Gravitational losses for the binary systems induced by the next-to-leading spin-orbit coupling effects*

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Abstract: The orbital energy and momentum of the compact binary systems will loss due to gravitational radiation. Based on the mass and mass-current multipole moments of the binary system with the spin vector defined by Bohé et al. [Class. Quantum Grav. **30**, 075017 (2013)], we calculate the loss rates of energy, angular momentum and linear momentum induced by the next-to-leading spin-orbit effects. For the case of circular orbit, the formulations for these losses are given in terms of the orbital frequency.

Keywords: gravitational wave radiation, gravitational losses, spin-orbit coupling, post-Newtonian approximation

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I. INTRODUCTION

The compact binary systems such as black hole-black hole pairs, black hole-dense star (neutron star, white dwarf, or others) pairs, or double dense stars, are the best candidates for the gravitational-wave sources, which have been detected by LIGO/VIRGO [1−4], and also will be detected by the space-based gravitational-wave detectors such as LISA, TianQin, and Taiji in the near future [5−9].

Gravitational radiation will carry away the orbital energy, angular momentum, and linear momentum of the compact binary systems. The analytic calculations for the gravitational losses can only be achieved via the post-Newtonian (PN) approximations, which have been extensively studied in the literature. The references where the results were first presented are summarized in the following tables.

Table 1 presents the references for the loss rates of the non-spinning binary system's energy E, angular momentum $\boldsymbol{J}$, and linear momentum $\boldsymbol{P}$ to the different PN orders including the tail contributions.

For the case of the spinning binary systems, the situation is somewhat more complicated as the definitions of the spin vector and the supplementary spin condition (SSC) are not unique. Although they describe the same physical phenomena, the formulations of the gravitational loss rates are dependent on these definitions. Table 2 provides the reference for the formulations for the gravitational loss rates of the spinning binary system's energy, angular momentum, and linear momentum induced by the spin-orbit (SO) coupling based on the spin vector defined by Barker and O'Connell [23] under the SSC given by Pirani [24].

Table 3 presents the references for the formulations of the gravitational losses of the spinning binary system's energy, angular momentum, and linear momentum due to SO contributions and spin-spin (SS) contributions based on the spin vector defined by Faye, Blanchet and Buonanno [26] and under the SSC given by Tulczyjew [27]. Table 4 provides the references for the case of the spin vector defined by Bohé et al. [28].

In this work, we make use of the 2.5PN dynamic equation of the center-of-mass and the 1PN precession equations of the spin vector given by Bohé et al. [28], and the mass moment given by Marsat et al. [29], to calculate the gravitational loss of the spinning binary systems induced by the next-to-leading SO coupling effects.

The rest of this paper is organized as follows. Section 2 introduces the 2.5PN acceleration for the relative motion of the spinning binary systems, which will be used in later derivations. In Section 3 we give the formulas for calculating gravitational losses. Section 4 we derive the

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Table 1. References for the gravitational loss rates of the non-spinning binary systems in the Newtonian, 1PN, 2PN, 2.5PN, 3PN and 3.5PN approximation. The superscript "·" denotes the time derivative.

gravitational loss rates	circular orbit	general orbit
$\dot{E}_N$	Peters [10]	Peters [10]
$\dot{E}_{1PN}$	Will & Wagoner [11]	Will & Wagoner [11]
$\dot{E}_{2PN}$	Blanchet, Damour & Iyer [12]	Will & Wiseman [13]
$\dot{E}_{2.5tail}$	Blanchet [14]	Arun et al. [15]
$\dot{E}_{3PN}$	Blanchet, Iyer & Joguet [16]	Arun et al. [15]
$\dot{E}_{3.5tail}$	Blanchet [17]	N/A
$\dot{\boldsymbol{J}}_N$, $\dot{\boldsymbol{J}}_{1PN}$	Junker & Schäfer [18]	Junker & Schäfer [18]
$\dot{\boldsymbol{J}}_{2PN}$	Gopakumar & Iyer [19]	Gopakumar & Iyer [19]
$\dot{\boldsymbol{J}}_{2.5tail}$, $\dot{\boldsymbol{J}}_{3PN}$	Arun et al. [20]	Arun et al. [20]
$\dot{\boldsymbol{P}}_N$	Junker & Schäfer [18]	Junker & Schäfer [18]
$\dot{\boldsymbol{P}}_{1PN}$, $\dot{\boldsymbol{P}}_{2PN}$	Racine, Buonanno & Kidder [21]	Racine, Buonanno & Kidder [21]
$\dot{\boldsymbol{P}}_{2.5tail}$	Kastha [22]	Kastha [22]

Table 2. Reference for the gravitational loss rates of the spinning binary systems with the spin vector defined by Barker and O'Connell and under the Pirani's SSC

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{1.5SO}$, $\dot{\boldsymbol{J}}_{1.5SO}$, $\dot{\boldsymbol{P}}_{1.5SO}$	Kidder [25]	Kidder [25]

Table 3. References for the gravitational loss rates of the spinning binary system base on the spin vector defined by Faye, Blanchet and Buonanno and under the Tulczyjew's SSC

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{2.5SO}$	Blanchet, Buonanno & Faye [30]	Blanchet, Buonanno & Faye [30]
$\dot{\boldsymbol{J}}_{1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{J}}_{2.5SO}$	**This work**	**This work**
$\dot{\boldsymbol{P}}_{0.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{2SS}$, $\dot{\boldsymbol{J}}_{2SS}$, $\dot{\boldsymbol{P}}_{1.5SO}$, $\dot{\boldsymbol{P}}_{2SS}$	Racine, Buonanno & Kidder [21]	Racine, Buonanno & Kidder [21]

Table 4. References for the gravitational loss rates of the spinning binary systems based on the spin vector defined by Bohé et al. and under the Tulczyjew's SSC

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{2.5SO}$	Marsat et al. [29]	**This work**
$\dot{E}_{3SOtail}$, $\dot{E}_{3.5SO}$, $\dot{E}_{4SOtail}$	Marsat et al. [29]	N/A
$\dot{E}_{2SS}$, $\dot{E}_{3SS}$	Bohé et al. [31]	Bohé et al. [31]
$\dot{\boldsymbol{J}}_{1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{J}}_{2.5SO}$	**This work**	**This work**
$\dot{\boldsymbol{P}}_{0.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{P}}_{1.5SO}$	**This work**	**This work**

gravitational losses induced by the next-to-leading spin-orbit coupling effects. For completeness and as a comparison, we also present the 2.5PN angular momentum loss in terms of the spin vector defined by Faye, Blanchet and

Buonanno [26] in Appendix B. Section 5 gives these loss rates for the circular orbit. Summary is given in Section 6. In this article, small Greek alphabet represents 0, 1, 2, 3, and small letter represents 1, 2, 3.

2. THE MOTION FOR THE BINARY SYSTEM IN THE 2.5PN APPROXIMATION

We assume the spinning compact binary has masses M_1 and M_2. The position vectors of the bodies are $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, and the corresponding velocities are $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, the spins of the two bodies are $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. The precession equations of the spinning binary systems can be written as [28]

$$\frac{d\boldsymbol{S}}{dt}=\frac{GM}{c^2R^2}\eta\Big\{\boldsymbol{V}\Big[\frac{7}{2}(\boldsymbol{n}\cdot\boldsymbol{S})+\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]-\mathbf{n}\Big[\frac{7}{2}(\boldsymbol{V}\cdot\boldsymbol{S})+\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}\,, \tag{1}$$

$$\frac{d\boldsymbol{\Delta}}{dt}=\frac{GM}{c^2R^2}\Big\{\boldsymbol{V}\Big[\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+\Big(\frac{3}{2}-\frac{5}{2}\eta\Big)(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]-\mathbf{n}\Big[\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})-\Big(\frac{3}{2}-\frac{5}{2}\eta\Big)(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}, \tag{2}$$

