3, and c_3 for $N_c=3$ as [23]

$$D=\frac{1}{2}a_1+\frac{1}{6}b_3,\tag{34a}$$

$$F=\frac{1}{3}a_1+\frac{1}{6}b_2+\frac{1}{9}b_3,\tag{34b}$$

$$C=-a_1-\frac{1}{2}c_3,\tag{34c}$$

$$\mathcal{H}=-\frac{3}{2}a_1-\frac{3}{2}b_2-\frac{5}{2}b_3.\tag{34d}$$

A further simplification can be achieved if the power series expansion in Δ of the function

$$t_1\frac{k^0}{k^0-\Delta}+t_2\frac{k^0}{k^0+\Delta}=t_1+t_2+(t_1-t_2)\frac{\Delta}{k^0}+(t_1+t_2)\frac{\Delta^2}{k^{02}}+(t_1-t_2)\frac{\Delta^3}{k^{03}}+O\left[\frac{\Delta^4}{k^{04}}\right],\tag{35}$$

where t_k are some coefficients, is substituted into Eqs. (30)−(33) to rewrite the final forms of the scattering amplitudes as

$$\begin{aligned}
f^2k^0\mathcal{A}_{\rm LO}(p+\pi^+\to p+\pi^+)=&\left[-\frac{1}{2}(D+F)^2+\frac{1}{9}\left[-\frac{k^0}{k^0-\Delta}+3\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}'\\
&+\left[\frac{1}{2}(D+F)^2-\frac{1}{18}\left[\frac{k^0}{k^0-\Delta}+3\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3=f^2k^0\mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-),
\end{aligned}\tag{36}$$

$$\begin{aligned}
f^2k^0\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-)=&\left[\frac{1}{2}(D+F)^2-\frac{1}{9}\left[3\frac{k^0}{k^0-\Delta}-\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}'\\
&+\left[\frac{1}{2}(D+F)^2-\frac{1}{18}\left[3\frac{k^0}{k^0-\Delta}+\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3=f^2k^0\mathcal{A}_{\rm LO}(n+\pi^+\to n+\pi^+),
\end{aligned}\tag{37}$$

$$f^2k^0\mathcal{A}_{\rm LO}(p+\pi^0\to p+\pi^0) = -\frac{2}{9}\left[\frac{k^0}{k^0-\Delta}-\frac{k^0}{k^0+\Delta}\right]C^2\mathbf{k}\cdot\mathbf{k}' + \left[\frac{1}{2}(D+F)^2-\frac{1}{9}\left[\frac{k^0}{k^0-\Delta}+\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3 = f^2k^0\mathcal{A}_{\rm LO}(n+\pi^0\to n+\pi^0), \tag{38}$$

$$\sqrt{2}f^2k^0\mathcal{A}_{\rm LO}(p+\pi^-\to n+\pi^0) = \left[-(D+F)^2+\frac{2}{9}\left[\frac{k^0}{k^0-\Delta}+\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}' + \frac{1}{9}\left[\frac{k^0}{k^0-\Delta}-\frac{k^0}{k^0+\Delta}\right]C^2\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3 = \sqrt{2}f^2k^0\mathcal{A}_{\rm LO}(n+\pi^+\to p+\pi^0), \tag{39}$$

which are valid to order $O(\Delta^3/k^{0^3})$.

A glance at the above expression shows that scattering amplitudes for $N\pi\to N\pi$ processes are written in terms of the $SU(3)$ invariants F, D, and C [25, 26], which is a totally expected and consistent result because F and D come along $BB\pi$ vertices, where $g_A=D+F$ represents the axial coupling for neutron beta decay in the limit of exact $SU(3)$ symmetry, whereas C comes along $TB\pi$ vertices. Further, in the limit $\Delta\to 0$, the coefficients of the C^2 terms do not vanish.

As for the $\mathcal{A}_{\rm LO}^{(3/2)}$ and $\mathcal{A}_{\rm LO}^{(1/2)}$ amplitudes, they are found to be

$$f^2k^0\mathcal{A}_{\rm LO}^{(3/2)} = \left[-\frac{25}{72}a_1^2-\frac{5}{36}a_1b_2-\frac{25}{108}a_1b_3 -\frac{1}{72}b_2^2-\frac{5}{108}b_2b_3-\frac{25}{648}b_3^2\right. \tag{40}$$

$$\left.+\frac{2}{9}\left[1-\frac{2\Delta}{k^0}+\frac{\Delta^2}{k^{0^2}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathbf{k}\cdot\mathbf{k}' + \left[\frac{25}{72}a_1^2+\frac{5}{36}a_1b_2+\frac{25}{108}a_1b_3+\frac{1}{72}b_2^2+\frac{5}{108}b_2b_3+\frac{25}{648}b_3^2 -\frac{2}{9}\left[1-\frac{1}{2}\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{0^2}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta^3}{k^{0^3}}\right], \tag{41}$$

and

$$f^2k^0\mathcal{A}_{\rm LO}^{(1/2)} = \left[\frac{25}{36}a_1^2+\frac{5}{18}a_1b_2+\frac{25}{54}a_1b_3 +\frac{1}{36}b_2^2+\frac{5}{54}b_2b_3+\frac{25}{324}b_3^2\right. \tag{42}$$

$$\left.-\frac{4}{9}\left[1+\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{0^2}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathbf{k}\cdot\mathbf{k}' + \left[\frac{25}{72}a_1^2+\frac{5}{36}a_1b_2+\frac{25}{108}a_1b_3+\frac{1}{72}b_2^2+\frac{5}{108}b_2b_3+\frac{25}{648}b_3^2 -\frac{2}{9}\left[1+\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{0^2}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta^3}{k^{0^3}}\right], \tag{43}$$

or equivalently,

$$f^2k^0\mathcal{A}_{\rm LO}^{(3/2)} = \left[-\frac{1}{2}(D+F)^2+\frac{1}{9}\left[-\frac{k^0}{k^0-\Delta}+3\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}' + \left[\frac{1}{2}(D+F)^2-\frac{1}{18}\left[\frac{k^0}{k^0-\Delta}+3\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3, \tag{44}$$

and

$$f^2k^0\mathcal{A}_{\rm LO}^{(1/2)} = \left[(D+F)^2-\frac{4}{9}\frac{k^0}{k^0-\Delta}C^2\right]\mathbf{k}\cdot\mathbf{k}' + \left[\frac{1}{2}(D+F)^2-\frac{2}{9}\frac{k^0}{k^0-\Delta}C^2\right]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3, \tag{45}$$

which are valid to order $O(\Delta^3/k^{0^3})$.

1. *Isospin relations*

The $N\pi\to N\pi$ scattering amplitudes satisfy the following isospin relations:

$$\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-)-\mathcal{A}_{\rm LO}(p+\pi^0\to p+\pi^0) + \frac{1}{\sqrt{2}}\mathcal{A}_{\rm LO}(p+\pi^-\to n+\pi^0)=0, \tag{46}$$

$$\mathcal{A}_{\rm LO}(p+\pi^+\to p+\pi^+)-\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-) - \sqrt{2}\mathcal{A}_{\rm LO}(p+\pi^-\to n+\pi^0)=0, \tag{47}$$

$$\mathcal{A}_{\rm LO}(p+\pi^+\to p+\pi^+)+\mathcal{A}_{\rm LO}(p+\pi^-\to p+\pi^-) -2\mathcal{A}_{\rm LO}(p+\pi^0\to p+\pi^0)=0, \tag{48}$$

$$\mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-)-\mathcal{A}_{\rm LO}(n+\pi^0\to n+\pi^0) - \frac{1}{\sqrt{2}}\mathcal{A}_{\rm LO}(n+\pi^+\to p+\pi^0)=0, \tag{49}$$

$$\mathcal{A}_{\rm LO}(n+\pi^+\to n+\pi^+)-\mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-) + \sqrt{2}\mathcal{A}_{\rm LO}(n+\pi^+\to p+\pi^0)=0, \tag{50}$$

$$\mathcal{A}_{\rm LO}(n+\pi^+\to n+\pi^+)+\mathcal{A}_{\rm LO}(n+\pi^-\to n+\pi^-) -2\mathcal{A}_{\rm LO}(n+\pi^0\to n+\pi^0)=0. \tag{51}$$

B. Scattering amplitude from Fig. 1(c)

Following the lines of Eq. (18), for the $N\pi$ system, the scattering amplitudes arising from Fig. 1(c) read

$$\mathcal{A}_{\rm vertex}(p+\pi^+\to p+\pi^+)=\frac{1}{4}\frac{k^0}{f^2} =\mathcal{A}_{\rm vertex}(n+\pi^-\to n+\pi^-), \tag{52}$$

$$\mathcal{A}_{\rm vertex}(p+\pi^-\to p+\pi^-)=-\frac{1}{4}\frac{k^0}{f^2} =\mathcal{A}_{\rm vertex}(n+\pi^+\to n+\pi^+), \tag{53}$$

$$\mathcal{A}_{\rm vertex}(p+\pi^0\to p+\pi^0)=0 =\mathcal{A}_{\rm vertex}(n+\pi^0\to n+\pi^0), \tag{54}$$

$$\mathcal{A}_{\rm vertex}(p+\pi^-\to n+\pi^0)=\frac{1}{2\sqrt{2}}\frac{k^0}{f^2} =\mathcal{A}_{\rm vertex}(n+\pi^+\to p+\pi^0), \tag{55}$$

from which the following amplitudes can be obtained,

$$\mathcal{A}^{(3/2)}_{\rm vertex}=\frac{1}{4}\frac{k^0}{f^2}, \tag{56}$$

and

$$\mathcal{A}^{(1/2)}_{\rm vertex}=-\frac{1}{2}\frac{k^0}{f^2}. \tag{57}$$

1. *Isospin relations*

In a close analogy of the previous case, the isospin relations between these scattering amplitudes are

$$\mathcal{A}_{\rm vertex}(p+\pi^-\to p+\pi^-)-\mathcal{A}_{\rm vertex}(p+\pi^0\to p+\pi^0) +\frac{1}{\sqrt{2}}\mathcal{A}_{\rm vertex}(p+\pi^-\to n+\pi^0)=0, \tag{58}$$

$$\mathcal{A}_{\rm vertex}(p+\pi^+\to p+\pi^+)-\mathcal{A}_{\rm vertex}(p+\pi^-\to p+\pi^-) -\sqrt{2}\mathcal{A}_{\rm vertex}(p+\pi^-\to n+\pi^0)=0, \tag{59}$$

$$\mathcal{A}_{\rm vertex}(p+\pi^+\to p+\pi^+)+\mathcal{A}_{\rm vertex}(p+\pi^-\to p+\pi^-) -2\mathcal{A}_{\rm vertex}(p+\pi^0\to n+\pi^0)=0, \tag{60}$$

$$\mathcal{A}_{\rm vertex}(n+\pi^-\to n+\pi^-)-\mathcal{A}_{\rm vertex}(n+\pi^0\to n+\pi^0) -\frac{1}{\sqrt{2}}\mathcal{A}_{\rm vertex}(n+\pi^+\to p+\pi^0)=0, \tag{61}$$

$$\mathcal{A}_{\rm vertex}(n+\pi^+\to n+\pi^+)-\mathcal{A}_{\rm vertex}(n+\pi^-\to n+\pi^-) +\sqrt{2}\mathcal{A}_{\rm vertex}(n+\pi^+\to p+\pi^0)=0, \tag{62}$$

$$\mathcal{A}_{\rm vertex}(n+\pi^+\to n+\pi^+)+\mathcal{A}_{\rm vertex}(n+\pi^-\to n+\pi^-) -2\mathcal{A}_{\rm vertex}(n+\pi^0\to n+\pi^0)=0. \tag{63}$$

