Predictions for the isospin-violating decays of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ *

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Abstract: In this study, we investigate the isospin-violating decays of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$, which may provide additional information for the determination of the properties of $B_c(1P)^+$, the first orbital excitation states of the B_c meson. By assuming a dual relation between the $U(1)$ anomaly soft-gluon coupling for $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ and the intermediate meson loop transitions, we can quantify the isospin-violating decay effects for these four P-wave states. We observe that the partial decay width of $B_{c0}^{*+} \to B_c^+\pi^0$ is approximately three orders of magnitude larger than that for $B_{c2}^{*+} \to B_c^+\pi^0$. This indicates that B_{c0}^{*+} can be established in the $B_c^+\pi^0$ decay channel as a single state. Meanwhile, the two axial-vector states $B_{c1}^+/B_{c1}'^+$ can be possibly identified in $B_{c1}^+/B_{c1}'^+ \to B_c^{*+}\pi^0$ with comparable strengths. Although these isospin-violating decays are observed to be small, the theoretical predictions should be useful for guiding future experimental efforts.

Keywords: isospin-violating decay, heavy flavor meson, long-distance mechanism

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I. INTRODUCTION

As the only meson containing two different heavy quarks, the B_c meson provides a unique perspective for studying the properties of the non-perturbative regime of quantum chromodynamics (QCD). The mass spectrum of B_c mesons has been predicted by several theoretical approaches [1−13]. However, experimentally, only the ground state B_c^+ [14] and the first radial excitation $B_c(2S)^+$ [15−18] have been observed so far. In addition, the mass and decay modes of B_c^+ have been measured with high precision [19−26]. However, for excited B_c states beyond the ground state, experimental information on their decay modes remains scarce.

Recently, the first observation of $B_c(1P)^+$, the first orbital excitation of the B_c meson, has been reported by the LHCb Collaboration in the invariant mass spectrum of the $B_c^+\gamma$ final state [27, 28]. The $B_c(1P)^+$ multiplet consists of four physical states: $B_{c0}^{*+}(1^3P_0)$, $B_{c1}^+(1P_1)$, $B_{c1}'^+(1P_1')$, and $B_{c2}^{*+}(1^3P_2)$, where $1P_1$ and $1P_1'$ are mixtures of the 1^1P_1 and 1^3P_1 states [13]. All the four states can decay into $B_c^{*+}\gamma$ with $B_c^{*+} \to B_c^+\gamma$ to be detected in the final state. Meanwhile, the $1P_1$ and $1P_1'$ states can reach the $B_c^+\gamma$ final state directly through an S-wave transition. However, owing to the limited statistics and energy resolution, only one photon associated with the observed two peaks in the $B_c^+\gamma$ spectrum is effective. Namely, LHCb only measured a B_c^+ plus a photon in the final state. Thus, it is likely that the data contain contributions from all the four states.

In addition to the radiative decays, by analogy with the isospin-violating decays of D_s^{*+} [29, 30], the $B_c(1P)^+$ states can decay to the final states $B_c^{(*)+}\pi^0$ via isospin-violating processes ($B_{c0}^{*+}/B_{c2}^{*+} \to B_c^+\pi^0$, $B_{c1}^+/B_{c1}'^+ \to B_c^{*+}\pi^0$). Although B_c^{*+} has not yet been observed experimentally, it is rather reliable that the B_c^{*+} mass is below the $B_c^+\pi^0$ threshold through theoretical studies [13]. Therefore, the decays of $B_{c0}^{*+}/B_{c2}^{*+} \to B_c^+\pi^0$ and $B_{c1}^+/B_{c1}'^+ \to B_c^{*+}\pi^0$ can provide more information about these four P-wave states.