where $M\equiv M_1+M_2$ denotes the total mass of the system, $\delta M\equiv M_1-M_2$ and $\eta\equiv M_1M_2/M^2$. $\boldsymbol{S}\equiv\boldsymbol{S}_1+\boldsymbol{S}_2$ denotes the total spin. $\boldsymbol{\Delta}=M(\boldsymbol{S}_2/M_2-\boldsymbol{S}_1/M_1)$. $\boldsymbol{V}\equiv\boldsymbol{V}_1-\boldsymbol{V}_2$ is the binary's relative velocity. $\boldsymbol{n}\equiv\boldsymbol{R}/R$ is the unit vector, with $\boldsymbol{R}=\boldsymbol{X}_1-\boldsymbol{X}_2$ being the separation vector between the two bodies and $R\equiv|\boldsymbol{R}|$. We also need the 2.5PN acceleration of the spinning binary system's relative motion, which can be written as [28,25]

$$\frac{d\boldsymbol{V}}{dt}=\boldsymbol{A}_N+\boldsymbol{A}_{1PN}+\boldsymbol{A}_{1.5SO}+\boldsymbol{A}_{2PN}+\boldsymbol{A}_{2.5SO}\,, \tag{3}$$

where

$$\boldsymbol{A}_N=-\frac{GM}{R^2}\boldsymbol{n}\,, \tag{4}$$

$$\boldsymbol{A}_{1PN}=-\frac{GM}{c^2R^2}\Big\{\boldsymbol{n}\Big[(1+3\eta)V^2-(4+2\eta)\frac{GM}{R}-\frac{3}{2}\eta\dot{R}^2\Big]-2(2-\eta)\dot{R}\boldsymbol{V}\Big\}\,, \tag{5}$$

$$\boldsymbol{A}_{1.5SO}=\frac{G}{c^3R^3}\Big\{6\boldsymbol{n}\Big[2(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{\delta m}{m}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]-\Big[7(\boldsymbol{V}\times\boldsymbol{S})+3\frac{\delta m}{m}(\boldsymbol{V}\times\boldsymbol{\Delta})\Big]+3\dot{R}\Big(3\boldsymbol{n}\times\boldsymbol{S}+\frac{\delta m}{m}\boldsymbol{n}\times\boldsymbol{\Delta}\Big)\Big\}\,, \tag{6}$$

$$\begin{aligned}\boldsymbol{A}_{2PN}=&-\frac{GM}{c^4R^2}\Big\{\boldsymbol{n}\Big[\frac{3}{4}(12+29\eta)\frac{(GM)^2}{R^2}+\eta(3-4\eta)V^4+\frac{15}{8}\eta(1-3\eta)\dot{R}^4-\frac{3}{2}\eta(3-4\eta)V^2\dot{R}^2-\frac{1}{2}\eta(13-4\eta)\frac{GM}{R}V^2\\&-(2+25\eta+2\eta^2)\frac{GM}{R}\dot{R}^2\Big]-\frac{1}{2}\dot{R}\boldsymbol{V}\Big[\eta(15+4\eta)V^2-(4+41\eta+8\eta^2)\frac{GM}{R}-3\eta(3+2\eta)\dot{R}^2\Big]\Big\}\,,\end{aligned} \tag{7}$$

$$\begin{aligned}\boldsymbol{A}_{2.5SO}=&\frac{G}{c^5R^3}\Big\{\Big[(24+19\eta)\frac{GM}{R}+\frac{3}{2}(1+10\eta)\dot{R}^2-14\eta V^2\Big](\boldsymbol{V}\times\boldsymbol{S})-\Big[(28+29\eta)\frac{GM}{R}+\frac{45}{2}\eta\dot{R}^2+\frac{3}{2}(1-15\eta)V^2\Big]\dot{R}(\boldsymbol{n}\times\boldsymbol{S})\\&+\Big[\frac{1}{2}(24+19\eta)\frac{GM}{R}+\frac{3}{2}(1+6\eta)\dot{R}^2-7\eta V^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})-\Big[\frac{1}{2}(24+31\eta)\frac{GM}{R}+15\eta\dot{R}^2+\frac{3}{2}(1-8\eta)V^2\Big]\frac{\delta M}{M}\dot{R}^2(\boldsymbol{n}\times\boldsymbol{\Delta})\\&-\boldsymbol{n}\Big[(44+33\eta)\frac{GM}{R}+30\eta\dot{R}^2-24\eta V^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}-\boldsymbol{n}\Big[\frac{1}{2}(48+37\eta)\frac{GM}{R}+15\eta\dot{R}^2-12\eta V^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\\&-\Big[\frac{21}{2}(1-\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{3}{2}(3-4\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]\dot{R}\boldsymbol{V}\Big\}\,,\end{aligned} \tag{8}$$

where the Tulczyjew's SSC has been employed. $\dot{R}=\boldsymbol{n}\cdot\boldsymbol{V}$.

III. FORMULAS FOR CALCULATING THE GRAVITATIONAL LOSSES

The isolated system's energy, angular momentum, and linear momentum losses due to the gravitational-wave radiation can be written in terms of the symmetric and tracefree (STF)-multipole moments as follows [32]:

$$\frac{dE}{dt}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}\overset{(l+1)}{\mathcal{I}}_{A_l}\overset{(l+1)}{\mathcal{I}}_{A_l}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!}\overset{(l+1)}{\mathcal{J}}_{A_l}\overset{(l+1)}{\mathcal{J}}_{A_l}\Big\}\,, \tag{9}$$

$$\frac{dJ_j}{dt}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{(l+1)(l+2)}{(l-1)l!(2l+1)!!}\epsilon_{jpq}\overset{(l)}{\mathcal{I}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{I}}_{qA_{l-1}}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{4l^2(l+2)}{(l-1)(l+1)!(2l+1)!!}\epsilon_{jpq}\overset{(l)}{\mathcal{J}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{J}}_{qA_{l-1}}\Big\}\,, \tag{10}$$

$$\begin{aligned}\frac{dP_j}{dt}=&-\frac{G}{c^7}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{2(l+2)(l+3)}{l(l+1)!(2l+3)!!}\overset{(l+2)}{\mathcal{I}}_{jA_l}\overset{(l+1)}{\mathcal{I}}_{A_l}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{8(l+3)}{(l+1)!(2l+3)!!}\overset{(l+2)}{\mathcal{J}}_{jA_l}\overset{(l+1)}{\mathcal{J}}_{jA_l}\\&+\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{8(l+2)}{(l-1)(l+1)!(2l+1)!!}\epsilon_{jpq}\overset{(l+1)}{\mathcal{I}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{J}}_{qA_{l-1}}\Big\}\,,\end{aligned} \tag{11}$$

where E, J_j and P_j are the orbital energy, angular momentum and linear one of the system. $\mathcal{I}_{A_l}$ and $\mathcal{J}_{A_l}$ are the STF radiative mass and current multipole moments, respectively. A_l denotes a multi-index of length l, i.e. $A_l=a_1a_2...a_l$, where a_i with $1\leq i\leq l$ takes the indies of $1,2,3$. $\overset{(l)}{\mathcal{I}}\equiv d^l\mathcal{I}/dt^l$ and $\overset{(l)}{\mathcal{J}}\equiv d^l\mathcal{J}/dt^l$.