IV. PROCESSES WITH STRANGENESS: TWO CASE STUDIES

To test the applicability of the approach, two processes including strangeness have been selected with no specific criteria. They are two case studies: $\Lambda+K^+\to p+\pi^0$ and $\Xi^0+K^0\to\Lambda+\eta$. The respective scattering amplitudes from Fig. 1(a,b) read

$$\begin{aligned}
&4\sqrt{3}f^2k^0\mathcal{A}_{\rm LO}(\Lambda+K^+\to p+\pi^0)\\
&=\left[-\frac{17}{12}a_1^2-\frac{1}{2}a_1b_2-\frac{17}{18}a_1b_3-\frac{1}{12}b_2^2-\frac{1}{6}b_2b_3-\frac{17}{108}b_3^2\right.\\
&\quad+\frac{2}{3}\left[1+\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\Bigg]\,\mathbf{k}\cdot\mathbf{k}'\\
&\quad+\left[-\frac{13}{12}a_1^2-\frac{5}{6}a_1b_2-\frac{13}{18}a_1b_3-\frac{1}{12}b_2^2-\frac{5}{18}b_2b_3-\frac{13}{108}b_3^2\right.\\
&\quad+\frac{1}{3}\left[1+\frac{\Delta}{k^0}+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\Bigg](\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta}{k^0}\right]^3,
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
&4\sqrt{3}f^2k^0\mathcal{A}_{\rm LO}(\Xi^0+K^0\to\Lambda+\eta)\\
&=\left[-\frac{3}{4}a_1^2-\frac{1}{6}a_1b_2-\frac{1}{2}a_1b_3-\frac{1}{12}b_2^2-\frac{1}{18}b_2b_3-\frac{1}{12}b_3^2\right.\\
&\quad+\frac{4}{3}\frac{\Delta}{k^0}\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\Bigg]\,\mathbf{k}\cdot\mathbf{k}'\\
&\quad+\left[-\frac{11}{12}a_1^2-\frac{1}{6}a_1b_2-\frac{11}{18}a_1b_3+\frac{1}{12}b_2^2-\frac{1}{18}b_2b_3-\frac{11}{108}b_3^2\right.
\end{aligned}$$

$$+\frac{2}{3}\left[1+\frac{\Delta^2}{k^{02}}\right]\left[a_1^2+a_1c_3+\frac{1}{4}c_3^2\right]\right](\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta}{k^0}\right]^3, \tag{65}$$

or equivalently,

$$\begin{aligned}&4\sqrt{3}f^2k^0\mathcal{A}_{\rm LO}(\Lambda+K^+\to p+\pi^0)\\ &=\left[-3D^2-2DF-3F^2+\frac{2}{3}\left[\frac{k^0}{k^0-\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}'\\ &+\left[D^2-6DF-3F^2+\frac{1}{3}\left[\frac{k^0}{k^0-\Delta}\right]C^2\right](\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{\Delta}{k^0}\right]^3,\end{aligned} \tag{66}$$

and

$$\begin{aligned}&4\sqrt{3}f^2k^0\mathcal{A}_{\rm LO}(\Xi^0+K^0\to\Lambda+\eta)\\ &=\left[-3D^2+2DF-3F^2+\frac{2}{3}\left[\frac{k^0}{k^0-\Delta}-\frac{k^0}{k^0+\Delta}\right]C^2\right]\mathbf{k}\cdot\mathbf{k}'\\ &+\left[-D^2-6DF+3F^2+\frac{1}{3}\left[\frac{k^0}{k^0-\Delta}+\frac{k^0}{k^0+\Delta}\right]C^2\right](\mathbf{k}\times\mathbf{k}')_3\\ &+O\left[\frac{\Delta}{k^0}\right]^3.\end{aligned} \tag{67}$$

The above expressions have been obtained in a complete parallelism to the nucleon-pion processes, and therefore, no additional details are required.

V. FIRST-ORDER *SU*(3) SYMMETRY BREAKING IN THE SCATTERING AMPLITUDE

The $SU(3)$ flavor symmetry is not an exact symmetry and is actually broken. Flavor SB and strong IB refer to the deviation of the strong force from the ideal symmetric limit where all quark flavors are treated on an equal footing (flavor symmetry) and where the up and down quarks are considered identical (isospin symmetry).

Two major sources of $SU(3)$ symmetry breaking are identified. The first one is caused by the light quark masses, and the perturbation transforms as the adjoint (octet) irreducible representation of $SU(3)$,

$$\epsilon\mathcal{H}^8+\epsilon'\mathcal{H}^3. \tag{68}$$

The first term in Eq. (68) is considered the dominant $SU(3)$ breaking and transforms as the eighth component of a flavor octet, where $\epsilon\sim m_s/\Lambda_{\rm QCD}$ represents a (dimensionless) measure of SB; $\epsilon\sim 0.3$, which is comparable to an $1/N_c$ effect. The second term represents the leading QCD isospin breaking effect, i.e., the one associated with the difference of the up and down quark masses and transforms as the third component of a flavor octet, where $\epsilon'\sim(m_d-m_u)/\Lambda_{\rm QCD}$; therefore, $\epsilon'\ll\epsilon$. This isospin breaking mechanism is referred to as strong isospin breaking.

The second source of symmetry breaking is induced by electromagnetic interactions. Second-order electromagnetic mass splittings in the quark charge matrix can obtain a suppression factor of $\epsilon''\sim\alpha_{\rm em}/4\pi$. To a good approximation,

$$\frac{m_d-m_u}{\Lambda_{\rm QCD}}\sim\frac{\alpha_{\rm em}}{4\pi}. \tag{69}$$

In this section, effects caused by first-order SB and IB to the scattering amplitude are discussed by extending the projection operator technique applied to the diagrams displayed in Figs. 1(a,b) and 1(c) separately as they involve different operator structures. These effects are added to the lowest-order results $\mathcal{A}_{\rm LO}$ to obtain more accurate expressions. Loop graphs that complement the analysis will be attempted elsewhere in the framework of large-N_c chiral perturbation theory.

A. Flavor projection operators for the product of three adjoints

First-order flavor symmetry breaking contributions to the scattering amplitude are computed from the tensor product of the scattering amplitude itself, which transforms under the spin-flavor symmetry $SU(2)\times SU(3)$ as $(2,8\otimes 8)$, and the perturbation, which transforms as $(0,8)$. The tensor product of three adjoint representations $8\otimes 8\otimes 8$ decomposes as

$$8\otimes 8\otimes 8=2(1)\oplus 8(8)\oplus 4(10\oplus\overline{10})\oplus 6(27)\oplus 2(35\oplus\overline{35})\oplus 64. \tag{70}$$

Thus, the effects of SB can be evaluated by constructing the $1/N_c$ expansions of the pieces of the scattering amplitude transforming as $(2,1)$, $(2,8)$, $(2,10\oplus\overline{10})$, $(2,27)$, $(2,35\oplus\overline{35})$, and $(2,64)$ under $SU(2)\times SU(3)$. These $1/N_c$ expansions need to be expressed in terms of a complete basis of linearly independent operators $\{R_k^{(ij)(a_1a_2a_3)}\}$, where a generic operator $R_k^{(ij)(a_1a_2a_3)}$ represents a spin-2 object with three adjoint indices. For $N_c=3$, up to three-body operators should be retained in the series. Accordingly, first-order SB can be accounted for by setting one of the flavor indices to 8, *v.gr.*, $a_3=8$, whereas first-order strong IB can be accounted for by setting one of the flavor indices to 3, *v.gr.*, $a_3=3$. For completeness, the set of up to three-body operators used as a basis is listed in the Online Resource. The set contains 170 linearly independent operators, where $R^{(ij)(a_1a_28)}$ and $R^{(ij)(a_1a_23)}$ represent operators with $I=0$ and $I=1$, respectively. Naively, isospin breaking induced by electromagnetism should appear from operators with $I=2$ and $I=3$, which emerge from the tensor product of four and five adjoint presentations, respectively. These tensor products are not treated here.

The task of constructing operators that yield SB effects is facilitated by the implementation of the projection operator technique presented in Ref. [22], extended to the decomposition (Eq. (70)). The projection operators can be constructed as

$$\left[\mathcal{P}^{(m)}\right]^{c_1c_2c_3b_1b_2b_3} = \left[\left(\frac{C-c^{n_1}\mathcal{I}}{c^m-c^{n_1}}\right)\left(\frac{C-c^{n_2}\mathcal{I}}{c^m-c^{n_2}}\right)\left(\frac{C-c^{n_3}\mathcal{I}}{c^m-c^{n_3}}\right)\times\left(\frac{C-c^{n_4}\mathcal{I}}{c^m-c^{n_4}}\right)\left(\frac{C-c^{n_5}\mathcal{I}}{c^m-c^{n_5}}\right)\right]^{c_1c_2c_3b_1b_2b_3}, \tag{71}$$

where m indicates the flavor representation of each projector and the n_i indicates flavor representations besides m. The quadratic Casimir operator reads

$$[C]^{c_1c_2c_3b_1b_2b_3} = 6\delta^{c_1b_1}\delta^{c_2b_2}\delta^{c_3b_3} - 2\delta^{c_1b_1}f^{ac_2b_2}f^{ac_3b_3} - 2\delta^{c_2b_2}f^{ac_1b_1}f^{ac_3b_3} - 2\delta^{c_3b_3}f^{ac_1b_1}f^{ac_2b_2}, \tag{72}$$

and

$$c^1=0,\ c^8=3,\ c^{10+\overline{10}}=6,\ c^{27}=8,\ c^{35+\overline{35}}=12,\ c^{64}=15, \tag{73}$$

are its corresponding eigenvalues.

Therefore, the product $\left[\mathcal{P}^{(\dim)}R_k^{(ij)}\right]^{c_1c_2c_3}$ effectively provides the component of the operator $R_k^{(ij)(c_1c_2c_3)}$ transforming in the irreducible representation of dimension dim according to decomposition (Eq. (70)).

However, the explicit analytic construction of $\left[\mathcal{P}^{(\dim)}\right]^{c_1c_2c_3b_1b_2b_3}$ faces several algebraic challenges. The most evident one is dealing with the products of up to ten f symbols contained in the C^5 operator, which cannot be reduced in terms containing fewer f or d symbols. Thus, the algebraic forms of $\left[\mathcal{P}^{(\dim)}\right]^{c_1c_2c_3b_1b_2b_3}$ contain hundreds of terms, which, in practice, become unmanageable. A more pragmatic approach such as the matrix method should be adopted to solve this problem.

To start with, each projection operator (or quadratic Casimir operator) is an object with six adjoint indices, each one with eight possible values, and therefore, all these objects have 8^6 elements. However, Casimir operators have all or half of their indices contracted, and the projectors are applied on three-body operators with three adjoint indices; therefore, half of the projector indices are always contracted. Thus, it is possible to collect the first three indices (c_1,c_2,c_3) and last three indices (b_1,b_2,b_3) of both the Casimir and projectors in only two indices, one for each set. These new indices have $8^3=512$ values. In this way, a matrix representation for the projectors can be constructed. They comprise 512×512 matrices. Similarly, the three-body operators with three adjoint indices can be represented as vectors with 512 entries. Therefore, instead of performing the index contractions $\left[\mathcal{P}^{(\dim)}R_k^{(ij)}\right]^{c_1c_2c_3}$, the problem reduces to ordinary matrix multiplications. The whole procedure is very reliable and effectively simplifies the analysis.

Let $\mathsf{P}^{(\dim)}$ represent the matrix corresponding to the projection operator $\left[\mathcal{P}^{(\dim)}\right]^{c_1c_2c_3b_1b_2b_3}$. With the method implemented, a series of consistency checks have been performed, namely,

$$\mathsf{P}^{(m)}\mathsf{P}^{(m)}=\mathsf{P}^{(m)}, \qquad \mathsf{P}^{(m)}\mathsf{P}^{(n)}=0, \quad n\neq m, \tag{74}$$

along with

$$\mathsf{P}^{(1)}+\mathsf{P}^{(8)}+\mathsf{P}^{(10+\overline{10})}+\mathsf{P}^{(27)}+\mathsf{P}^{(35+\overline{35})}+\mathsf{P}^{(64)}=\mathsf{I}_{512}, \tag{75}$$

where I_{512} represents the identity matrix of order 512. The above relations are the usual properties that projection operators must satisfy. No further details on the method are presented here.

The matrix method to construct projection operators can be extended to the tensor products of four and five adjoint representations. In the first case,

$$8\otimes8\otimes8\otimes8 = 8(1)\oplus32(8)\oplus33(27)\oplus12(64)\oplus125\oplus20(10\oplus\overline{10})\oplus2(28\oplus\overline{28})\oplus15(35\oplus\overline{35})\oplus3(81\oplus\overline{81}). \tag{76}$$

This decomposition (Eq. (76)) contains operators with four flavor indices, two of which can be fixed to $\{8,8\}$, $\{3,8\}$, and $\{3,3\}$, which will help identify operators with $I=0$, $I=1$, and $I=2$, respectively. Numerically, the procedure to construct projections operators would be rather involved, requiring a considerable amount of computing time; however, this procedure can still be performed.