In this study, we employ the effective Lagrangian approach to calculate the decay widths of the isospin-violating processes $B_c(1P)^+ \to B_c^{(*)+}\pi^0$. At the quark level, the isospin-violating decay $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ can be described by the soft-gluon coupling to the light $u\bar{u}$ and $d\bar{d}$ via the $U(1)$ anomaly term. Meanwhile, as π^0 is an isov-

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ector with the flavor wavefunction $(u\bar{u}-d\bar{d})/\sqrt{2}$, the two terms will cancel each other. Considering the property of the $U(1)$ anomaly coupling, which is proportional to the quark mass and only partially conserves the axial current, the isospin-violating coupling is proportional to the term m_d-m_u, which will be convoluted by the transition form factor for $B_c(1P)^+ \to B_c^{(*)+}$. According to the concept of quark-hadron duality [31], this process can be interpreted as the $B_c(1P)^+$ meson producing intermediate $\mathcal{B}^{(*)}$ and $\mathcal{D}^{(*)}$ states, which then rescatter via the exchange of a $\mathcal{D}^{(*)}$ meson to produce the $B_c^{(*)+}\pi^0$ final state. The two loop amplitudes involving the intermediate mesons with u and d can have a constructive phase to produce the η meson or a destructive phase to produce π^0 owing to isospin conservation. This mechanism can be quantified by the effective Lagrangian approach, where the vertex couplings can be determined by the $^3P_0$ model [32–34]. This method has been broadly applied to the studies of various Okubo-Zweig-Iizuka (OZI) rule [35] evading processes [36–38], isospin symmetry breaking processes [29, 38–42], and helicity selection rule violation processes [36, 43, 44] in the literature.

The remainder of this paper is organized as follows. We first introduce our formalism in Sec. II. The numerical results and discussions are presented in Sec. III. A brief summary and conclusion are given in Sec. IV.

II. FORMALISM

In this section, we first present the effective Lagrangians, and derive the loop amplitudes for $B_c(1P)^+ \to B_c^{(*)+}\pi^0$. We then determine the coupling constants appearing in the effective Lagrangians within the framework of the $^3P_0$ model [32].

A. Effective Lagrangians

The effective Lagrangians for the B_{cJ} couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ are as follows [45, 46]:

$$\mathcal{L}_{B_c} = \mathrm{i}g_{B_c\mathcal{B}^*\mathcal{D}} B_c \mathcal{B}^{*i}_\mu \partial^\mu \mathcal{D}_i + \mathrm{i}g_{B_c\mathcal{B}\mathcal{D}^*} B_c \partial^\mu \mathcal{B}_i \mathcal{D}^{*i}_\mu + g_{B_c\mathcal{B}^*\mathcal{D}^*}\varepsilon_{\mu\nu\rho\sigma} B_c \partial^\mu \mathcal{B}^{*i\nu}\partial^\rho \mathcal{D}^{*\sigma}_i + \text{h.c.}, \tag{1}$$

$$\mathcal{L}_{B^*_{c0}} = g_{B^*_{c0}\mathcal{B}\mathcal{D}} B^*_{c0}\mathcal{B}^i\mathcal{D}_i + g_{B^*_{c0}\mathcal{B}^*\mathcal{D}^*} B^*_{c0}\mathcal{B}^{*i}_\mu \mathcal{D}^{*\mu}_i + \text{h.c.}, \tag{2}$$

$$\mathcal{L}_{B^*_{c2}} = g_{B^*_{c2}\mathcal{B}\mathcal{D}}\partial_\mu \mathcal{B}^i \partial_\nu \mathcal{D}_i B^{*\mu\nu}_{c2} + g_{B^*_{c2}\mathcal{B}^*\mathcal{D}^*} B^{*\mu\nu}_{c2}\mathcal{B}^{*i}_\mu \mathcal{D}^*_{i\nu} + \text{h.c.}, \tag{3}$$

$$\begin{aligned}\mathcal{L}_{B^*_c} = {} & \mathrm{i}g_{B^*_c\mathcal{B}\mathcal{D}} B^{*\mu}_c \partial_\mu \mathcal{B}_i \mathcal{D}^i + g_{B^*_c\mathcal{B}^*\mathcal{D}}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu B^{*\nu}_c \partial^\rho \mathcal{B}^{*\sigma}_i \mathcal{D}^i \\ & + g_{B^*_c\mathcal{B}\mathcal{D}^*}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu B^{*\nu}_c\partial^\rho\mathcal{D}^{*\sigma}_i \mathcal{B}^i \\ & + \mathrm{i}g_{B^*_c\mathcal{B}^*\mathcal{D}^*}(\partial^\mu B^{*\nu}_c - \partial^\nu B^{*\mu}_c)\mathcal{B}^*_{i\mu}\mathcal{D}^{*i}_\nu + \text{h.c.},\end{aligned} \tag{4}$$