Following the calculation method given in Ref. [18], we take all possible values for the parameter l in Eqs. (9)-(11) to ensure the accuracy of $\frac{dE}{dt}$ and $\frac{dJ_j}{dt}$ to the 2.5PN order, and that of $\frac{dP_j}{dt}$ to the 1.5PN order, as follows

$$\frac{dE}{dt}=-\frac{G}{c^5}\Big[\frac{1}{5}\overset{(3)}{\mathcal{I}}_{kl}\overset{(3)}{\mathcal{I}}_{kl}+\frac{1}{c^2}\frac{16}{45}\overset{(3)}{\mathcal{J}}_{kl}\overset{(3)}{\mathcal{J}}_{kl}+\frac{1}{c^2}\frac{1}{189}\overset{(4)}{\mathcal{I}}_{klm}\overset{(4)}{\mathcal{I}}_{klm}+\frac{1}{c^4}\frac{1}{84}\overset{(4)}{\mathcal{J}}_{klm}\overset{(4)}{\mathcal{J}}_{klm}+\frac{1}{c^4}\frac{1}{9072}\overset{(5)}{\mathcal{I}}_{klmn}\overset{(5)}{\mathcal{I}}_{klmn}\Big]\,, \tag{12}$$

$$\frac{dJ_j}{dt}=-\frac{G}{c^5}\epsilon_{jpq}\Big[\frac{2}{5}\overset{(2)}{\mathcal{I}}_{pk}\overset{(3)}{\mathcal{I}}_{qk}+\frac{1}{c^2}\frac{32}{45}\overset{(2)}{\mathcal{J}}_{pk}\overset{(3)}{\mathcal{J}}_{qk}+\frac{1}{c^2}\frac{1}{63}\overset{(3)}{\mathcal{I}}_{pkl}\overset{(4)}{\mathcal{I}}_{qkl}+\frac{1}{c^4}\frac{1}{28}\overset{(3)}{\mathcal{J}}_{pkl}\overset{(4)}{\mathcal{J}}_{qkl}+\frac{1}{c^4}\frac{1}{2268}\overset{(4)}{\mathcal{I}}_{pklm}\overset{(4)}{\mathcal{I}}_{qklm}\Big], \tag{13}$$

$$\frac{dP_j}{dt}=-\frac{G}{c^7}\Big[\frac{2}{63}\overset{(4)}{\mathcal{I}}_{jkl}\overset{(3)}{\mathcal{I}}_{kl}+\frac{1}{c^2}\frac{4}{63}\overset{(4)}{\mathcal{J}}_{jkl}\overset{(3)}{\mathcal{J}}_{kl}+\frac{16}{45}\epsilon_{jpq}\overset{(3)}{\mathcal{I}}_{pk}\overset{(3)}{\mathcal{J}}_{qk}+\frac{1}{c^2}\frac{1}{1134}\overset{(5)}{\mathcal{I}}_{jklm}\overset{(4)}{\mathcal{I}}_{klm}+\frac{1}{c^2}\frac{1}{126}\epsilon_{jpq}\overset{(4)}{\mathcal{I}}_{pkl}\overset{(4)}{\mathcal{J}}_{qkl}\Big]\,. \tag{14}$$

The non-spin mass and mass-current multipole moments for the binary system can be written as [13,33]

$$\begin{aligned}\underset{NS}{\mathcal{I}_{ij}}=&\,\mu R^2\Big\{1+\frac{1}{c^2}\Big[\frac{29}{42}(1-3\eta)V^2-\frac{1}{7}(5-8\eta)\frac{GM}{R}\Big]+\frac{1}{c^4}\Big[\frac{1}{756}(2021-5947\eta-4883\eta^2)\frac{GM}{R}V^2\\&-\frac{1}{252}(355+1906\eta-337\eta^2)\frac{(GM)^2}{R^2}-\frac{1}{756}(131-907\eta+1273\eta^2)\frac{GM}{R}\dot{R}^2+\frac{1}{504}(253-1835\eta+3545\eta^2)V^4\Big]\Big\}n_{<ij>}\\&+\mu R^2\Big\{\frac{11}{21c^2}(1-3\eta)+\frac{1}{c^4}\Big[\frac{1}{189}(742-335\eta-985\eta^2)\frac{GM}{R}+\frac{1}{126}(41-337\eta+733\eta^2)V^2+\frac{5}{63}(1-5\eta+5\eta^2)\dot{R}^2\Big]\Big\}V_{<ij>}\\&-\mu\dot{R}R^2\Big\{\frac{4}{7c^2}(1-3\eta)+\frac{2}{c^4}\Big[\frac{1}{63}(13-101\eta+209\eta^2)V^2+\frac{1}{756}(1085-4057\eta-1463\eta^2)\frac{GM}{R}\Big]\Big\}n_{<i}V_{j>}\,,\end{aligned} \tag{15}$$

$$\underset{NS}{\mathcal{J}_{ij}}=\mu R^2\frac{\delta M}{M}\Big\{1+\frac{1}{c^2}\Big[\frac{3}{14}(9+10\eta)\frac{GM}{R}+\frac{1}{28}(13-68\eta)V^2\Big]\Big\}\epsilon_{pq<i}n_{j>}n_pV_q-\mu\frac{\delta M}{M}\frac{1}{c^2}\frac{5}{28}(1-2\eta)\dot{R}R^2\epsilon_{pq<i}V_{j>}n_pV_q\,, \tag{16}$$

$$\underset{NS}{\mathcal{I}}_{ijk} = \mu R^3 \frac{\delta M}{M}\Big\{-1+\frac{1}{c^2}\Big[\frac{1}{6}(5-13\eta)\frac{GM}{R}-\frac{1}{6}(5-19\eta)V^2\Big]\Big\}n_{<ijk>}+\mu\frac{\delta M}{M}\frac{1}{c^2}(1-2\eta)\dot{R}R^3 n_{<ij}V_{k>} - \mu\frac{\delta M}{M}\frac{1}{c^2}(1-2\eta)R^3 n_{<i}V_{jk>}\,, \tag{17}$$

$$\underset{NS}{\mathcal{J}}_{ijk} = \mu(1-3\eta)R^3\epsilon_{pq<i}n_{jk>}n_pV_q\,, \tag{18}$$

$$\underset{NS}{\mathcal{I}}_{ijkl} = \mu(1-3\eta)R^4 n_{<ijkl>}\,, \tag{19}$$

where n_i and V_i denote the i component of $\boldsymbol{n}$ and $\boldsymbol{V}$. $\mu=\eta M$ is the reduce mass of the binary system. $n_{<ij>}=n_{<i}n_{j>}$, etc. denote the STF section of the tensors, as in Ref. [18].

The spin mass and mass-current multipole moments for the binary system with the spin vector defined by Bohé et al. can be written as [29]

$$\begin{aligned}\underset{S}{\mathcal{I}}_{ij} =& \frac{R\eta}{c^3}\Big\{\frac{8}{3}(\boldsymbol{V}\times\boldsymbol{S})_{<i}n_{j>}+\frac{8}{3}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_{j>}-\frac{4}{3}(\boldsymbol{n}\times\boldsymbol{S})_{<i}V_{j>}-\frac{4}{3}\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}V_{j>}\Big\}+\frac{R\eta}{c^5}\Big\{\Big[\frac{1}{21}(33+125\eta)\frac{GM}{R}\\ &+\frac{26}{21}(1-3\eta)V^2\Big](\boldsymbol{V}\times\boldsymbol{S})_{<i}n_{j>}+\Big[\frac{1}{3}(1+16\eta)\frac{GM}{R}+\frac{1}{21}(26-116\eta)V^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_{j>}\\ &+\Big[-\frac{1}{3}(22+10\eta)\frac{GM}{R}-\frac{4}{21}(1-3\eta)V^2\Big](\boldsymbol{n}\times\boldsymbol{S})_{<i}V_{j>}+\Big[-\frac{1}{21}(56+34\eta)\frac{GM}{R}\\ &+\frac{1}{21}(-4+36\eta)V^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}V_{j>}-\frac{4}{21}(1-3\eta)\dot{R}(\boldsymbol{V}\times\boldsymbol{S})_{<i}V_{j>}-\frac{4}{21}(1-5\eta)\dot{R}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}V_{j>}\\ &+\Big[\frac{3}{7}(1-3\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{1}{21}(9-40\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]V_{<i}V_{j>}+\frac{GM}{R}\Big[\frac{38}{21}(1+12\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}\\ &+\frac{2}{21}(24-13\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]n_{<i}n_{j>}+\frac{GM}{R}\Big[\frac{1}{21}(17+61\eta)\dot{R}(\boldsymbol{n}\times\boldsymbol{S})_{<i}n_{j>}+\frac{1}{21}(21+34\eta)\dot{R}\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}n_{j>}\Big]\\ &+\frac{GM}{R}\Big[-\frac{1}{3}(6-10\eta)(\boldsymbol{n}\cdot\boldsymbol{S})-\frac{2}{3}(3-2\eta)\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big](\boldsymbol{n}\times\boldsymbol{V})_{<i}n_{j>}\Big\}\,,\end{aligned} \tag{20}$$