1. *Flavor SB effects on the scattering amplitude from Fig. 1(a,b)*

The mechanism of flavor projection operators can be better understood through a few examples. The operator $\{T^a,\{T^b,T^c\}\}$ contributes to the scattering amplitude of the process $n+\pi^+\to n+\pi^+$ through components with flavor indices $a=(1-\mathrm{i}2)/\sqrt{2}$, $b=(1-\mathrm{i}2)/\sqrt{2}$, and $c=8$. Using the matrix method, the $\{1,1,8\}$ component of the flavor 8 piece becomes

$$
\begin{aligned}
[\mathcal{P}^{(8)}]^{118cde}\{T^c,\{T^d,T^e\}\} = {} & \frac{1}{15}T^1T^1T^8 + \frac{1}{30\sqrt{3}}T^1T^4T^6 + \frac{1}{30\sqrt{3}}T^1T^5T^7 + \frac{1}{30\sqrt{3}}T^1T^6T^4 + \frac{1}{30\sqrt{3}}T^1T^7T^5 + \frac{4}{15}T^1T^8T^1 \\
& + \frac{1}{15}T^2T^2T^8 - \frac{1}{30\sqrt{3}}T^2T^4T^7 + \frac{1}{30\sqrt{3}}T^2T^5T^6 + \frac{1}{30\sqrt{3}}T^2T^6T^5 - \frac{1}{30\sqrt{3}}T^2T^7T^4 + \frac{4}{15}T^2T^8T^2 \\
& + \frac{1}{15}T^3T^3T^8 + \frac{1}{30\sqrt{3}}T^3T^4T^4 + \frac{1}{30\sqrt{3}}T^3T^5T^5 - \frac{1}{30\sqrt{3}}T^3T^6T^6 - \frac{1}{30\sqrt{3}}T^3T^7T^7 + \frac{4}{15}T^3T^8T^3 \\
& - \frac{1}{15\sqrt{3}}T^4T^1T^6 + \frac{1}{15\sqrt{3}}T^4T^2T^7 - \frac{1}{15\sqrt{3}}T^4T^3T^4 + \frac{1}{30\sqrt{3}}T^4T^4T^3 + \frac{1}{10}T^4T^4T^8 + \frac{1}{30\sqrt{3}}T^4T^6T^1 \\
& - \frac{1}{30\sqrt{3}}T^4T^7T^2 + \frac{1}{5}T^4T^8T^4 - \frac{1}{15\sqrt{3}}T^5T^1T^7 - \frac{1}{15\sqrt{3}}T^5T^2T^6 - \frac{1}{15\sqrt{3}}T^5T^3T^5 + \frac{1}{30\sqrt{3}}T^5T^5T^3 \\
& + \frac{1}{10}T^5T^5T^8 + \frac{1}{30\sqrt{3}}T^5T^6T^2 + \frac{1}{30\sqrt{3}}T^5T^7T^1 + \frac{1}{5}T^5T^8T^5 - \frac{1}{15\sqrt{3}}T^6T^1T^4 - \frac{1}{15\sqrt{3}}T^6T^2T^5 \\
& + \frac{1}{15\sqrt{3}}T^6T^3T^6 + \frac{1}{30\sqrt{3}}T^6T^4T^1 + \frac{1}{30\sqrt{3}}T^6T^5T^2 - \frac{1}{30\sqrt{3}}T^6T^6T^3 + \frac{1}{10}T^6T^6T^8 + \frac{1}{5}T^6T^8T^6 \\
& - \frac{1}{15\sqrt{3}}T^7T^1T^5 + \frac{1}{15\sqrt{3}}T^7T^2T^4 + \frac{1}{15\sqrt{3}}T^7T^3T^7 - \frac{1}{30\sqrt{3}}T^7T^4T^2 + \frac{1}{30\sqrt{3}}T^7T^5T^1 \\
& - \frac{1}{30\sqrt{3}}T^7T^7T^3 + \frac{1}{10}T^7T^7T^8 + \frac{1}{5}T^7T^8T^7 + \frac{1}{15}T^8T^1T^1 + \frac{1}{15}T^8T^2T^2 + \frac{1}{15}T^8T^3T^3 + \frac{1}{10}T^8T^4T^4 \\
& + \frac{1}{10}T^8T^5T^5 + \frac{1}{10}T^8T^6T^6 + \frac{1}{10}T^8T^7T^7 + \frac{2}{5}T^8T^8T^8.
\end{aligned} \tag{77}
$$

Similar expressions to Eq. (77) can be found for the {2,2,8}, {1,2,8}, and {2,1,8} components required in the example. Therefore, it can be shown that

$$
\begin{aligned}
&[\mathcal{P}^{(1)} + \mathcal{P}^{(8)} + \mathcal{P}^{(10+\overline{10})} + \mathcal{P}^{(27)} + \mathcal{P}^{(35+\overline{35})} \\
&\quad + \mathcal{P}^{(64)}]^{118cde}\{T^c,\{T^d,T^e\}\} = \{T^1,\{T^1,T^8\}\},
\end{aligned} \tag{78}
$$

which is the expected result. Computing the matrix elements of the operator (Eq. (77)) is straightforward; therefore,

$$
\langle \pi^+ n|[\mathcal{P}^{(8)}]^{118cde}\{T^c,\{T^d,T^e\}\}|\pi^+ n\rangle = \frac{1}{2}\sqrt{3}, \tag{79}
$$

and

$$
\langle \pi^+ n|[\mathcal{P}^{(r)}]^{118cde}\{T^c,\{T^d,T^e\}\}|\pi^+ n\rangle = 0, \tag{80}
$$

for $r \neq 8$.

The procedure can be repeated for each flavor combination so that the different contributions of the operator $\{T^a,\{T^b,T^c\}\}$ to the scattering amplitude of the process $n+\pi^+ \to n+\pi^+$ can be made available. For the canonical example, the final expression can be summarized as

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}k^i k'^j \delta^{ij}\langle \pi^+ n|[\mathcal{P}^{(8)}]^{(1-i2)(1-i2)8cde}\{T^c,\{T^d,T^e\}\}|\pi^+ n\rangle \\
&= \frac{1}{2}\sqrt{3}\mathbf{k}\cdot\mathbf{k}',
\end{aligned} \tag{81}
$$

and

$$
\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}k^i k'^j \delta^{ij}\left[\langle \pi^+ n|[P^{(r)}]^{(1-i2)(1-i2)8cde}\right]\{T^c,\{T^d,T^e\}\}|\pi^+ n\rangle = 0 \tag{82}
$$

for $r \neq 8$.

Gathering partial results, the first-order SB to the scattering amplitude $\mathcal{A}_{\rm LO}$ Eq. (19) (denoted hereafter by $\delta\mathcal{A}_{\rm SB}$ and for which $I = 0$) can be organized as

$$
\begin{aligned}
& f^2 k^0 \delta\mathcal{A}_{\rm SB}(B+\pi^a \to B'+\pi^b) \\
&= \sum_{\rm dim}\Big[N_c g_1^{(\rm dim)} k^i k'^j \langle \pi^b B'|[\mathcal{P}^{(\rm dim)} R_1^{(ij)}]^{(ab8)}|\pi^a B\rangle \\
&\quad + N_c g_2^{(\rm dim)} k^i k'^j \langle \pi^b B'|[\mathcal{P}^{(\rm dim)} R_2^{(ij)}]^{(ab8)}|\pi^a B\rangle \\
&\quad + \sum_{r=3}^{16} g_r^{(\rm dim)} k^i k'^j \langle \pi^b B'|[\mathcal{P}^{(\rm dim)} R_r^{(ij)}]^{(ab8)}|\pi^a B\rangle
\end{aligned}
$$

$$+\frac{1}{N_c}\sum_{r=17}^{71} g_r^{(\mathrm{dim})} k^i k'^j \langle \pi^b B' | [\mathcal{P}^{(\mathrm{dim})} R_r^{(ij)}]^{(ab8)} | \pi^a B\rangle$$
$$+\frac{1}{N_c^2}\sum_{r=72}^{170} g_r^{(\mathrm{dim})} k^i k'^j \langle \pi^b B' | [\mathcal{P}^{(\mathrm{dim})} R_r^{(ij)}]^{(ab8)} | \pi^a B\rangle \Big], \tag{83}$$

where $g_r^{(\mathrm{dim})}$, $r = 1,\ldots,170$, are undetermined coefficients expected to be of order one. The sum over dim covers all six irreducible representations indicated in the relation (Eq. (70)), and the sums over i and j are implicit.

For example, the flavor 1 piece of $\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^+ \to n+\pi^+)$ using the corresponding matrix elements of the operators listed in the Online Resource becomes

$$\begin{aligned}2\sqrt{3}f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^+\to n+\pi^+) = {} & \Big[6g_2^{(1)}+\frac{1}{3}g_{18}^{(1)}+\frac{1}{2}g_{20}^{(1)}+\frac{1}{2}g_{52}^{(1)}+\frac{1}{2}g_{53}^{(1)}+\frac{1}{2}g_{54}^{(1)}+\frac{1}{18}g_{95}^{(1)}+\frac{1}{18}g_{96}^{(1)}+\frac{1}{18}g_{97}^{(1)}+\frac{1}{18}g_{98}^{(1)}\\ & +\frac{1}{18}g_{99}^{(1)}+\frac{1}{18}g_{100}^{(1)}+\frac{1}{3}g_{110}^{(1)}+\frac{1}{3}g_{111}^{(1)}+\frac{1}{3}g_{112}^{(1)}+\frac{1}{9}g_{116}^{(1)}+\frac{1}{9}g_{117}^{(1)}+\frac{1}{9}g_{118}^{(1)}+\frac{1}{9}g_{119}^{(1)}\\ & +\frac{1}{9}g_{120}^{(1)}+\frac{1}{9}g_{121}^{(1)}+\frac{1}{6}g_{134}^{(1)}+\frac{1}{6}g_{135}^{(1)}+\frac{1}{6}g_{136}^{(1)}+\frac{1}{6}g_{137}^{(1)}+\frac{1}{6}g_{138}^{(1)}+\frac{1}{6}g_{139}^{(1)}+\frac{1}{6}g_{140}^{(1)}+\frac{1}{6}g_{141}^{(1)}\\ & +\frac{1}{6}g_{142}^{(1)}+\frac{1}{6}g_{143}^{(1)}+\frac{1}{6}g_{144}^{(1)}+\frac{1}{6}g_{145}^{(1)}-\frac{1}{18}g_{146}^{(1)}-\frac{1}{18}g_{147}^{(1)}-\frac{1}{18}g_{148}^{(1)}-\frac{1}{18}g_{149}^{(1)}\Big]\mathbf{k}\cdot\mathbf{k}'\\ & +\Big[g_4^{(1)}+\frac{1}{6}g_{63}^{(1)}+\frac{1}{6}g_{64}^{(1)}+\frac{1}{6}g_{65}^{(1)}+\frac{1}{6}g_{66}^{(1)}+\frac{1}{6}g_{67}^{(1)}+\frac{1}{6}g_{68}^{(1)}+\frac{1}{6}g_{73}^{(1)}+\frac{1}{6}g_{80}^{(1)}+\frac{1}{6}g_{81}^{(1)}+\frac{1}{6}g_{82}^{(1)}\\ & +\frac{1}{12}g_{122}^{(1)}+\frac{1}{12}g_{123}^{(1)}+\frac{1}{12}g_{124}^{(1)}+\frac{1}{8}g_{125}^{(1)}+\frac{1}{8}g_{126}^{(1)}+\frac{1}{8}g_{127}^{(1)}-\frac{1}{24}g_{128}^{(1)}-\frac{1}{24}g_{129}^{(1)}-\frac{1}{24}g_{130}^{(1)}\\ & -\frac{1}{16}g_{150}^{(1)}-\frac{1}{16}g_{151}^{(1)}-\frac{1}{16}g_{152}^{(1)}-\frac{1}{16}g_{153}^{(1)}-\frac{1}{16}g_{154}^{(1)}-\frac{1}{16}g_{155}^{(1)}-\frac{1}{16}g_{156}^{(1)}-\frac{1}{16}g_{157}^{(1)}-\frac{1}{16}g_{158}^{(1)}\\ & -\frac{1}{16}g_{159}^{(1)}-\frac{1}{16}g_{160}^{(1)}-\frac{1}{16}g_{161}^{(1)}-\frac{1}{16}g_{162}^{(1)}-\frac{1}{16}g_{163}^{(1)}-\frac{1}{16}g_{164}^{(1)}+\frac{1}{16}g_{165}^{(1)}+\frac{1}{16}g_{166}^{(1)}+\frac{1}{16}g_{167}^{(1)}\\ & +\frac{1}{16}g_{168}^{(1)}+\frac{1}{16}g_{169}^{(1)}+\frac{1}{16}g_{170}^{(1)}\Big]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3.\end{aligned} \tag{84}$$

However, the applicability of expressions such as Eq. (84) is hindered by several disadvantages. The obvious one is the impossibility of determining all free parameters. For the $N\pi\to N\pi$ process, simpler expressions are obtained by defining effective coefficients expressed in terms of linear combinations of the $g_r^{(\mathrm{dim})}$ ones. In view of this, Eq. (84) can be written as

$$f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^+\to n+\pi^+) = d_1^{(1)}\mathbf{k}\cdot\mathbf{k}' + e_1^{(1)}\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3. \tag{85}$$

where the $d_1^{(1)}$ and $e_1^{(1)}$ coefficients are easily read off using Eq. (84).

Thus, the final expressions obtained for first-order SB effects to the scattering amplitudes for the $N+\pi\to N+\pi$ process are given by

$$\begin{aligned}& f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(p+\pi^+\to p+\pi^+)\\ &= (d_1^{(1)}+d_1^{(8)}+d_1^{(10+\overline{10})}+d_1^{(27)})\mathbf{k}\cdot\mathbf{k}'\\ &\quad +(e_1^{(1)}+e_1^{(8)}+e_1^{(10+\overline{10})}+e_1^{(27)})\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3\\ &= f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^-\to n+\pi^-),\end{aligned} \tag{86}$$

$$\begin{aligned}& f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(p+\pi^-\to p+\pi^-)\\ &= (d_1^{(1)}+d_1^{(8)}-d_1^{(10+\overline{10})}-d_1^{(27)}+d_2^{(8)}+d_2^{(27)})\mathbf{k}\cdot\mathbf{k}'\\ &\quad +(e_1^{(1)}+e_1^{(8)}-e_1^{(10+\overline{10})}+e_2^{(8)})\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3\\ &= f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^+\to n+\pi^+),\end{aligned} \tag{87}$$

$$\begin{aligned}& f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(p+\pi^0\to p+\pi^0)\\ &= \frac{1}{2}(2d_1^{(1)}+2d_1^{(8)}+d_2^{(8)}+d_2^{(27)})\mathbf{k}\cdot\mathbf{k}'\\ &\quad +\frac{1}{2}(2e_1^{(1)}+2e_1^{(8)}+e_1^{(27)}+e_2^{(8)})\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3\\ &= f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^0\to n+\pi^0),\end{aligned} \tag{88}$$

$$\begin{aligned}& \sqrt{2}f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(p+\pi^-\to n+\pi^0)\\ &= (2d_1^{(10+\overline{10})}+2d_1^{(27)}-d_2^{(8)}-d_2^{(27)})\mathbf{k}\cdot\mathbf{k}'\\ &\quad +(2e_1^{(10+\overline{10})}+e_1^{(27)}-e_2^{(8)})\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3\\ &= \sqrt{2}f^2k^0\delta\mathcal{A}_{\mathrm{SB}}(n+\pi^+\to p+\pi^0).\end{aligned} \tag{89}$$

Expressions (86)−(89) are written in terms of 11 unknown parameters that contain implicit suppression factors in N_c; thus, they are expected to be $O(N_c^0)$, $O(N_c^{-1})$, and $O(N_c^{-2})$ for coefficients coming from 1, 8 and $10+\overline{10}$, and 27 representations, respectively. Neither the flavor $35+\overline{35}$ nor the flavor 64 representation participates in the final expressions.

The isospin relations (Eqs. (46)−(51)) are satisfied by corrections to scattering amplitudes (Eqs. (86)−(89)), which is a completely expected result.

Furthermore,

$$f^2k^0\delta\mathcal{A}_{\rm SB}^{(3/2)} = (d_1^{(1)}+d_1^{(8)}+d_1^{(10+\overline{10})}+d_1^{(27)})\mathbf{k}\cdot\mathbf{k}' + (e_1^{(1)}+e_1^{(8)}+e_1^{(10+\overline{10})}+e_1^{(27)}){\rm i}(\mathbf{k}\times\mathbf{k}')_3, \tag{90}$$

and

$$f^2k^0\delta\mathcal{A}_{\rm SB}^{(1/2)} = \left[d_1^{(1)}+d_1^{(8)}-2d_1^{(10+\overline{10})}-2d_1^{(27)}+\frac{3}{2}d_2^{(8)}+\frac{3}{2}d_2^{(27)}\right]\mathbf{k}\cdot\mathbf{k}' + \left[e_1^{(1)}+e_1^{(8)}-2e_1^{(10+\overline{10})}-\frac{1}{2}e_1^{(27)}+\frac{3}{2}d_2^{(8)}\right]{\rm i}(\mathbf{k}\times\mathbf{k}')_3. \tag{91}$$

2. *SB effects to the scattering amplitude from Fig. 1(c)*

The SB effects to the scattering amplitude from Fig. 1(c) are obtained following the lines of the previous section. In this case, $A_{\rm vertex}^{ab}$ in Eq. (20) is a spin-zero object and contains two adjoint indices. A straightforward way to obtain the spin-0 operators with three adjoint indices to account for SB is forming tensor products of $R_k^{(ij)(abcd)}$ listed in the Online Resource with δ^{ij} to saturate spin indices. With this procedure, out of the 170 original operators, only 59 remain. The corresponding operator basis $\{V^{abc}\}$ is also listed in the Online Resource. However, after repeating the computation of the action of flavor projectors on these 59 operators, computing matrix elements, and gathering together partial results, only one unknown parameter is required to parametrize SB effects from Fig. 1(c). The final forms of the amplitudes read

$$\delta\mathcal{A}_{\rm vertex}(p+\pi^+\to p+\pi^+) = -\frac{1}{4}\frac{k^0}{f^2}h_1 = \delta\mathcal{A}_{\rm vertex}(n+\pi^-\to n+\pi^-), \tag{92}$$

$$\delta\mathcal{A}_{\rm vertex}(p+\pi^-\to p+\pi^-) = -\frac{1}{4}\frac{k^0}{f^2}h_1 = \delta\mathcal{A}_{\rm vertex}(n+\pi^+\to n+\pi^+), \tag{93}$$

$$\delta\mathcal{A}_{\rm vertex}(p+\pi^0\to p+\pi^0) = -\frac{1}{4}\frac{k^0}{f^2}h_1 = \delta\mathcal{A}_{\rm vertex}(n+\pi^0\to n+\pi^0), \tag{94}$$

$$\delta\mathcal{A}_{\rm vertex}(p+\pi^-\to n+\pi^0) = 0 = \delta\mathcal{A}_{\rm vertex}(n+\pi^+\to p+\pi^0), \tag{95}$$

where h_1 represents a new unknown parameter, which is a linear combination of 1, 8, and 27 operator coefficients. Note that $\mathcal{A}_{\rm vertex}(p+\pi^0\to p+\pi^0)$ and $\mathcal{A}_{\rm vertex}(n+\pi^0\to n+\pi^0)$ no longer vanish because of SB, whereas $\mathcal{A}_{\rm vertex}(p+\pi^-\to n+\pi^0)$ and $\mathcal{A}_{\rm vertex}(n+\pi^+\to p+\pi^0)$ remain unchanged. Further, the isospin relations (Eqs. (52)−(55)) are unaffected by SB effects, as expected.

Similarly,

$$\delta\mathcal{A}_{\rm SB,vertex}^{(3/2)} = \frac{1}{4}\frac{k^0}{f^2}h_1, \tag{96}$$

and

$$\delta\mathcal{A}_{\rm SB,vertex}^{(1/2)} = -\frac{1}{2}\frac{k^0}{f^2}h_1. \tag{97}$$

3. *Strong isospin breaking to the scattering amplitude from Fig. 1(a,b)*

The evaluation of IB corrections to the scattering amplitudes, hereafter denoted by $\delta\mathcal{A}_{\rm IB}$, can be performed in a manner similar to that for the flavor SB described in the previous sections, except that the free flavor index is now fixed to 3. The corresponding $1/N_c$ expansion for which $I=1$ reads

$$\begin{aligned} &f^2k^0\delta\mathcal{A}_{\rm IB}(B+\pi^a\to B'+\pi^b) = \\ &\sum_{\rm dim}\Big[N_c s_1^{(\rm dim)}k^ik'^j\langle\pi^bB'|[\mathcal{P}^{(\rm dim)}R_1^{(ij)}]^{(ab3)}|\pi^aB\rangle \\ &+N_c s_2^{(\rm dim)}k^ik'^j\langle\pi^bB'|[\mathcal{P}^{(\rm dim)}R_2^{(ij)}]^{(ab3)}|\pi^aB\rangle \\ &+\sum_{r=3}^{16}s_r^{(\rm dim)}k^ik'^j\langle\pi^bB'|[\mathcal{P}^{(\rm dim)}R_r^{(ij)}]^{(ab3)}|\pi^aB\rangle \\ &+\frac{1}{N_c}\sum_{r=17}^{71}s_r^{(\rm dim)}k^ik'^j\langle\pi^bB'|[\mathcal{P}^{(\rm dim)}R_r^{(ij)}]^{(ab3)}|\pi^aB\rangle \\ &+\frac{1}{N_c^2}\sum_{r=72}^{170}s_r^{(\rm dim)}k^ik'^j\langle\pi^bB'|[\mathcal{P}^{(\rm dim)}R_r^{(ij)}]^{(ab3)}|\pi^aB\rangle\Big], \end{aligned} \tag{98}$$

where $s_r^{(\rm dim)}$, $r=1,\ldots,170$, are undetermined coefficients, which are expected to be of order one. The sum over dim covers all six irreducible representations indicated in the relation (Eq. (70)), and the sums over i and j are implicit.

The matrix elements of Eq. (98) can be straightforwardly obtained following the lines of the previous sections. This enables one to obtain violations to isospin relations (46)−(51) as

$$
\begin{aligned}
&f^2k^0\Big[\delta\mathcal{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-\delta\mathcal{A}_{\rm IB}(p+\pi^0\to p+\pi^0)+\frac{1}{\sqrt{2}}\delta\mathcal{A}_{\rm IB}(p+\pi^-\to n+\pi^0)\Big]\\
=&\Big[\Big[-N_cw_1^{(1)}-\frac{1}{N_c}w_3^{(1)}-\frac{1}{N_c}w_4^{(1)}\Big]+\Big[\frac{3}{4}w_1^{(8)}-w_2^{(8)}+w_3^{(8)}+\frac{3}{4N_c}w_{19}^{(8)}-\frac{1}{N_c}w_{20}^{(8)}+\frac{1}{N_c}w_{21}^{(8)}+\frac{1}{2N_c}w_{22}^{(8)}+\frac{1}{2N_c}w_{23}^{(8)}+\frac{3}{4N_c}w_{24}^{(8)}\\
&-\frac{1}{N_c}w_{25}^{(8)}+\frac{3}{4N_c}w_{26}^{(8)}-\frac{1}{N_c}w_{27}^{(8)}+\frac{1}{N_c}w_{28}^{(8)}-\frac{1}{N_c}w_{29}^{(8)}+\frac{3}{4N_c}w_{30}^{(8)}-\frac{1}{N_c}w_{31}^{(8)}+\frac{1}{N_c}w_{32}^{(8)}\Big]+\Big[-\frac{1}{N_c}w_1^{(27)}+\frac{3}{4N_c}w_2^{(27)}-\frac{1}{N_c}w_3^{(27)}\\
&+\frac{1}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[-w_2^{(1)}-\frac{1}{N_c}w_4^{(1)}\Big]+\Big[\frac{3}{4}w_4^{(8)}-w_5^{(8)}+w_6^{(8)}+\frac{1}{2}w_7^{(8)}+\frac{1}{2}w_8^{(8)}+\frac{3}{4}w_9^{(8)}-w_{10}^{(8)}+\frac{3}{4N_c}w_{11}^{(8)}-\frac{1}{N_c}w_{12}^{(8)}\\
&+\frac{1}{N_c}w_{13}^{(8)}+\frac{1}{2N_c}w_{14}^{(8)}+\frac{1}{2N_c}w_{15}^{(8)}+\frac{3}{4N_c}w_{16}^{(8)}-\frac{1}{N_c}w_{17}^{(8)}-\frac{1}{N_c}w_{18}^{(8)}-\frac{1}{N_c}w_{33}^{(8)}+\frac{3}{4N_c}w_{34}^{(8)}+\frac{1}{N_c}w_{35}^{(8)}-\frac{1}{N_c}w_{36}^{(8)}\Big]\\
&+\Big[\frac{3}{4N_c}w_1^{(10+\overline{10})}-\frac{1}{N_c}w_2^{(10+\overline{10})}+\frac{1}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[-\frac{1}{N_c}w_5^{(27)}+\frac{3}{4N_c}w_6^{(27)}+\frac{1}{N_c}w_7^{(27)}-\frac{1}{N_c}w_8^{(27)}+\frac{3}{4N_c}w_9^{(27)}\\
&+\frac{1}{N_c}w_{10}^{(27)}\Big]\Big]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3,+O\Big[\frac{1}{N_c^2}\Big]
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
&f^2k^0\Big[\delta\mathcal{A}_{\rm IB}(p+\pi^+\to p+\pi^+)-\delta\mathcal{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-\sqrt{2}\delta\mathcal{A}_{\rm IB}(p+\pi^-\to n+\pi^0)\Big]\\
=&\Big[\Big[2N_cw_1^{(1)}+\frac{2}{N_c}w_3^{(1)}+\frac{2}{N_c}w_4^{(1)}\Big]+\Big[-4w_3^{(8)}-\frac{4}{N_c}w_{21}^{(8)}-\frac{1}{N_c}w_{22}^{(8)}+\frac{1}{N_c}w_{23}^{(8)}+\frac{2}{N_c}w_{25}^{(8)}-\frac{4}{N_c}w_{28}^{(8)}+\frac{2}{N_c}w_{29}^{(8)}-\frac{4}{N_c}w_{32}^{(8)}\Big]\\
&+\Big[\frac{2}{N_c}w_1^{(27)}-\frac{4}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[2w_2^{(1)}+\frac{2}{N_c}w_4^{(1)}\Big]+\Big[-4w_6^{(8)}-w_7^{(8)}+w_8^{(8)}+2w_{10}^{(8)}-\frac{4}{N_c}w_{13}^{(8)}-\frac{1}{N_c}w_{14}^{(8)}\\
&+\frac{1}{N_c}w_{15}^{(8)}+\frac{2}{N_c}w_{17}^{(8)}+\frac{2}{N_c}w_{18}^{(8)}+\frac{2}{N_c}w_{33}^{(8)}+\frac{4}{N_c}w_{36}^{(8)}\Big]+\Big[-\frac{4}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[\frac{2}{N_c}w_5^{(27)}+\frac{4}{N_c}w_8^{(27)}\Big]\Big]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\Big[\frac{1}{N_c^2}\Big],
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
&f^2k^0\Big[\delta\mathcal{A}_{\rm IB}(p+\pi^+\to p+\pi^+)+\delta\mathcal{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-2\delta\mathcal{A}_{\rm IB}(p+\pi^0\to p+\pi^0)\Big]\\
=&\Big[\Big[\frac{3}{2}w_1^{(8)}-2w_2^{(8)}-2w_3^{(8)}+\frac{3}{2N_c}w_{19}^{(8)}-\frac{2}{N_c}w_{20}^{(8)}-\frac{2}{N_c}w_{21}^{(8)}+\frac{2}{N_c}w_{23}^{(8)}+\frac{3}{2N_c}w_{24}^{(8)}+\frac{3}{2N_c}w_{26}^{(8)}-\frac{2}{N_c}w_{27}^{(8)}-\frac{2}{N_c}w_{28}^{(8)}+\frac{3}{2N_c}w_{30}^{(8)}\\
&-\frac{2}{N_c}w_{31}^{(8)}-\frac{2}{N_c}w_{32}^{(8)}\Big]+\Big[\frac{3}{2N_c}w_2^{(27)}-\frac{2}{N_c}w_3^{(27)}-\frac{2}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[\frac{3}{2}w_4^{(8)}-2w_5^{(8)}-2w_6^{(8)}+2w_8^{(8)}+\frac{3}{2}w_9^{(8)}+\frac{3}{2N_c}w_{11}^{(8)}\\
&-\frac{2}{N_c}w_{12}^{(8)}-\frac{2}{N_c}w_{13}^{(8)}+\frac{2}{N_c}w_{15}^{(8)}+\frac{3}{2N_c}w_{16}^{(8)}+\frac{3}{2N_c}w_{34}^{(8)}+\frac{2}{N_c}w_{35}^{(8)}+\frac{2}{N_c}w_{36}^{(8)}\Big]+\Big[\frac{3}{2N_c}w_1^{(10+\overline{10})}-\frac{2}{N_c}w_2^{(10+\overline{10})}\\
&-\frac{2}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[\frac{3}{2N_c}w_6^{(27)}+\frac{2}{N_c}w_7^{(27)}+\frac{2}{N_c}w_8^{(27)}+\frac{3}{2N_c}w_9^{(27)}+\frac{2}{N_c}w_{10}^{(27)}\Big]\Big]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\Big[\frac{1}{N_c^2}\Big],
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
&f^2k^0\Big[\delta\mathcal{A}_{\rm IB}(n+\pi^-\to n+\pi^-)-\delta\mathcal{A}_{\rm IB}(n+\pi^0\to n+\pi^0)-\frac{1}{\sqrt{2}}\delta\mathcal{A}_{\rm IB}(n+\pi^+\to p+\pi^0)\Big]\\
=&\Big[\Big[-N_cw_1^{(1)}-\frac{1}{N_c}w_3^{(1)}-\frac{1}{N_c}w_4^{(1)}\Big]+\Big[-\frac{3}{4}w_1^{(8)}+w_2^{(8)}-w_3^{(8)}-\frac{3}{4N_c}w_{19}^{(8)}+\frac{1}{N_c}w_{20}^{(8)}-\frac{1}{N_c}w_{21}^{(8)}-\frac{1}{2N_c}w_{22}^{(8)}-\frac{1}{2N_c}w_{23}^{(8)}-\frac{3}{4N_c}w_{24}^{(8)}\\
&-\frac{1}{N_c}w_{25}^{(8)}-\frac{3}{4N_c}w_{26}^{(8)}+\frac{1}{N_c}w_{27}^{(8)}-\frac{1}{N_c}w_{28}^{(8)}-\frac{1}{N_c}w_{29}^{(8)}-\frac{3}{4N_c}w_{30}^{(8)}+\frac{1}{N_c}w_{31}^{(8)}-\frac{1}{N_c}w_{32}^{(8)}\Big]+\Big[-\frac{1}{N_c}w_1^{(27)}-\frac{3}{4N_c}w_2^{(27)}+\frac{1}{N_c}w_3^{(27)}\\
&-\frac{1}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[-w_2^{(1)}-\frac{1}{N_c}w_4^{(1)}\Big]+\Big[-\frac{3}{4}w_4^{(8)}+w_5^{(8)}-w_6^{(8)}-\frac{1}{2}w_7^{(8)}-\frac{1}{2}w_8^{(8)}-\frac{3}{4}w_9^{(8)}-w_{10}^{(8)}-\frac{3}{4N_c}w_{11}^{(8)}+\frac{1}{N_c}w_{12}^{(8)}\\
&-\frac{1}{N_c}w_{13}^{(8)}-\frac{1}{2N_c}w_{14}^{(8)}-\frac{1}{2N_c}w_{15}^{(8)}-\frac{3}{4N_c}w_{16}^{(8)}-\frac{1}{N_c}w_{17}^{(8)}-\frac{1}{N_c}w_{18}^{(8)}-\frac{1}{N_c}w_{33}^{(8)}-\frac{3}{4N_c}w_{34}^{(8)}-\frac{1}{N_c}w_{35}^{(8)}+\frac{1}{N_c}w_{36}^{(8)}\Big]\\
&+\Big[-\frac{3}{4N_c}w_1^{(10+\overline{10})}+\frac{1}{N_c}w_2^{(10+\overline{10})}-\frac{1}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[-\frac{1}{N_c}w_5^{(27)}-\frac{3}{4N_c}w_6^{(27)}-\frac{1}{N_c}w_7^{(27)}+\frac{1}{N_c}w_8^{(27)}-\frac{3}{4N_c}w_9^{(27)}\\
&-\frac{1}{N_c}w_{10}^{(27)}\Big]\Big]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\Big[\frac{1}{N_c^2}\Big],
\end{aligned}
\tag{102}
$$