$$\begin{aligned}\underset{S}{\mathcal{J}}_{ij} =& -\frac{3}{2}\frac{R\eta}{c}n_{<i}\Delta_{j>}+\frac{R\eta}{c^3}\Big\{\Big[-\frac{2}{7}\frac{\delta M}{M}V^2+\frac{10}{7}\frac{\delta M}{M}\frac{GM}{R}\Big]n_{<i}S_{j>}+\Big[-\frac{1}{28}(29-143\eta)V^2+\frac{1}{28}(61-71)\frac{GM}{R}\Big]n_{<i}\Delta_{j>}\\ &+\frac{1}{28}\Big[33\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})+(33-155\eta)(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]n_{<i}V_{j>}+\frac{1}{7}\dot{R}\Big[3\frac{\delta M}{M}V_{<i}S_{j>}+(3-16\eta)V_{<i}\Delta_{j>}\Big]-\frac{1}{14}\Big[(11-47\eta)(\boldsymbol{n}\cdot\boldsymbol{\Delta})\\ &+11\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})\Big]V_{<i}V_{j>}-\frac{1}{14}\frac{GM}{R}\Big[29\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+(8-31\eta)(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]n_{<i}n_{j>}\Big\},\end{aligned} \tag{21}$$

$$\underset{S}{\mathcal{I}}_{ijk} = \frac{R^2\eta}{c^3}\Big\{-\frac{9}{2}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{S})_{<i}n_jn_{k>}-\frac{3}{2}(3-11\eta)(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_jn_{k>}+3\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{S})_{<i}n_jV_{k>}+3(1-3\eta)(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}n_jV_{k>}\Big\}\,, \tag{22}$$

$$\underset{S}{\mathcal{J}}_{ijk} = \frac{R^2\eta}{c}\Big[2n_{<ij}S_{k>}+2\frac{\delta M}{M}n_{<ij}\Delta_{k>}\Big]\,. \tag{23}$$

The STF tensors used in our derivations are provided in the Appendix A.

IV. GRAVITATIONAL LOSSES INDUCED BY THE NEXT-TO-LEADING SPIN-ORBIT COUPLING EFFECTS

Substituting Eqs. (15)-(23) into Eqs. (12)-(14), making use of Eqs. (4)-(8) to calculate the time derivative of velocity, and keeping the results to include the contributions of the next-to-leading spin-orbit coupling effects, we can obtain the loss rates of the orbital energy, angular momentum, and linear momentum as follows

$$\frac{dE}{dt} = \dot{E}_N + \dot{E}_{1PN} + \dot{E}_{1.5SO} + \dot{E}_{2PN} + \dot{E}_{2.5SO} \,, \tag{24}$$

$$\frac{d\boldsymbol{J}}{dt} = \dot{\boldsymbol{J}}_N + \dot{\boldsymbol{J}}_{1PN} + \dot{\boldsymbol{J}}_{1.5SO} + \dot{\boldsymbol{J}}_{2PN} + \dot{\boldsymbol{J}}_{2.5SO} \,, \tag{25}$$

$$\frac{d\boldsymbol{P}}{dt} = \dot{\boldsymbol{P}}_N + \dot{\boldsymbol{P}}_{0.5SO} + \dot{\boldsymbol{P}}_{1PN} + \dot{\boldsymbol{P}}_{1.5SO} \,, \tag{26}$$

where

$$\dot{E}_N = -\frac{8}{15}\frac{G^3M^2\mu^2}{c^5R^4}\left[12V^2 - 11\dot{R}^2\right], \tag{27}$$

$$\dot{E}_{1PN} = -\frac{2}{105}\frac{G^3M^2\mu^2}{c^7R^4}\Big[(785-825\eta)V^4 - 2(1487-1392\eta)V^2\dot{R}^2 + 3(687-620\eta)\dot{R}^4 - 160(17-\eta)\frac{GM}{R}V^2 + 8(367-15\eta)\frac{GM}{R}\dot{R}^2 + 16(1-4\eta)\frac{(GM)^2}{R^2}\Big] \,, \tag{28}$$

$$\dot{E}_{1.5SO} = -\frac{8}{15}\frac{G^3M\mu^2}{c^8R^5}\left\{\left[78\dot{R}^2 - 8\frac{GM}{R} - 80V^2\right](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S} + \left[4\frac{GM}{R} - 43V^2 + 51\dot{R}^2\right]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\right\} \,, \tag{29}$$

$$\begin{aligned}\dot{E}_{2PN} = &-\frac{2}{2835}\frac{G^3M^2\mu^2}{c^9R^4}\Big\{\Big[18(1692-5497\eta+4430\eta^2)V^6 - 24(253-1026\eta+56\eta^2)\frac{(GM)^3}{R^3} - 54(1719-10278\eta \\ &+6292\eta^2)V^4\dot{R}^2 + 108(4987-8513\eta+2165\eta^2)\frac{GM}{R}V^2\dot{R}^2 - 3(106319+9798\eta+5376\eta^2)\frac{(GM)^2}{R^2}\dot{R}^2 \\ &+54(2018-15207\eta+7572\eta^2)V^2\dot{R}^4 - 12(33510-60971\eta+14290\eta^2)\frac{GM}{R}\dot{R}^4 - 36(4446-5237\eta+1393\eta^2)\frac{GM}{R}V^4 \\ &-18(2501-20234\eta+8404\eta^2)\dot{R}^6 + (281473+81828\eta+4368\eta^2)\frac{(GM)^2}{R^2}V^2\Big]\Big\} \,,\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{E}_{2.5SO} = &-\frac{2}{105}\frac{G^3M\mu^2}{c^{10}R^5}\Big\{\Big[(3776+1560\eta)\frac{G^2M^2}{R^2} + (15220-896\eta)\frac{GM}{R}V^2 - (12892-2024\eta)\frac{GM}{R}\dot{R}^2 - (4828-7240\eta)V^4 \\ &+(14076-20016\eta)V^2\dot{R}^2 - (8976-12576\eta)\dot{R}^4\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S} + \Big[(10774-1988\eta)\frac{GM}{R}V^2 \\ &-(14654-4796\eta)\frac{GM}{R}\dot{R}^2 - (2603-4160\eta)V^4 + (9456-14484\eta)V^2\dot{R}^2 - (548-952\eta)\frac{G^2M^2}{R^2} \\ &-(7941-10704\eta)\dot{R}^4\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\} \,,\end{aligned} \tag{31}$$

$$\dot{\boldsymbol{J}}_N = -\frac{8}{5}\frac{G^2M\mu^2}{c^5R^2}(\boldsymbol{n}\times\boldsymbol{V})\left\{2V^2 - 3\dot{R}^2 + 2\frac{GM}{R}\right\} \,, \tag{32}$$

$$\dot{\boldsymbol{J}}_{1PN} = -\frac{2}{105}\frac{G^2M\mu^2}{c^7R^2}(\boldsymbol{n}\times\boldsymbol{V})\Big[(307-548\eta)V^4-6(74-277\eta)V^2\dot{R}^2+15(19-72\eta)\dot{R}^4-4(58+95\eta)\frac{GM}{R}V^2 +2(372+197\eta)\frac{GM}{R}\dot{R}^2-2(745-2\eta)\frac{(GM)^2}{R^2}\Big], \tag{33}$$