$$\begin{aligned}
&f^2k^0\left[\delta\mathcal{A}_{\rm IB}(n+\pi^+\to n+\pi^+)-\delta\mathcal{A}_{\rm IB}(n+\pi^-\to n+\pi^-)+\sqrt{2}\delta\mathcal{A}_{\rm IB}(n+\pi^+\to p+\pi^0)\right]\\
&=\Big[\Big[2N_cw_1^{(1)}+\frac{2}{N_c}w_3^{(1)}+\frac{2}{N_c}w_4^{(1)}\Big]+\Big[4w_3^{(8)}+\frac{4}{N_c}w_{21}^{(8)}+\frac{1}{N_c}w_{22}^{(8)}-\frac{1}{N_c}w_{23}^{(8)}+\frac{2}{N_c}w_{25}^{(8)}+\frac{4}{N_c}w_{28}^{(8)}+\frac{2}{N_c}w_{29}^{(8)}+\frac{4}{N_c}w_{32}^{(8)}\Big]\\
&+\Big[\frac{2}{N_c}w_1^{(27)}+\frac{4}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[2w_2^{(1)}+\frac{2}{N_c}w_4^{(1)}\Big]+\Big[4w_6^{(8)}+w_7^{(8)}-w_8^{(8)}+2w_{10}^{(8)}+\frac{4}{N_c}w_{13}^{(8)}+\frac{1}{N_c}w_{14}^{(8)}-\frac{1}{N_c}w_{15}^{(8)}+\frac{2}{N_c}w_{17}^{(8)}\\
&+\frac{2}{N_c}w_{18}^{(8)}+\frac{2}{N_c}w_{33}^{(8)}-\frac{4}{N_c}w_{36}^{(8)}\Big]+\Big[\frac{4}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[\frac{2}{N_c}w_5^{(27)}-\frac{4}{N_c}w_8^{(27)}\Big]\Big]{\rm i}(\mathbf{k}\times\mathbf{k}')_3+\mathcal{O}\left[\frac{1}{N_c^2}\right],
\end{aligned}\tag{103}$$

$$\begin{aligned}
&f^2k^0\left[\delta\mathcal{A}_{\rm IB}(n+\pi^+\to n+\pi^+)+\delta\mathcal{A}_{\rm IB}(n+\pi^-\to n+\pi^-)-2\delta\mathcal{A}_{\rm IB}(n+\pi^0n\to\pi^0)\right.\\
&=\Big[\Big[-\frac{3}{2}w_1^{(8)}+2w_2^{(8)}+2w_3^{(8)}-\frac{3}{2N_c}w_{19}^{(8)}+\frac{2}{N_c}w_{20}^{(8)}+\frac{2}{N_c}w_{21}^{(8)}-\frac{2}{N_c}w_{23}^{(8)}-\frac{3}{2N_c}w_{24}^{(8)}-\frac{3}{2N_c}w_{26}^{(8)}+\frac{2}{N_c}w_{27}^{(8)}+\frac{2}{N_c}w_{28}^{(8)}-\frac{3}{2N_c}w_{30}^{(8)}\\
&+\frac{2}{N_c}w_{31}^{(8)}+\frac{2}{N_c}w_{32}^{(8)}\Big]+\Big[-\frac{3}{2N_c}w_2^{(27)}+\frac{2}{N_c}w_3^{(27)}+\frac{2}{N_c}w_4^{(27)}\Big]\Big]\mathbf{k}\cdot\mathbf{k}'+\Big[\Big[-\frac{3}{2}w_4^{(8)}+2w_5^{(8)}+2w_6^{(8)}-2w_8^{(8)}-\frac{3}{2}w_9^{(8)}-\frac{3}{2N_c}w_{11}^{(8)}\\
&+\frac{2}{N_c}w_{12}^{(8)}+\frac{2}{N_c}w_{13}^{(8)}-\frac{2}{N_c}w_{15}^{(8)}-\frac{3}{2N_c}w_{16}^{(8)}-\frac{3}{2N_c}w_{34}^{(8)}-\frac{2}{N_c}w_{35}^{(8)}-\frac{2}{N_c}w_{36}^{(8)}\Big]+\Big[-\frac{3}{2N_c}w_1^{(10+\overline{10})}+\frac{2}{N_c}w_2^{(10+\overline{10})}+\frac{2}{N_c}w_3^{(10+\overline{10})}\Big]\\
&+\Big[-\frac{3}{2N_c}w_6^{(27)}-\frac{2}{N_c}w_7^{(27)}-\frac{2}{N_c}w_8^{(27)}-\frac{3}{2N_c}w_9^{(27)}-\frac{2}{N_c}w_{10}^{(27)}\Big]\Big]{\rm i}(\mathbf{k}\times\mathbf{k}')_3+\mathcal{O}\left[\frac{1}{N_c^2}\right].
\end{aligned}\tag{104}$$

The effective coefficients $w_m^{(\rm dim)}$ can be written in terms of the original ones as

$$w_1^{(1)}=-2s_1^{(1)},\tag{105}$$

$$w_2^{(1)}=s_3^{(1)},\tag{106}$$

$$w_3^{(1)}=-s_{17}^{(1)}-\frac{3}{2}s_{19}^{(1)},\tag{107}$$

$$w_4^{(1)}=-\frac{3}{2}s_{51}^{(1)},\tag{108}$$

$$w_1^{(8)}=s_5^{(8)},\tag{109}$$

$$w_2^{(8)}=\frac{1}{4}s_6^{(8)},\tag{110}$$

$$w_3^{(8)}=\frac{1}{4}s_7^{(8)},\tag{111}$$

$$w_4^{(8)}=\frac{5}{6}s_8^{(8)},\tag{112}$$

$$w_5^{(8)}=\frac{5}{24}s_9^{(8)},\tag{113}$$

$$w_6^{(8)}=\frac{5}{24}s_{10}^{(8)},\tag{114}$$

$$w_7^{(8)}=\frac{5}{6}s_{11}^{(8)},\tag{115}$$

$$w_8^{(8)}=-\frac{5}{6}s_{12}^{(8)},\tag{116}$$

$$w_9^{(8)}=\frac{5}{18}s_{13}^{(8)},\tag{117}$$

$$w_{10}^{(8)}=\frac{1}{6}(s_{14}^{(8)}-s_{15}^{(8)}),\tag{118}$$

$$w_{11}^{(8)}=\frac{1}{2}s_{21}^{(8)},\tag{119}$$

$$w_{12}^{(8)}=\frac{1}{8}s_{22}^{(8)},\tag{120}$$

$$w_{13}^{(8)}=\frac{1}{8}s_{23}^{(8)},\tag{121}$$

$$w_{14}^{(8)}=\frac{1}{2}s_{24}^{(8)},\tag{122}$$

$$w_{15}^{(8)}=-\frac{1}{2}s_{25}^{(8)},\tag{123}$$

$$w_{16}^{(8)} = \frac{1}{6}s_{26}^{(8)}, \tag{124}$$

$$w_{17}^{(8)} = \frac{1}{2}(s_{27}^{(8)} - s_{28}^{(8)}), \tag{125}$$

$$w_{18}^{(8)} = \frac{1}{2}s_{29}^{(8)}, \tag{126}$$

$$w_{19}^{(8)} = \frac{5}{6}(s_{30}^{(8)} + s_{33}^{(8)}), \tag{127}$$

$$w_{20}^{(8)} = \frac{5}{24}(s_{31}^{(8)} + s_{34}^{(8)}), \tag{128}$$

$$w_{21}^{(8)} = \frac{5}{24}(s_{32}^{(8)} + s_{35}^{(8)}), \tag{129}$$

$$w_{22}^{(8)} = \frac{5}{6}(s_{36}^{(8)} + s_{38}^{(8)}), \tag{130}$$

$$w_{23}^{(8)} = -\frac{5}{6}(s_{37}^{(8)} + s_{39}^{(8)}), \tag{131}$$

$$w_{24}^{(8)} = \frac{5}{18}(s_{40}^{(8)} + s_{41}^{(8)}), \tag{132}$$

$$w_{25}^{(8)} = \frac{1}{6}(-s_{42}^{(8)} + s_{43}^{(8)} - s_{44}^{(8)} - s_{45}^{(8)} - s_{46}^{(8)} - s_{47}^{(8)}), \tag{133}$$