$$\begin{aligned}\dot{\boldsymbol{J}}_{1.5SO} = {}& \frac{4}{15}\frac{G^2\mu^2}{c^8R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{\Big[163\frac{GM}{R}+111V^2-195\dot{R}^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\Big[71\frac{GM}{M}+57V^2-105\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\}\\ &+\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\boldsymbol{V}\Big\{\Big[12(\boldsymbol{n}\cdot\boldsymbol{S})+5\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]\frac{GM}{M}\dot{R}+\Big[50\frac{GM}{R}+71V^2-108\dot{R}^2\Big](\boldsymbol{V}\cdot\boldsymbol{S})\\ &+\Big[27\frac{GM}{R}+35V^2-54\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\}+\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\boldsymbol{n}\Big\{\Big[41\frac{GM}{R}V^2-4\frac{(GM)^2}{R^2}-45\frac{GM}{R}\dot{R}^2\Big](\boldsymbol{n}\cdot\boldsymbol{S})+\Big[165\dot{R}^2-132V^2\\ &-54\frac{GM}{R}\Big]\dot{R}(\boldsymbol{V}\cdot\boldsymbol{S})+\Big[2\frac{(GM)^2}{R^2}+24\frac{GM}{R}V^2-27\frac{GM}{R}\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})-\Big[25\frac{GM}{R}+60V^2-75\dot{R}^2\Big]\frac{\delta M}{M}\dot{R}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\}\\ &+\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\Big\{\boldsymbol{S}\Big[4\frac{G^2M^2}{R^2}-91\frac{GM}{R}V^2-71V^4+87\frac{GM}{R}\dot{R}^2+240V^2\dot{R}^2-165\dot{R}^4\Big]\\ &+\boldsymbol{\Delta}\Big[2\frac{G^2M^2}{R^2}+49\frac{GM}{R}V^2+35V^4-45\frac{GM}{R}\dot{R}^2-114V^2\dot{R}^2+75\dot{R}^4\Big]\Big\},\end{aligned} \tag{34}$$

$$\begin{aligned}\dot{\boldsymbol{J}}_{2PN} = {}& -\frac{1}{2835}\frac{G^2M\mu^2}{c^9R^2}(\boldsymbol{n}\times\boldsymbol{V})\Big[(340724+140922\eta+2772\eta^2)\frac{(GM)^3}{R^3}-(49140-205380\eta+122220\eta^2)\dot{R}^6\\ &+(23985-111195\eta+116046\eta^2)V^6+(151848-451836\eta+82566\eta^2)\frac{(GM)^2}{R^2}\dot{R}^2-(200808-372582\eta\\ &+87255\eta^2)\frac{GM}{R}\dot{R}^4+(96525-453735\eta+423360\eta^2)V^2\dot{R}^4-(191718-183222\eta+61704\eta^2)\frac{(GM)^2}{R^2}V^2\\ &+(196677-194427\eta+22959\eta^2)\frac{GM}{R}V^2\dot{R}^2-(60642-341631\eta+422199\eta^2)V^4\dot{R}^2\\ &+(1485-4419\eta+36198\eta^2)\frac{GM}{R}V^4\Big],\end{aligned} \tag{35}$$

$$\begin{aligned}\dot{\boldsymbol{J}}_{2.5SO} = {}& -\frac{2}{315}\frac{G^2\mu^2}{c^{10}R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}\Big[(60751+4021\eta)\frac{(GM)^2}{R^2}-(3996-12501\eta)V^4+(11340-10710\eta)\dot{R}^4\\ &+(5276+18000\eta)\frac{GM}{R}V^2-(8640+4995\eta)V^2\dot{R}^2-(23799+19353\eta)\frac{GM}{R}\dot{R}^2\Big]+\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\\ &\times\Big[(30042+1376\eta)\frac{G^2M^2}{R^2}+(6030+6041\eta)\frac{GM}{R}V^2-(17850+4164\eta)\frac{GM}{R}\dot{R}^2-(1638-7335\eta)V^4\\ &-(8820+11160\eta)V^2\dot{R}^2+(11970+2205\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{n}\Big\{(\boldsymbol{n}\cdot\boldsymbol{S})\Big[(6360+744\eta)\frac{(GM)^3}{R^3}\\ &-(4013+1388\eta)\frac{GM}{R}V^2\dot{R}^2-(35306+1742\eta)\frac{(GM)^2}{R^2}V^2+(61380+4956\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(3024-5040\eta)V^6\\ &+(12429+5166\eta)\frac{GM}{R}\dot{R}^4+(5645+1254\eta)\frac{GM}{R}V^4+(26460-44100\eta)V^4\dot{R}^2-(26460-44100\eta)V^2\dot{R}^4\Big]\\ &+\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big[(30942+5082\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(17094+2194\eta)\frac{(GM)^2}{R^2}V^2-(720-420\eta)\frac{(GM)^3}{R^3}\\ &+(5058-247\eta)\frac{GM}{R}V^4-(18615+1338\eta)\frac{GM}{R}V^2\dot{R}^2+(7263-9\eta)\frac{GM}{R}\dot{R}^4-(3024-2016\eta)V^6\\ &+(26460-17640\eta)V^4\dot{R}^2-(26460-17640\eta)V^2\dot{R}^4\Big]+(\boldsymbol{V}\cdot\boldsymbol{S})\dot{R}\Big[(51034+5742\eta)\frac{GM}{R}V^2\end{aligned}$$

$$
\begin{aligned}
&-(84174+8610\eta)\frac{GM}{R}\dot{R}^2-(17190-8460\eta)V^2\dot{R}^2-(3681-18324\eta)V^4+(37556+6938\eta)\frac{(GM)^2}{R^2}+(22365-34020\eta)\dot{R}^4\Big]\\
&+(\boldsymbol{V}\cdot\boldsymbol{\Delta})\dot{R}\Big[(14634+3748\eta)\frac{(GM)^2}{R^2}+(29595+5218\eta)\frac{GM}{R}V^2-(41397+7176\eta)\frac{GM}{R}\dot{R}^2-(5913-7011\eta)V^4\\
&-(5850-6210\eta)V^2\dot{R}^2+(16065-16065\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{V}\Big\{(\boldsymbol{n}\cdot\boldsymbol{S})\dot{R}\Big[(3118+10176\eta)\frac{GM}{R}V^2\\
&-(27946+1222\eta)\frac{(GM)^2}{R^2}+(1902-8124\eta)\frac{GM}{R}\dot{R}^2+(3024-5040\eta)V^4-(26460-44100\eta)V^2\dot{R}^2\\
&+(26460-44100\eta)\dot{R}^4\Big]-(\boldsymbol{V}\cdot\boldsymbol{S})\Big[(20890+10614\eta)\frac{GM}{R}V^2+(33212+4178\eta)\frac{(GM)^2}{R^2}-(47262+10002\eta)\frac{GM}{R}\dot{R}^2\\
&-(2247-10974\eta)V^4-(22752-20448\eta)V^2\dot{R}^2+(25965-38970\eta)\dot{R}^4\Big]+\frac{\delta M}{M}\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\\
&\times\Big[(5313-2496\eta)\frac{GM}{R}\dot{R}^2-(26460-17640\eta)V^2\dot{R}^2+(885+3847\eta)\frac{GM}{R}V^2+(26460-17640\eta)\dot{R}^4-(13896+1136\eta)\frac{(GM)^2}{R^2}\\
&+(3024-2016\eta)V^4\Big]-\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big[(10995+6622\eta)\frac{GM}{R}V^2+(14970+1552\eta)\frac{(GM)^2}{R^2}-(20661+6012\eta)\frac{GM}{R}\dot{R}^2\\
&-(663-6147\eta)V^4-(19908-4122\eta)V^2\dot{R}^2+(24345-13545\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\Big\{\boldsymbol{S}\Big[(14731+14916\eta)\frac{GM}{R}V^4\\
&-(75236+6410\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(72764+1658\eta)\frac{(GM)^2}{R^2}V^2-(80308+28326\eta)\frac{GM}{R}V^2\dot{R}^2-(6360+744\eta)\frac{(GM)^3}{R^3}\\
&+(73329+16746\eta)\frac{GM}{R}\dot{R}^4-(2247-10974\eta)V^6-(19071-2124\eta)V^4\dot{R}^2+(43155-47430\eta)V^2\dot{R}^4-(22365-34020\eta)\dot{R}^6\Big]\\
&+\frac{\delta M}{M}\boldsymbol{\Delta}\Big[(720-420\eta)\frac{(GM)^3}{R^3}+(36228+2050\eta)\frac{(GM)^2}{R^2}V^2-(35844+5998\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(34089+11577\eta)\frac{GM}{R}\dot{R}^4\\
&+(6915+8983\eta)\frac{GM}{R}V^4-(38772+17776\eta)\frac{GM}{R}V^2\dot{R}^2-(663-6147\eta)V^6-(13995+2889\eta)V^4\dot{R}^2\\
&+(30195-19755\eta)V^2\dot{R}^4-(16065-16065\eta)\dot{R}^6\Big]\Big\},
\end{aligned}
\tag{36}
$$