$$w_{26}^{(8)} = 3s_{48}^{(8)}, \tag{134}$$

$$w_{27}^{(8)} = \frac{3}{4}s_{49}^{(8)}, \tag{135}$$

$$w_{28}^{(8)} = \frac{3}{4}s_{50}^{(8)}, \tag{136}$$

$$w_{29}^{(8)} = \frac{3}{5}s_{51}^{(8)}, \tag{137}$$

$$w_{30}^{(8)} = \frac{3}{5}s_{52}^{(8)}, \tag{138}$$

$$w_{31}^{(8)} = \frac{3}{20}s_{53}^{(8)}, \tag{139}$$

$$w_{32}^{(8)} = \frac{3}{20}s_{54}^{(8)}, \tag{140}$$

$$w_{33}^{(8)} = -\frac{3}{10}(s_{57}^{(8)} - s_{58}^{(8)} + s_{59}^{(8)} + s_{60}^{(8)} - s_{61}^{(8)} + s_{62}^{(8)}), \tag{141}$$

$$w_{34}^{(8)} = \frac{11}{30}(s_{63}^{(8)} + s_{66}^{(8)}), \tag{142}$$

$$w_{35}^{(8)} = -\frac{11}{120}(s_{64}^{(8)} + s_{67}^{(8)}), \tag{143}$$

$$w_{36}^{(8)} = -\frac{11}{120}(s_{65}^{(8)} + s_{68}^{(8)}), \tag{144}$$

$$w_{1}^{(10+\overline{10})} = -\frac{1}{3}s_{66}^{(10+\overline{10})}, \tag{145}$$

$$w_{2}^{(10+\overline{10})} = -\frac{1}{12}s_{67}^{(10+\overline{10})}, \tag{146}$$

$$w_{3}^{(10+\overline{10})} = -\frac{1}{12}(s_{65}^{(10+\overline{10})} - s_{68}^{(10+\overline{10})}), \tag{147}$$

$$w_{1}^{(27)} = -\frac{1}{10}s_{51}^{(27)}, \tag{148}$$

$$w_{2}^{(27)} = \frac{2}{5}s_{52}^{(27)}, \tag{149}$$

$$w_{3}^{(27)} = \frac{1}{10}s_{53}^{(27)}, \tag{150}$$

$$w_{4}^{(27)} = \frac{1}{10}s_{54}^{(27)}, \tag{151}$$

$$w_{5}^{(27)} = \frac{1}{30}(-s_{57}^{(27)} + s_{58}^{(27)} - s_{59}^{(27)} - s_{60}^{(27)} + s_{61}^{(27)} - s_{62}^{(27)}), \tag{152}$$

$$w_{6}^{(27)} = \frac{2}{15}s_{63}^{(27)}, \tag{153}$$

$$w_{7}^{(27)} = -\frac{1}{30}s_{64}^{(27)}, \tag{154}$$

$$w_{8}^{(27)} = -\frac{1}{30}(s_{65}^{(27)} + s_{68}^{(27)}), \tag{155}$$

$$w_{9}^{(27)} = \frac{2}{15}s_{66}^{(27)}, \tag{156}$$

$$w_{10}^{(27)} = -\frac{1}{30}s_{67}^{(27)}. \tag{157}$$

Unfortunately, unlike flavor SB corrections, strong IB corrections cannot be further simplified in terms of fewer effective operator coefficients. The reason is considerably simple. The operator basis $\{R^{(ij)(abc)}\}$ is constituted by 170 linearly independent operators so that they all contribute to the $1/N_c$ expansion (Eq. (98)). This is not a single rule to eliminate some of them. In Eqs. (99)−(104), the explicit dependence on N_c that comes along the operators involved in the $1/N_c$ expansion have been kept. Those expressions are evaluated at $N_c=3$, which is a useful artifact to identify leading and subleading terms in them. Recall that $f^2\sim O(N_c)$, and therefore, the unitarity of the scattering amplitudes is not compromised.

Therefore, the usefulness of relations (Eqs. (99)−(104)) can be better appreciated by retaining leading and subleading terms in N_c. Specifically, Eqs. (101) and (104) obtain leading corrections from the 8 representation, whereas $10+\overline{10}$ and 27 representations are $1/N_c$ suppressed. The relevant operators for the symmetric part are $\delta^{ij}\delta^{ab}T^3$, $\delta^{ij}\delta^{a3}T^b$, and $\delta^{ij}\delta^{b3}T^a$, whereas for the antisymmetric part, the relevant operators are $\mathrm{i}\epsilon^{ijm}\delta^{ab}G^{m3}$, $\mathrm{i}\epsilon^{ijm}\delta^{a3}G^{mb}$, $\mathrm{i}\epsilon^{ijm}\delta^{b3}G^{ma}$, $\mathrm{i}\epsilon^{ijm}f^{a3e}f^{beg}G^{mg}$, and $\mathrm{i}\epsilon^{ijm}d^{abe}d^{3eg}G^{mg}$.

Equations (99), (100), (102), and (103) obtain important corrections from the singlet and octet representations, whereas $10+\overline{10}$ and 27 representations obtain $1/N_c$-suppressed factors.

In addition,

$$
\begin{aligned}
&f^2k^0\left[\delta\mathcal{A}_{\rm IB}(p+\pi^0\to p+\pi^0)-\delta\mathcal{A}_{\rm IB}(n+\pi^0\to n+\pi^0)\right]\\
&=\Bigg[\Big[\frac{1}{2}w_1^{(8)}+2w_2^{(8)}+2w_3^{(8)}+\frac{1}{2N_c}w_{19}^{(8)}+\frac{2}{N_c}w_{20}^{(8)}+\frac{2}{N_c}w_{21}^{(8)}+\frac{1}{2N_c}w_{24}^{(8)}+\frac{1}{2N_c}w_{26}^{(8)}+\frac{2}{N_c}w_{27}^{(8)}+\frac{2}{N_c}w_{28}^{(8)}+\frac{1}{2N_c}w_{30}^{(8)}+\frac{2}{N_c}w_{31}^{(8)}+\frac{2}{N_c}w_{32}^{(8)}\Big]\\
&\quad+\Big[\frac{1}{2N_c}w_2^{(27)}+\frac{2}{N_c}w_3^{(27)}+\frac{2}{N_c}w_4^{(27)}\Big]\Bigg]\mathbf{k}\cdot\mathbf{k}'+\Bigg[\Big[\frac{1}{2}w_4^{(8)}+2w_5^{(8)}+2w_6^{(8)}+\frac{1}{2}w_9^{(8)}+\frac{1}{2N_c}w_{11}^{(8)}+\frac{2}{N_c}w_{12}^{(8)}+\frac{2}{N_c}w_{13}^{(8)}+\frac{1}{2N_c}w_{16}^{(8)}+\frac{1}{2N_c}w_{34}^{(8)}\\
&\quad-\frac{2}{N_c}w_{35}^{(8)}-\frac{2}{N_c}w_{36}^{(8)}\Big]+\Big[\frac{1}{2N_c}w_1^{(10+\overline{10})}+\frac{2}{N_c}w_2^{(10+\overline{10})}+\frac{2}{N_c}w_3^{(10+\overline{10})}\Big]+\Big[\frac{1}{2N_c}w_6^{(27)}-\frac{2}{N_c}w_7^{(27)}-\frac{2}{N_c}w_8^{(27)}+\frac{1}{2N_c}w_9^{(27)}\\
&\quad-\frac{2}{N_c}w_{10}^{(27)}\Big]\Bigg]\mathrm{i}(\mathbf{k}\times\mathbf{k}')_3+O\left[\frac{1}{N_c^2}\right].
\end{aligned}
\tag{158}
$$

In this case, there is a type of octet dominance because the $10+\overline{10}$ representation starts contributing at order $O(1/N_c^2)$ and the 27 representation is at least one factor of $1/N_c$ suppressed relative to the octet representation.

4. *Strong isospin breaking to the scattering amplitude from Fig. 1(c)*

Strong IB corrections emerging from Fig. 1(c) can be cast into

$$
\begin{aligned}
&f^2k^0[\delta\mathrm{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-\delta\mathrm{A}_{\rm IB}(p+\pi^0\to p+\pi^0)+\frac{1}{\sqrt{2}}\delta\mathrm{A}_{\rm IB}(p+\pi^-\to n+\pi^0)]\\
&=\Big[2N_cv_1^{(1)}+\frac{3}{2N_c}v_6^{(1)}\Big]+\Big[\frac{3}{4}v_3^{(8)}-\frac{1}{4}v_4^{(8)}+\frac{1}{4}v_5^{(8)}+\frac{15}{8N_c}v_8^{(8)}-\frac{5}{8N_c}v_9^{(8)}+\frac{5}{8N_c}v_{10}^{(8)}+\frac{5}{4N_c}v_{11}^{(8)}-\frac{5}{4N_c}v_{12}^{(8)}\\
&\quad+\frac{5}{8N_c}v_{13}^{(8)}+\frac{1}{2N_c}v_{14}^{(8)}-\frac{1}{2N_c}v_{15}^{(8)}+\frac{1}{2N_c}v_{16}^{(8)}\Big]+O\left[\frac{1}{N_c^2}\right],
\end{aligned}
\tag{159}
$$

$$
\begin{aligned}
&f^2k^0[\delta\mathrm{A}_{\rm IB}(p+\pi^+\to p+\pi^+)-\delta\mathrm{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-\sqrt{2}\delta\mathrm{A}_{\rm IB}(p+\pi^-\to n+\pi^0)]\\
&=\Big[-4N_cv_1^{(1)}-\frac{3}{N_c}v_6^{(1)}\Big]+\Big[-v_5^{(8)}-\frac{5}{2N_c}v_{10}^{(8)}-\frac{5}{2N_c}v_{11}^{(8)}-\frac{5}{2N_c}v_{12}^{(8)}-\frac{1}{N_c}v_{14}^{(8)}+\frac{1}{N_c}v_{15}^{(8)}-\frac{1}{N_c}v_{16}^{(8)}\Big]+O\left[\frac{1}{N_c^2}\right],
\end{aligned}
\tag{160}
$$

$$
\begin{aligned}
&f^2k^0[\delta\mathrm{A}_{\rm IB}(p+\pi^+\to p+\pi^+)+\delta\mathrm{A}_{\rm IB}(p+\pi^-\to p+\pi^-)-2\delta\mathrm{A}_{\rm IB}(p+\pi^0\to p+\pi^0)]\\
&=\frac{3}{2}v_3^{(8)}-\frac{1}{2}v_4^{(8)}-\frac{1}{2}v_5^{(8)}+\frac{15}{4N_c}v_8^{(8)}-\frac{5}{4N_c}v_9^{(8)}-\frac{5}{4N_c}v_{10}^{(8)}-\frac{5}{N_c}v_{12}^{(8)}+\frac{5}{4N_c}v_{13}^{(8)}+O\left[\frac{1}{N_c^2}\right],
\end{aligned}
\tag{161}
$$

$$f^2k^0[\delta A_{IB}(n+\pi^-\to n+\pi^-)-\delta A_{IB}(n+\pi^0\to n+\pi^0)-\frac{1}{\sqrt{2}}\delta A_{IB}(n+\pi^+\to p+\pi^0)]$$
$$=\left[2N_cv_1^{(1)}+\frac{3}{2N_c}v_6^{(1)}\right]+\left[\frac{5}{4}v_3^{(8)}+\frac{1}{4}v_4^{(8)}-\frac{1}{4}v_5^{(8)}+\frac{25}{8N_c}v_8^{(8)}+\frac{5}{8N_c}v_9^{(8)}-\frac{5}{8N_c}v_{10}^{(8)}-\frac{5}{4N_c}v_{11}^{(8)}-\frac{15}{4N_c}v_{12}^{(8)}+\frac{25}{24N_c}v_{13}^{(8)}+\frac{1}{2N_c}v_{14}^{(8)}-\frac{1}{2N_c}v_{15}^{(8)}+\frac{1}{2N_c}v_{16}^{(8)}\right]+O\left[\frac{1}{N_c^2}\right], \tag{162}$$

$$f^2k^0[\delta A_{IB}(n+\pi^+\to n+\pi^+)-\delta A_{IB}(n+\pi^-\to n+\pi^-)+\sqrt{2}\delta A_{IB}(n+\pi^+\to p+\pi^0)]$$
$$=\left[-4N_cv_1^{(1)}-\frac{3}{N_c}v_6^{(1)}\right]+\left[-2v_3^{(8)}+v_5^{(8)}-\frac{5}{N_c}v_8^{(8)}+\frac{5}{2N_c}v_{10}^{(8)}+\frac{5}{2N_c}v_{11}^{(8)}+\frac{15}{2N_c}v_{12}^{(8)}-\frac{5}{3N_c}v_{13}^{(8)}-\frac{1}{N_c}v_{14}^{(8)}+\frac{1}{N_c}v_{15}^{(8)}-\frac{1}{N_c}v_{16}^{(8)}\right]+O\left[\frac{1}{N_c^2}\right], \tag{163}$$

$$f^2k^0[\delta A_{IB}(n+\pi^+\to n+\pi^+)+\delta A_{IB}(n+\pi^-\to n+\pi^-)-2\delta A_{IB}(n+\pi^0\to n+\pi^0)]$$
$$=\frac{1}{2}v_3^{(8)}+\frac{1}{2}v_4^{(8)}+\frac{1}{2}v_5^{(8)}+\frac{5}{4N_c}v_8^{(8)}+\frac{5}{4N_c}v_9^{(8)}+\frac{5}{4N_c}v_{10}^{(8)}+\frac{5}{12N_c}v_{13}^{(8)}+O\left[\frac{1}{N_c^2}\right] \tag{164}$$

Equations (159)−(164) cannot be reduced further in terms of effective operator coefficients. In a manner similar to that in the previous section, Eqs. (161) and (164) are dominated by corrections from the octet representations, and numerically, they should be at least a factor of $1/N_c$ smaller than Eqs. (159), (160), (162), and (163), which are dominated by the singlet representation.