$$
\dot{\boldsymbol{P}}_N=-\frac{8}{105}\frac{G^3M^2\mu^2}{c^7R^4}(1-4\eta)^{\frac{1}{2}}\Big\{\boldsymbol{V}\Big[38\dot{R}^2-50V^2-8\frac{GM}{R}\Big]+\dot{R}\boldsymbol{n}\Big[55V^2+12\frac{GM}{R}-45\dot{R}^2\Big]\Big\}\,,
\tag{37}
$$

$$
\dot{\boldsymbol{P}}_{0.5SO}=-\frac{8}{15}\frac{G^3M\mu^2}{c^8R^5}\Big\{4\dot{R}(\boldsymbol{V}\times\boldsymbol{\Delta})-2V^2(\boldsymbol{n}\times\boldsymbol{\Delta})-(\boldsymbol{n}\times\boldsymbol{V})\Big[3\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})+2(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}\,,
\tag{38}
$$

$$
\begin{aligned}
\dot{\boldsymbol{P}}_{1PN}=&-\frac{1}{945}\frac{G^3M^2\mu^2}{c^9R^4}(1-4\eta)^{\frac{1}{2}}\times\Big\{\boldsymbol{V}\Big[32(189+17\eta)\frac{(GM)^2}{R^2}-12(2663-1394\eta)\dot{R}^4+36(907-162\eta)\frac{GM}{R}V^2\\
&+120(392-257\eta)V^2\dot{R}^2-444(25-28\eta)V^4-12(2699+10\eta)\frac{GM}{R}\dot{R}^2\Big]+\dot{R}\boldsymbol{n}\Big[4(12301-1168\eta)\frac{GM}{R}\dot{R}^2\\
&+24(851-779\eta)V^4-24(2834-1877\eta)V^2\dot{R}^2-12(590-4\eta)\frac{(GM)^2}{R^2}+24(1843-1036\eta)\dot{R}^4\\
&-12(4385-956\eta)\frac{GM}{R}V^2\Big]\Big\}\,,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\dot{\boldsymbol{P}}_{1.5SO}=&\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}(\boldsymbol{n}\times\boldsymbol{V})\Big\{\Big[21852\dot{R}^2-19536V^2+2166\frac{GM}{R}\Big]\dot{R}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+\Big[-272\frac{GM}{R}-4857\dot{R}^2+3314V^2\Big]\\
&\times\dot{R}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})+\Big[-(4902+2919\eta)\frac{GM}{R}+(6264-37062\eta)\dot{R}^2-(5172-35448\eta)V^2\Big]\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})+\Big[-(572-647\eta)\frac{GM}{R}\\
&-(4281-8556\eta)\dot{R}^2+(2024-5621\eta)V^2\Big](\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\}+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}\boldsymbol{n}\dot{R}\Big\{\Big[3180\frac{GM}{R}-17964\dot{R}^2+17592V^2\Big]
\end{aligned}
$$

$$\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}\Big\} + \Big[(1509-5451\eta)\frac{GM}{R} + (9492-35934\eta)V^2 - (9396-35514\eta)\dot{R}^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}$$
$$+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}V\Big\{-\Big[7985V^2+2176\frac{GM}{R}-8043\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\Big[(3822-14361\eta)\dot{R}^2-(3968-15017\eta)V^2$$
$$-(697-2503\eta)\frac{GM}{R}\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\}+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}\Big\{-\frac{\delta M}{M}\dot{R}\Big[608\frac{GM}{R}-5709\dot{R}^2+5431V^2\Big](\boldsymbol{V}\times\boldsymbol{S})$$
$$-\Big[(3585+4506\eta)\frac{GM}{R}\dot{R}^2+(10287-9810\eta)\dot{R}^4+(1417-3484\eta)\frac{GM}{R}V^2+(10287-9810\eta)V^2\dot{R}^2$$
$$+(1322-968\eta)V^4\Big](\boldsymbol{n}\times\boldsymbol{\Delta})-\dot{R}\Big[(5677-15868\eta)V^2-(8355-18048\eta)\dot{R}^2-(6478-184\eta)\frac{GM}{R}\Big](\boldsymbol{V}\times\boldsymbol{\Delta})$$
$$+\Big[1812\frac{GM}{R}\dot{R}^2-6300\dot{R}^4-664\frac{GM}{R}V^2+7491V^2\dot{R}^2-1469V^4+72\frac{(GM)^2}{R^2}\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{S})\Big\}. \tag{40}$$

Among the above results, the loss rate of the system's energy given in Eqs. (27)-(30) has been achieved in Ref. [13]. The Newtonian and 1PN contributions to the loss rate of the system's angular momentum, see Eqs. (32) and (33), as well as the Newtonian contribution to the linear momentum loss rate, see Eq. (37), have been obtained in Ref. [18]. The 2PN contributions to the loss rate of the system's angular momentum and the 1PN contributions the linear momentum loss rate, see Eqs. (35) and (39), have been obtained in Ref [20,21]. The 1.5PN SO contributions to the loss rates of the energy and angular momentum, see Eqs. (29) and (34), as well as the 0.5PN SO contribution to the linear momentum loss rate, see (38), have been achieved in Ref [25]. Here we include them for the completeness.

The core results of this work are the loss rates of the system's energy and angular momentum induced by the 2.5PN SO coupling effects, see Eqs. (31) and (36), and the linear momentum's loss rate induced by the 1.5PN SO coupling effect, see Eq. (40). Notice that all these losses can be called as the next-to-leading spin-orbit coupling effects, since the leading spin-orbit coupling effects to the system's energy and angular momentum are the 1.5PN SO coupling contribution, while the leading spin-orbit coupling effects to the system's linear momentum is the 0.5PN SO coupling contribution.

V. LOSS RATES OF GRAVITATIONAL RADIATION IN THE CASE OF CIRCULAR ORBIT

When the binary systems loss their orbital energy, angular momentum and linear momentum due to the gravitational-wave radiation, their orbit will shrink and their eccentricity will decrease. In the final stage of binary inspiral, their orbit can be approximated as circular one [25,34]. In this case, we have $\dot{R}=0$. Following Ref. [16], we introduce the PN parameter

$$x=\Big(\frac{GM\omega}{c^3}\Big)^{\frac{2}{3}}, \tag{41}$$

with ω being the orbital frequency

$$\omega^2=\frac{GM}{R^3}\Big\{1-\frac{GM}{c^2R}(3-\eta)+\Big(\frac{GM}{c^2R}\Big)^2\Big(6+\frac{41}{4}\eta+\eta^2\Big)-\Big(\frac{GM}{c^2R}\Big)^{\frac{3}{2}}\Big(5\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+3\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)+\Big(\frac{GM}{c^2R}\Big)^{\frac{5}{2}}\Big[\Big(\frac{45}{2}-\frac{27}{2}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}$$
$$+\Big(\frac{27}{2}-\frac{13}{2}\eta\Big)\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}, \tag{42}$$

where $\boldsymbol{l}\equiv\boldsymbol{n}\times\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}\equiv\boldsymbol{V}/V$ denotes the unit vector for the angular momentum. Then, we achieve

$$\frac{dE}{dt}=-\frac{32}{5}\frac{c^5}{G}x^5\eta^2\Big\{1-x\Big(\frac{1247}{336}+\frac{35}{12}\eta\Big)-x^2\Big(\frac{44711}{9072}-\frac{9271}{504}\eta-\frac{65}{18}\eta^2\Big)-x^{\frac{3}{2}}\frac{1}{G}\Big(4\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{5}{4}\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)$$
$$-x^{\frac{5}{2}}\frac{1}{G}\Big[\Big(\frac{9}{2}-\frac{272}{9}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\Big(\frac{13}{16}-\frac{43}{4}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}, \tag{43}$$

$$\frac{d\boldsymbol{J}}{dt}=-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{1-x\Big(\frac{1247}{336}+\frac{35\eta}{12}\Big)-x^2\Big(\frac{44711}{9072}-\frac{9271}{504}\eta-\frac{65}{18}\eta^2\Big)-x^{\frac{3}{2}}\frac{1}{G}\Big(4\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\frac{5}{4}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)$$

$$-x^{\frac{5}{2}}\frac{1}{G}\Big[\Big(\frac{9}{2}-\frac{272}{9}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\Big(\frac{13}{16}-\frac{43}{4}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}\boldsymbol{l}-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{x^{\frac{3}{2}}\Big(\frac{121}{24}\frac{\boldsymbol{n}\cdot\boldsymbol{S}}{GM^2}+\frac{5}{2}\frac{\delta M}{M}\frac{\Delta_{\boldsymbol{n}}}{GM^2}\Big)$$
$$-x^{\frac{5}{2}}\Big[\Big(\frac{2387}{96}+\frac{2057}{126}\eta\Big)\frac{\boldsymbol{n}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{545}{42}+\frac{5725}{672}\eta\Big)\frac{\delta M}{M}\frac{\Delta_{\boldsymbol{n}}}{GM^2}\Big]\Big\}\boldsymbol{n}-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{x^{\frac{3}{2}}\Big(\frac{37}{24}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big)$$
$$-x^{\frac{5}{2}}\Big[\Big(\frac{2425}{224}+\frac{1387}{1008}\eta\Big)\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{2227}{336}+\frac{439}{224}\eta\Big)\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big]\Big\}\boldsymbol{\lambda}\,, \tag{44}$$

$$\frac{d\boldsymbol{P}}{dt}=\frac{464}{105}\frac{c^4}{G}x^{\frac{11}{2}}\eta^2(1-4\eta)^{\frac{1}{2}}\Big\{1-\Big(\frac{452}{87}+\frac{1139}{522}\eta\Big)x-\frac{7}{29}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{GM^2}x^{\frac{1}{2}}-\Big[\frac{470}{87}\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{67}{58}-\frac{206}{29}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{GM^2}\Big]x^{\frac{3}{2}}\Big\}\boldsymbol{\lambda}$$
$$+\frac{464}{105}\frac{c^4}{G}x^6\eta^2(1-4\eta)^{\frac{1}{2}}\Big\{\frac{14}{29}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}+\Big[\frac{109}{116}\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{25}{116}-\frac{57}{58}\eta\Big)\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big]x\Big\}\boldsymbol{l}. \tag{45}$$

VI. SUMMARY

Based on the spin vector defined by Bohé et al. and under the Tulczyjew's SSC, we have calculated the loss rates for the binary system's energy, angular momentum and linear momentum induced by the next-to-leading spin-orbit coupling effects in the case of general orbit. For comparison, we have also adopted the spin vector defined by Faye, Blanchet and Buonanno to calculate the angular momentum's loss rate to the same PN order. For the case of circular orbit, we formulate these gravitational losses in terms of the orbital frequency. The achieved results are useful in determining the time change of the orbital parameters for the general motion when the spin vector defined by Bohé et al. is adopted.

APPENDIX A: THE STF TENSORS USED IN THE DERIVATIONS

For the readers' convenience, we give the STF tensors used in the derivations for the compact binary systems' gravitational losses.

$$A_{<i}B_{j>}=\frac{1}{2}(A_iB_j+B_iA_j)-\frac{1}{3}\delta_{ij}A_aB^a\,, \tag{A1}$$

$$A_{<i}B_jC_{k>}=\frac{1}{6}(A_iB_jC_k+A_iB_kC_j+A_jB_iC_k+A_jB_kC_i+A_kB_iC_j+A_kB_jC_i)-\frac{1}{15}[(\delta_{ij}C_k+\delta_{jk}C_i+\delta_{ik}C_j)A_aB^a$$
$$+(\delta_{ij}B_k+\delta_{jk}B_i+\delta_{ik}B_j)A_aC^a+(\delta_{ij}A_k+\delta_{jk}A_i+\delta_{ik}A_j)B_aC^a]\,, \tag{A2}$$

$$A_{<ijkl>}=A_iA_jA_kA_l-\frac{1}{7}A_aA^a(\delta_{ij}A_kA_l+\delta_{ik}A_iA_l+\delta_{il}A_jA_k+\delta_{jk}A_iA_l+\delta_{jl}A_iA_k+\delta_{kl}A_iA_j)$$
$$+\frac{1}{35}A_aA^aA_bA^b(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,, \tag{A3}$$

$$\epsilon_{ab<i}A_{j>}B_aC_b=\frac{1}{2}(\epsilon_{ab<i}A_{j>}+\epsilon_{ab<j}A_{i>})B_aC_b-\frac{1}{3}\delta_{ij}\epsilon_{abk}A_kB_aC_b\,, \tag{A4}$$

$$\epsilon_{ab<i}A_jB_{k>}C_aD_b=\Big\{\frac{1}{6}(\epsilon_{abi}A_jB_k+\epsilon_{abi}A_kB_j+\epsilon_{abj}A_iB_k+\epsilon_{abj}A_kB_i+\epsilon_{abk}A_iB_j+\epsilon_{abk}A_jB_i)-\frac{1}{15}[(\delta_{ij}B_k+\delta_{ik}B_j+\delta_{jk}B_i)\epsilon_{abl}A_l$$
$$+(\delta_{ij}A_k+\delta_{ik}A_j+\delta_{jk}A_i)\epsilon_{abl}B_l+(\delta_{ij}\epsilon_{abk}+\delta_{ik}\epsilon_{abj}+\delta_{jk}\epsilon_{abi})A_lB^l]\Big\}C_aD_b\,. \tag{A5}$$

APPENDIX B: THE ANGULAR MOMENTUM LOSS INDUCED BY THE 2.5PN SO EFFECTS IN TERMS OF THE SPIN VECTOR DEFINED BY FAYE, BLANCHET AND BUONANNO

Faye, Blanchet and Buonanno give a new definition for the spin vector, and calculate the energy loss of the spinning binary systems to the 2.5PN order [26]. Later, Racine, Buonanno & Kidder calculate the linear momentum loss to the 1.5PN order [21] using the same definition. For the angular momentum loss, we have checked that the result to the 1.5PN order is same as that with the spin vector defined by Barker and O'Connell [23], which is given by Kidder [25]. Notice that the results are same to the 1.5PN order among the different SSCs.

For completeness and comparison with the angular momentum loss formulated in terms of the spin vector defined by Bohé et al [28], here we derive the angular momentum loss in terms of the spin vector defined by Faye, Blanchet and Buonanno to the 2.5PN order.