In addition, for the relation

$$f^2k^0[\delta A_{IB}(p+\pi^0\to p+\pi^0)-\delta A_{IB}(n+\pi^0\to n+\pi^0)]$$
$$=\frac{1}{2}v_3^{(8)}+\frac{1}{2}v_4^{(8)}+\frac{1}{2}v_5^{(8)}+\frac{5}{4N_c}v_8^{(8)}+\frac{5}{4N_c}v_9^{(8)}+\frac{5}{4N_c}v_{10}^{(8)}+\frac{5}{12N_c}v_{13}^{(8)}+O\left[\frac{1}{N_c^2}\right]. \tag{165}$$

a kind of octet dominance is found in the sense that flavor $10+\overline{10}$ and 27 representations start contributing at a relative order $O(1/N_c^2)$, so they can be safely ignored.

B. Some remarks about a comparison with HBChPT expressions

Scattering amplitudes for the $N\pi$ system obtained here through the use of $SU(3)$ flavor projection operators are (partially) compared with HBChPT theory results at the tree-level order. At this point, the three terms retained in Eq. (8) for $N_c=3$ can be completely evaluated. The success of $SU(2)$ HBChPT to investigate the low-energy processes of pions and nucleons is undeniable. However, the inclusion of particles with strangeness requires the use of $SU(3)$ HBChPT. For example, s-wave pseudoscalar meson octet-baryon scattering lengths to the third chiral order in that framework have been studied with only baryon octet contributions [32] and both baryon octet and decuplet contributions [33]. The latter reference decuplet contributions to the threshold T-matrices are found to vanish in complete opposition to the present analysis where non-vanishing decuplet baryon contributions proportional to C are obtained, even in the degeneracy limit $\Delta\to 0$. In a more recent work [34], the T-matrices of pseudoscalar meson octet-baryon scattering to one-loop order are computed in HBChPT. For elastic meson-baryon scattering, the leading order $O(q)$ amplitudes resulting from tree diagrams for πN scattering contributing at the first chiral order are given in Eq. (10) and (11) of that reference, which can be compared to Eqs. (44) and (56) as well as (45) and (57) of this work in the limit $\Delta\to 0$ and by excluding decuplet baryon contributions. In addition to the kinematic factors relating the rest system of the initial baryon and the center of mass system, which can be linked through a Lorentz transformation, the Clebsch-Gordan structures coincide up to a global minus sign that might be traced back to the different conventions used. Other scattering processes such as $\pi\Sigma$, $\pi\Xi$, and KN discussed in Ref. [34] can be evaluated in the present formalism. A recent analysis with the inclusion of decuplet effects [35] reveals some interesting facts in the comparison with the present analysis. Except for some kinematic factors, the comparison is achieved for Δ replaced by $-\Delta$ in Eqs. (13)−(16) of that reference.

At the next-to-leading order, the explicit chiral symmetry breaking part of the meson-baryon effective chiral Lagrangian $\mathcal{L}^{(2,\mathrm{ct})}$ with no inclusion of decuplet baryon effects is presented in Eq. (8) of Ref. [34]. It yields the amplitudes $T_{\pi N}^{(I)}$ in terms of 11 LECs. For the πN system, they are given in Eqs. (64) and (65) of that reference. In principle, these LECs should (partially) correspond to the 12 operator coefficients contained in Eqs. (90) and (96) as well as (91) and (97), respectively. Although relationships among them should be linear, it is difficult to identify them, except for the vertex diagram for which $C_3 = -k_0 h_1/8$ for the fixed incident meson energy. A full identification requires including decuplet baryons in the framework of that reference and the computation of additional amplitudes in the framework discussed here. The latter will be attempted elsewhere.

VI. *S*-WAVE SCATTERING LENGTHS

The $N\pi$ forward scattering amplitude for a nucleon at rest can be readily obtained from Eqs. (45) and (44) at the threshold. Following the lines of Ref. [13], the *s*-wave scattering lengths including the baryon mass splitting and first-order SB are found to be

$$\begin{aligned} a^{(1/2)} &= \frac{1}{4\pi}\frac{m_\pi}{f^2}\left[1+\frac{m_\pi}{M_N}\right]^{-1}\left[(D+F)^2-\frac{4}{9}\left[1+\frac{\Delta}{m_\pi}+\frac{\Delta^2}{m_\pi^2}\right]C^2\right. \\ &\quad \left. + d_1^{(1)} + d_1^{(8)} + d_1^{(10+\overline{10})} + d_1^{(27)}\right] \\ &= a^+ + 2a^-, \end{aligned} \tag{166}$$

and

$$\begin{aligned} a^{(3/2)} &= \frac{1}{4\pi}\frac{m_\pi}{f^2}\left[1+\frac{m_\pi}{M_N}\right]^{-1}\left[-\frac{1}{2}(D+F)^2+\frac{2}{9}\left[1-\frac{2\Delta}{m_\pi}+\frac{\Delta^2}{m_\pi^2}\right]C^2\right. \\ &\quad \left. + d_1^{(1)} + d_1^{(8)} - 2d_1^{(10+\overline{10})} - 2d_1^{(27)} + \frac{3}{2}d_2^{(8)} + \frac{3}{2}d_2^{(27)}\right] \\ &= a^+ - a^-, \end{aligned} \tag{167}$$

which are valid to order $O(\Delta^3/m_\pi^3)$.

In the limit $\Delta \to 0$ and by removing SB effects,

$$a^{(1/2)} + 2a^{(3/2)} = 0, \tag{168}$$

which is a well-known result obtained in the context of current algebra [30, 31]. Equation (168) is fulfilled even in the presence of the C^2 term, which accounts for the contribution of decuplet baryons. Thus, violations to Eq. (168) arise not only from SB but also from a linear term in Δ.

The usefulness of Eqs. (166) and (167) relies entirely on the precise determination of the $SU(3)$ invariants D, F, and C and the six parameters $d_k^{(\mathrm{dim})}$ involved in those equations. For instance, these invariants can be extracted from baryon semileptonic decays. The latter set can be obtained by comparing the theoretical expressions with the available experimental data [20] via a least-squares fit. A detailed analysis requires additional theoretical expressions for which data are available and would involve processes including strangeness.

Isospin IB effects obtained here can also be incorporated into Eqs. (166) and (167) in a straightforward manner.

VII. CONCLUDING REMARKS

The material discussed in this work represents an enterprising program to understand the baryon-meson scattering processes in the context of the $1/N_c$ expansion. It presents new ideas, perspectives, or analytical frameworks that contribute to a more comprehensive understanding of the subject matter. The scattering amplitude for the process $B\pi \to B'\pi$, including the decuplet-octet baryon mass splitting and flavor symmetry breaking, has been computed, specialized to the process $N\pi \to N\pi$. Evidently, processes such as $\Delta\pi \to N\pi$ and $\Delta\pi \to \Delta\pi$ or those including strangeness can be evaluated because the formalism is sufficiently general to cover the cases when B and B' are any baryon states and π^a and π^b are any pseudo scalar mesons provided that the Gell-Mann–Nishijima scheme is fulfilled. The expressions for $N\pi \to N\pi$ scattering amplitudes obtained here get simple forms [Eqs. (36)−(39) and Eqs. (52)−(55)] once all ingredients are put together regardless of the original expressions such as Eq. (19). However, the inclusion of strong isospin breaking introduces a rather large number of operator coefficients such that the series have minimal utility, unless stringent suppressions in $1/N_c$ are performed to achieve only leading contributions. Violations to strong isospin breaking uncovered by relations (99)−(104) reveal which $SU(3)$ flavor representations dominate over the others.

One important result extracted from the present analysis is worth mentioning:It is evident that the spin-1/2 and spin-3/2 baryons are present from the outset because they together form an irreducible representation of the spin-flavor symmetry.

As mentioned in the introductory section, previous analyses about scattering amplitudes in the context of the $1/N_c$ expansion [9−11] focused their goals on some specific aspects of the theory. The analysis presented here with the extensive use of projection operators to classify operator structures contributes to the subject from a different perspective; the approaches complement them.

A comparison of the results obtained here with the HBChPT results obtained at the tree-level order can be made. Rewriting scattering amplitudes in terms of the $SU(3)$ invariant baryon-meson couplings D, F, C, and $\mathcal{H}$, Eqs. (36)−(39) enable a comparison with the tree-level values [in the $SU(3)$ exact limit] from HBChPT by dropping the mass difference Δ and possibly the C^2 terms, *i.e.*, under the degeneracy limit and with the decuplet baryon degrees of freedom integrated out, which is usually the common procedure advocated in literature. A full comparison will require the computation of loops in the combined formalism in $1/N_c$ and chiral corrections. This requires a formidable effort that will be attempted elsewhere.

VIII. SUPPLEMENTARY INFORMATION

This paper is complemented by some supplementary material where explicit reductions of baryon operators and their corresponding matrix elements (as tables) are presented. The pdf file can be obtained from authors by request.

A. Baryon operator basis used in baryon-meson scattering

The operators $S_m^{(ij)(ab)}$ that constitute the basis used in baryon-meson scattering at the lowest order, comprising up to seven-body operators, read

$$
\begin{aligned}
&S_1^{(ij)(ab)} = \mathrm{i}\delta^{ij} f^{abe} T^e, \quad S_2^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\delta^{ab} J^r, \quad S_3^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr} d^{abe} G^{re}, \quad S_4^{(ij)(ab)} = \delta^{ab}\{J^i,J^j\}, \\
&S_5^{(ij)(ab)} = \delta^{ij}\delta^{ab} J^2, \quad S_6^{(ij)(ab)} = \{G^{ia},G^{jb}\}, \quad S_7^{(ij)(ab)} = \{G^{ib},G^{ja}\}, \quad S_8^{(ij)(ab)} = \delta^{ij}\{G^{ra},G^{rb}\}, \\
&S_9^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{G^{ra},T^b\}, \quad S_{10}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{G^{rb},T^a\}, \quad S_{11}^{(ij)(ab)} = d^{abe}\{J^j,G^{ie}\}, \quad S_{12}^{(ij)(ab)} = \mathrm{i} f^{abe}\{J^i,G^{je}\}, \\
&S_{13}^{(ij)(ab)} = \mathrm{i} f^{abe}\{J^j,G^{ie}\}, \quad S_{14}^{(ij)(ab)} = \delta^{ij} d^{abe}\{J^r,G^{re}\}, \quad S_{15}^{(ij)(ab)} = \epsilon^{ijr} f^{abe}\mathcal{D}_2^{re}, \quad S_{16}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr} d^{abe}\mathcal{D}_3^{re}, \\
&S_{17}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr} d^{abe}\mathcal{O}_3^{re}, \quad S_{18}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^r,\{T^a,T^b\}\}, \quad S_{19}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^m,\{G^{ra},G^{rb}\}\}, \quad S_{20}^{(ij)(ab)} = \mathrm{i} f^{abe}\{T^e,\{J^i,J^j\}\}, \\
&S_{21}^{(ij)(ab)} = \mathrm{i}\delta^{ij} f^{abe}\{J^2,T^e\}, \quad S_{22}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\delta^{ab}\{J^2,J^r\}, \quad S_{23}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{G^{ja},\{J^m,G^{rb}\}\}, \\
&S_{24}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{G^{ia},\{J^m,G^{rb}\}\}, \quad S_{25}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{G^{jb},\{J^m,G^{ra}\}\}, \quad S_{26}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{G^{ib},\{J^m,G^{ra}\}\}, \\
&S_{27}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{G^{ma},\{J^r,G^{rb}\}\}, \quad S_{28}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{G^{mb},\{J^r,G^{ra}\}\}, \quad S_{29}^{(ij)(ab)} = \mathrm{i} f^{aeg} d^{beh}\{T^h,\{J^i,G^{jg}\}\}, \\
&S_{30}^{(ij)(ab)} = \mathrm{i} d^{aeg} f^{beh}\{T^g,\{J^j,G^{ih}\}\}, \quad S_{31}^{(ij)(ab)} = d^{abe}[J^2,\{J^i,G^{je}\}], \quad S_{32}^{(ij)(ab)} = d^{abe}[J^2,\{J^j,G^{ie}\}], \\
&S_{33}^{(ij)(ab)} = \mathrm{i} f^{abe}[J^2,\{J^i,G^{je}\}], \quad S_{34}^{(ij)(ab)} = \mathrm{i} f^{abe}[J^2,\{J^j,G^{ie}\}], \quad S_{35}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}[J^2,\{G^{ra},T^b\}], \\
&S_{36}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}[J^2,\{G^{rb},T^a\}], \quad S_{37}^{(ij)(ab)} = \epsilon^{ijr} f^{abe}\mathcal{D}_4^{re}, \quad S_{38}^{(ij)(ab)} = \mathrm{i} f^{abe}\{\{J^i,J^j\},\{J^r,G^{re}\}\}, \\
&S_{39}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{\mathcal{D}_2^{mb},\{J^r,G^{ra}\}\}, \quad S_{40}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{\mathcal{D}_2^{ma},\{J^r,G^{rb}\}\}, \quad S_{41}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,\{G^{ra},T^b\}\}, \\
&S_{42}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,\{G^{rb},T^a\}\}, \quad S_{43}^{(ij)(ab)} = \mathrm{i} f^{abe}\{J^2,\{J^i,G^{je}\}\}, \quad S_{44}^{(ij)(ab)} = \mathrm{i} f^{abe}\{J^2,\{J^j,G^{ie}\}\}, \\
&S_{45}^{(ij)(ab)} = \{J^2,\{G^{ia},G^{jb}\}\}, \quad S_{46}^{(ij)(ab)} = \{J^2,\{G^{ib},G^{ja}\}\}, \quad S_{47}^{(ij)(ab)} = d^{abe}\{\{J^i,J^j\},\{J^r,G^{re}\}\}, \\
&S_{48}^{(ij)(ab)} = \delta^{ab}\{J^2,\{J^i,J^j\}\}, \quad S_{49}^{(ij)(ab)} = \epsilon^{ijk}\epsilon^{rml}\{J^k,\{G^{ra},\{J^m,G^{lb}\}\}\}, \quad S_{50}^{(ij)(ab)} = \mathrm{i}\epsilon^{iml}[\{J^j,\{J^m,G^{la}\}\},\{J^r,G^{rb}\}], \\
&S_{51}^{(ij)(ab)} = \mathrm{i}\epsilon^{jml}[\{J^i,\{J^m,G^{la}\}\},\{J^r,G^{rb}\}], \quad S_{52}^{(ij)(ab)} = \mathrm{i}\epsilon^{jml}[\{J^i,\{J^m,G^{lb}\}\},\{J^r,G^{ra}\}], \\
&S_{53}^{(ij)(ab)} = \mathrm{i}\epsilon^{iml}[\{J^j,\{J^m,G^{lb}\}\},\{J^r,G^{ra}\}], \quad S_{54}^{(ij)(ab)} = \{G^{ia},\mathcal{O}_3^{jb}\}, \quad S_{55}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}[J^2,\{G^{mb},\{J^r,G^{ra}\}\}], \\
&S_{56}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,[G^{mb},\{J^r,G^{ra}\}]\}, \quad S_{57}^{(ij)(ab)} = \delta^{ij}\{J^2,\{G^{ra},G^{rb}\}\}, \quad S_{58}^{(ij)(ab)} = \delta^{ij} d^{abe}\{J^2,\{J^r,G^{re}\}\}, \\
&S_{59}^{(ij)(ab)} = \delta^{ij}\delta^{ab}\{J^2,J^2\}, \quad S_{60}^{(ij)(ab)} = \{[J^2,G^{ia}],[J^2,G^{jb}]\}, \quad S_{61}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}[J^2,\{G^{ib},\{J^m,G^{ra}\}\}], \\
&S_{62}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr} d^{abe}\mathcal{D}_5^{re}, \quad S_{63}^{(ij)(ab)} = \epsilon^{ijr} f^{abe}\mathcal{O}_5^{re}, \quad S_{64}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr} d^{abe}\mathcal{O}_5^{re}, \\
&S_{65}^{(ij)(ab)} = \{\mathcal{O}_3^{ia},\mathcal{D}_2^{jb}\}, \quad S_{66}^{(ij)(ab)} = \{\mathcal{D}_2^{ia},\mathcal{O}_3^{jb}\}, \quad S_{67}^{(ij)(ab)} = \{J^2,\{T^a,\{J^j,G^{ib}\}\}\}, \\
&S_{68}^{(ij)(ab)} = \{J^2,\{T^b,\{J^i,G^{ja}\}\}\}, \quad S_{69}^{(ij)(ab)} = \mathrm{i} f^{abe}\{J^2,\{T^e,\{J^i,J^j\}\}\}, \quad S_{70}^{(ij)(ab)} = \mathrm{i}\delta^{ij} f^{abe}\{J^2,\{J^2,T^e\}\}, \\
&S_{71}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\delta^{ab}\{J^2,\{J^2,J^r\}\}, \quad S_{72}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{J^m,\{G^{ra},G^{rb}\}\}\}, \quad S_{73}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,\{G^{ja},\{J^m,G^{rb}\}\}\}, \\
&S_{74}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{J^2,\{G^{ia},\{J^m,G^{rb}\}\}\}, \quad S_{75}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,\{G^{jb},\{J^m,G^{ra}\}\}\}, \quad S_{76}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{J^2,\{G^{ib},\{J^m,G^{ra}\}\}\},
\end{aligned}
$$

$$S_{77}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{G^{ma},\{J^r,G^{rb}\}\}\}, \qquad S_{78}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{G^{mb},\{J^r,G^{ra}\}\}\},$$
$$S_{79}^{(ij)(ab)} = \mathrm{i}f^{aeg}d^{beh}\{J^2,\{T^h,\{J^i,G^{jg}\}\}\}, \qquad S_{80}^{(ij)(ab)} = \mathrm{i}d^{aeg}f^{beh}\{J^2,\{T^g,\{J^j,G^{ih}\}\}\},$$
$$S_{81}^{(ij)(ab)} = d^{abe}\{J^2,[J^2,\{J^i,G^{je}\}]\}, \qquad S_{82}^{(ij)(ab)} = d^{abe}\{J^2,[J^2,\{J^j,G^{ie}\}]\},$$
$$S_{83}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijl}\{J^l,\{\{J^r,G^{ra}\},\{J^m,G^{mb}\}\}\}, \qquad S_{84}^{(ij)(ab)} = \{\{J^i,J^j\},\{T^a,\{J^r,G^{rb}\}\}\},$$
$$S_{85}^{(ij)(ab)} = \{\{J^i,J^j\},\{T^b,\{J^r,G^{ra}\}\}\}, \qquad S_{86}^{(ij)(ab)} = \mathrm{i}\epsilon^{mlr}\{\{J^i,J^j\},\{G^{mb},\{J^l,G^{ra}\}\}\},$$
$$S_{87}^{(ij)(ab)} = \mathrm{i}\epsilon^{jlm}\{\{J^i,\{J^l,G^{ma}\}\},\{J^r,G^{rb}\}\}, \qquad S_{88}^{(ij)(ab)} = \mathrm{i}\epsilon^{ilm}\{\{J^j,\{J^l,G^{ma}\}\},\{J^r,G^{rb}\}\},$$
$$S_{89}^{(ij)(ab)} = \mathrm{i}\epsilon^{jlm}\{\{J^i,\{J^l,G^{mb}\}\},\{J^r,G^{ra}\}\}, \qquad S_{90}^{(ij)(ab)} = \mathrm{i}\epsilon^{ilm}\{\{J^j,\{J^l,G^{mb}\}\},\{J^r,G^{ra}\}\},$$
$$S_{91}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{G^{jb},\{J^2,\{J^m,G^{ra}\}\}\}, \qquad S_{92}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,[J^2,\{G^{ra},T^b\}]\},$$
$$S_{93}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,[J^2,\{G^{rb},T^a\}]\}, \qquad S_{94}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,[J^2,\{J^i,G^{je}\}]\},$$
$$S_{95}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,[J^2,\{J^j,G^{ie}\}]\}, \qquad S_{96}^{(ij)(ab)} = \epsilon^{ijr}f^{abe}\mathcal{D}_6^{re},$$
$$S_{97}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,\{\{J^i,J^j\},\{J^r,G^{re}\}\}\}, \qquad S_{98}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{\mathcal{D}_2^{mb},\{J^r,G^{ra}\}\}\},$$
$$S_{99}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{\mathcal{D}_2^{ma},\{J^r,G^{rb}\}\}\}, \qquad S_{100}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,\{J^2,\{G^{ra},T^b\}\}\},$$
$$S_{101}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\{J^2,\{J^2,\{G^{rb},T^a\}\}\}, \qquad S_{102}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,\{J^2,\{J^i,G^{je}\}\}\},$$
$$S_{103}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,\{J^2,\{J^j,G^{ie}\}\}\}, \qquad S_{104}^{(ij)(ab)} = \{J^2,\{J^2,\{G^{ia},G^{jb}\}\}\},$$
$$S_{105}^{(ij)(ab)} = \{J^2,\{J^2,\{G^{ib},G^{ja}\}\}\}, \qquad S_{106}^{(ij)(ab)} = d^{abe}\{J^2,\{\{J^i,J^j\},\{J^r,G^{re}\}\}\},$$
$$S_{107}^{(ij)(ab)} = \delta^{ab}\{J^2,\{J^2,\{J^i,J^j\}\}\}, \qquad S_{108}^{(ij)(ab)} = \mathrm{i}\epsilon^{jml}\{J^2,[\{J^i,\{J^m,G^{la}\}\},\{J^r,G^{rb}\}]\},$$
$$S_{109}^{(ij)(ab)} = \mathrm{i}\epsilon^{jml}\{J^2,[\{J^i,\{J^m,G^{lb}\}\},\{J^r,G^{ra}\}]\}, \qquad S_{110}^{(ij)(ab)} = \mathrm{i}\epsilon^{iml}\{J^2,[\{J^j,\{J^m,G^{lb}\}\},\{J^r,G^{ra}\}]\},$$
$$S_{111}^{(ij)(ab)} = \{J^2,\{G^{ia},O_3^{jb}\}\}, \qquad S_{112}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,[J^2,\{G^{mb},\{J^r,G^{ra}\}\}]\},$$
$$S_{113}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{J^2,[G^{mb},\{J^r,G^{ra}\}]\}\}, \qquad S_{114}^{(ij)(ab)} = \delta^{ij}\{J^2,\{J^2,\{G^{ra},G^{rb}\}\}\},$$
$$S_{115}^{(ij)(ab)} = \delta^{ij}d^{abe}\{J^2,\{J^2,\{J^r,G^{re}\}\}\}, \qquad S_{116}^{(ij)(ab)} = \delta^{ij}\delta^{ab}\{J^2,\{J^2,J^2\}\},$$
$$S_{117}^{(ij)(ab)} = \{J^2,\{[J^2,G^{ia}],[J^2,G^{jb}]\}\}, \qquad S_{118}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,[J^2,\{G^{jb},\{J^m,G^{ra}\}\}]\},$$
$$S_{119}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{J^2,[J^2,\{G^{ib},\{J^m,G^{ra}\}\}]\}, \qquad S_{120}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}d^{abe}\mathcal{D}_7^{re},$$
$$S_{121}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}d^{abe}O_7^{re}, \qquad S_{122}^{(ij)(ab)} = \mathrm{i}f^{abe}\{J^2,\{J^2,\{T^e,\{J^i,J^j\}\}\}\},$$
$$S_{123}^{(ij)(ab)} = \mathrm{i}\delta^{ij}f^{abe}\{J^2,\{J^2,\{J^2,T^e\}\}\}, \qquad S_{124}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijr}\delta^{ab}\{J^2,\{J^2,\{J^2,J^r\}\}\},$$
$$S_{125}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,\{J^2,\{G^{ja},\{J^m,G^{rb}\}\}\}\}, \qquad S_{126}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{J^2,\{J^2,\{G^{ia},\{J^m,G^{rb}\}\}\}\},$$

$$S_{127}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,\{J^2,\{G^{jb},\{J^m,G^{ra}\}\}\}\}, \qquad S_{128}^{(ij)(ab)} = \mathrm{i}\epsilon^{jmr}\{J^2,\{J^2,\{G^{ib},\{J^m,G^{ra}\}\}\}\},$$
$$S_{129}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{J^2,\{G^{ma},\{J^r,G^{rb}\}\}\}\}, \qquad S_{130}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijm}\{J^2,\{J^2,\{G^{mb},\{J^r,G^{ra}\}\}\}\},$$
$$S_{131}^{(ij)(ab)} = \mathrm{i}f^{aeg}d^{beh}\{J^2,\{J^2,\{T^h,\{J^i,G^{jg}\}\}\}\}, \qquad S_{132}^{(ij)(ab)} = \mathrm{i}d^{aeg}f^{beh}\{J^2,\{J^2,\{T^g,\{J^j,G^{ih}\}\}\}\},$$
$$S_{133}^{(ij)(ab)} = d^{abe}\{J^2,\{J^2,[J^2,\{J^i,G^{je}\}]\}\}, \qquad S_{134}^{(ij)(ab)} = d^{abe}\{J^2,\{J^2,[J^2,\{J^j,G^{ie}\}]\}\},$$
$$S_{135}^{(ij)(ab)} = \mathrm{i}\epsilon^{ijl}\{J^2,\{J^l,\{\{J^r,G^{ra}\},\{J^m,G^{mb}\}\}\}\}, \qquad S_{136}^{(ij)(ab)} = \mathrm{i}\epsilon^{mlr}\{J^2,\{\{J^i,J^j\},\{G^{mb},\{J^l,G^{ra}\}\}\}\},$$
$$S_{137}^{(ij)(ab)} = \mathrm{i}\epsilon^{jlm}\{J^2,\{\{J^i,\{J^l,G^{ma}\}\},\{J^r,G^{rb}\}\}\}, \qquad S_{138}^{(ij)(ab)} = \mathrm{i}\epsilon^{jlm}\{J^2,\{\{J^i,\{J^l,G^{mb}\}\},\{J^r,G^{ra}\}\}\},$$
$$S_{139}^{(ij)(ab)} = \mathrm{i}\epsilon^{imr}\{J^2,\{G^{jb},\{J^2,\{J^m,G^{ra}\}\}\}\}.$$

For completeness, the operator coefficients $c_m^{(\mathrm{s})}$ and $c_m^{(\mathrm{a})}$ that accompany these operators are listed in the Online Resource for this paper.

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