The spins of the binary system are assumed as $\tilde{\boldsymbol{S}}_1$ and $\tilde{\boldsymbol{S}}_2$. Let $\tilde{\boldsymbol{S}}=\tilde{\boldsymbol{S}}_1+\tilde{\boldsymbol{S}}_2$ and $\tilde{\boldsymbol{\Delta}}=M\left(\frac{\tilde{\boldsymbol{S}}_2}{M_2}-\frac{\tilde{\boldsymbol{S}}_1}{M_1}\right)$. We achieve the angular momentum loss induced by the 2.5PN SO effect for general orbit, which can be written as

$$
\begin{aligned}
\dot{\boldsymbol{J}}_{2.5SO} =\ & \frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{(\boldsymbol{n}\times\boldsymbol{V})\cdot\tilde{\boldsymbol{S}}\Big[(1746-10800\eta)V^4-(106022+24242\eta)\frac{(GM)^2}{R^2}-(87570-163170\eta)\dot{R}^4 \\
& -(20458-7326\eta)\frac{GM}{R}V^2+(76320-121770\eta)V^2\dot{R}^2+(60612-33348\eta)\frac{GM}{R}\dot{R}^2\Big]+\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V}) \\
& \cdot\tilde{\boldsymbol{\Delta}}\Big[(40326-39300\eta)\frac{GM}{R}\dot{R}^2-(16194-15062\eta)\frac{GM}{R}V^2-(48876+13480\eta)\frac{G^2M^2}{R^2} \\
& +(1674-6822\eta)V^4+(30240-49320\eta)V^2\dot{R}^2-(34650-71190\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{n}\Big\{(\boldsymbol{n}\cdot\tilde{\boldsymbol{S}}) \\
& \times\Big[(5064+3792\eta)\frac{(GM)^3}{R^3}-(65118-101184\eta)\frac{GM}{R}V^2\dot{R}^2-(36782-12850\eta)\frac{(GM)^2}{R^2}V^2 \\
& +(62988-5160\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(4032-8064\eta)V^6+(48123-101916\eta)\frac{GM}{R}\dot{R}^4+(13769-16314\eta) \\
& \times\frac{GM}{R}V^4+(35280-70560\eta)V^4\dot{R}^2-(35280-70560\eta)V^2\dot{R}^4\Big]+\frac{\delta M}{M}(\boldsymbol{n}\cdot\tilde{\boldsymbol{\Delta}})\Big[(25290+5346\eta)\frac{(GM)^2}{R^2}\dot{R}^2 \\
& -(15270-3170\eta)\frac{(GM)^2}{R^2}V^2-(2280-2940\eta)\frac{(GM)^3}{R^3}+(8034-7567\eta)\frac{GM}{R}V^4-(31143-44670\eta) \\
& \times\frac{GM}{R}V^2\dot{R}^2+(18009-46449\eta)\frac{GM}{R}\dot{R}^4-(4032-4032\eta)V^6+(35280-35280\eta)V^4\dot{R}^2-(35280-35280\eta)V^2\dot{R}^4\Big] \\
& +(\boldsymbol{V}\cdot\tilde{\boldsymbol{S}})\dot{R}\Big[(46570-4842\eta)\frac{GM}{R}V^2-(78072-8868\eta)\frac{GM}{R}\dot{R}^2+(57510-219420\eta)V^2\dot{R}^2 \\
& -(26289-89172\eta)V^4+(36434-1780\eta)\frac{(GM)^2}{R^2}-(31185-126630\eta)\dot{R}^4\Big]+\frac{\delta M}{M}(\boldsymbol{V}\cdot\tilde{\boldsymbol{\Delta}})\dot{R}\Big[(12192+760\eta) \\
& \times\frac{(GM)^2}{R^2}+(24123+610\eta)\frac{GM}{R}V^2-(30939+1560\eta)\frac{GM}{R}\dot{R}^2-(14265-40995\eta)V^4+(15390-93510\eta)V^2\dot{R}^2 \\
& +(4095+49455\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{V}\Big\{(\boldsymbol{n}\cdot\tilde{\boldsymbol{S}})\dot{R}\Big[(25576-64002\eta)\frac{GM}{R}V^2-(29974-1166\eta)\frac{(GM)^2}{R^2} \\
& -(23046-80328\eta)\frac{GM}{R}\dot{R}^2+(4032-8064\eta)V^4-(35280-70560\eta)V^2\dot{R}^2+(35280-70560\eta)\dot{R}^4\Big]-(\boldsymbol{V}\cdot\tilde{\boldsymbol{S}}) \\
& \times\Big[(18730+504\eta)\frac{GM}{R}V^2+(33386-7516\eta)\frac{(GM)^2}{R^2}-(47052-18642\eta)\frac{GM}{R}\dot{R}^2-(8637-30144\eta)V^4 \\
& +(15066-89982\eta)V^2\dot{R}^2-(6345-54180\eta)\dot{R}^4\Big]-\frac{\delta M}{M}\dot{R}(\boldsymbol{n}\cdot\tilde{\boldsymbol{\Delta}})\Big[(6567-41892\eta)\frac{GM}{R}\dot{R}^2 \\
& +(35280-35280\eta)V^2\dot{R}^2-(11859-33338\eta)\frac{GM}{R}V^2-(35280-35280)\dot{R}^4+(12492+1772\eta)\frac{(GM)^2}{R^2} \\
& -(4032-3032\eta)V^4\Big]+\frac{\delta M}{M}(\boldsymbol{V}\cdot\tilde{\boldsymbol{\Delta}})\Big[(18075-9882\eta)\frac{GM}{R}\dot{R}^2-(14088-3956\eta)\frac{(GM)^2}{R^2}-(8835+400\eta)\frac{GM}{R}V^2
\end{aligned}
$$

$$+(3885-15759\eta)V^4+(4462+45918\eta)V^2\dot{R}^2-(13815+25515\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\Big\{\tilde{\boldsymbol{S}}\Big[(3901+31614\eta)\frac{GM}{R}V^4$$
$$-(73736-5218\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(5064+3792\eta)\frac{(GM)^3}{R^3}+(54591+37932\eta)\frac{GM}{R}\dot{R}^4-(8637-30144\eta)V^6$$
$$-(54796+53706\eta)\frac{GM}{R}V^2\dot{R}^2+(74456-19882\eta)\frac{(GM)^2}{R^2}V^2+(41355-179154\eta)V^4\dot{R}^2-(63855-273600\eta)V^2\dot{R}^4$$
$$+(31385-126630\eta)\dot{R}^6\Big]+\frac{\delta M}{M}\tilde{\boldsymbol{\Delta}}\Big[(2280-2940\eta)\frac{(GM)^3}{R^3}+(33564-6890\eta)\frac{(GM)^2}{R^2}V^2$$
$$-(29196+4570\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(22857+20109\eta)\frac{GM}{R}\dot{R}^4+(1233+15751\eta)\frac{GM}{R}V^4-(26724+23836\eta)$$
$$\times\frac{GM}{R}V^2\dot{R}^2-(3885-15759\eta)V^6+(9603-86913\eta)V^4\dot{R}^2-(1575-119025\eta)V^2\dot{R}^4-(4095+49455\eta)\dot{R}^6\Big]\Big\}\,. \tag{B1}$$

For the case of circular orbit, we have

$$\dot{\boldsymbol{J}}_{2.5SO}=\frac{32}{5}\frac{Mc^2\eta^2}{GM^2}\tilde{x}^6\Big\{\Big[\Big(\frac{95}{28}+\frac{239}{63}\eta\Big)\boldsymbol{l}\cdot\tilde{\boldsymbol{S}}+\frac{\delta M}{M}\Big(\frac{31}{16}-\frac{109}{28}\eta\Big)\boldsymbol{l}\cdot\tilde{\Delta}\Big]\boldsymbol{l}+\Big[\Big(\frac{4471}{224}+\frac{5911}{252}\eta\Big)\boldsymbol{n}\cdot\tilde{\boldsymbol{S}}$$
$$+\Big(\frac{383}{42}+\frac{1175}{96}\eta\Big)\frac{\delta M}{M}\boldsymbol{n}\cdot\tilde{\Delta}\Big]\boldsymbol{n}+\Big[\Big(\frac{5323}{672}+\frac{149}{18}\eta\Big)\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{S}}+\Big(\frac{229}{48}+\frac{1221}{224}\eta\Big)\frac{\delta M}{M}\boldsymbol{\lambda}\cdot\text{="707E}\tilde{\Delta}\Big]\boldsymbol{\lambda}\Big\}\,. \tag{B2}$$

where the parameter $\tilde{x}=\left(\frac{GM\tilde{\omega}}{c^3}\right)^{\frac{2}{3}}$ can be found in reference [30].

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