$$\begin{aligned}\mathcal{L}_{B_{c1}} = {} & g_{B_{c1}\mathcal{B}\mathcal{D}^*} B^\mu_{c1}\mathcal{D}^{*i}_\mu \mathcal{B}_i + g_{B_{c1}\mathcal{B}^*\mathcal{D}} B^\mu_{c1}\mathcal{B}^{*i}_\mu\mathcal{D}_i \\ & + \mathrm{i}g_{B_{c1}\mathcal{B}^*\mathcal{D}^*}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu B^\nu_{c1}\mathcal{B}^{*\rho}_i\mathcal{D}^{*i\sigma} + \text{h.c.},\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{L}_{B'_{c1}} = {} & g_{B'_{c1}\mathcal{B}\mathcal{D}^*} B'^\mu_{c1}\mathcal{D}^{*i}_\mu \mathcal{B}_i + g_{B'_{c1}\mathcal{B}^*\mathcal{D}} B'^\mu_{c1}\mathcal{B}^{*i}_\mu\mathcal{D}_i \\ & + \mathrm{i}g_{B'_{c1}\mathcal{B}^*\mathcal{D}^*}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu B'^\nu_{c1}\mathcal{B}^{*\rho}_i\mathcal{D}^{*i\sigma} + \text{h.c.},\end{aligned} \tag{6}$$

where $\mathcal{B} = (B^+, B^0, B^0_s)$ and $\mathcal{D} = (D^0, D^+, D^+_s)$. The coupling constants for the $B^{(*)}_{c(0,1,2)}\mathcal{B}^{(*)}\mathcal{D}$ vertices will be calculated using the $^3P_0$ model [32, 47].

The effective Lagrangian for the $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}$ coupling is given by [48, 49]

$$\begin{aligned}\mathcal{L}_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}} = {} & -\mathrm{i}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}(\mathcal{D}^i\partial^\mu \mathcal{P}_{ij}\mathcal{D}^{*j\dagger}_\mu - \mathcal{D}^{*i}_\mu\partial^\mu\mathcal{P}_{ij}\mathcal{D}^{j\dagger}) \\ & + \frac{1}{2} g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}}\varepsilon_{\mu\nu\alpha\beta}\mathcal{D}^{*\mu}_i\partial^\nu\mathcal{P}^{ij}\overleftrightarrow{\partial}^\alpha \mathcal{D}^{*\beta\dagger}_j,\end{aligned} \tag{7}$$

where

$$\mathcal{P} \equiv \begin{pmatrix} \frac{\sin\alpha_P\eta'+\cos\alpha_P\eta+\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\sin\alpha_P\eta'+\cos\alpha_P\eta-\pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \cos\alpha_P\eta'-\sin\alpha_P\eta \end{pmatrix}, \tag{8}$$

with $\alpha_P = 40.6^\circ$ adopted for the $\eta-\eta'$ mixing angle in the SU(3) flavor basis, and the value is an average obtained from the Particle Data Group (PDG) [50].

The coupling constants for the effective interaction vertices $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\pi^0$ can be obtained from heavy quark effective theory [41, 48, 49, 51]:

$$g_{D^*D^*\pi} = \frac{g_{D^*D\pi}}{\sqrt{m_D m_{D^*}}} = \frac{2}{f_\pi}g,\quad g_{D^{(*)}_s D^{(*)}_s \mathcal{P}} = \sqrt{\frac{m_{D^{(*)}_s} m_{D^{(*)}_s}}{m_{D^{(*)}} m_{D^{(*)}}}}\, g_{D^{(*)}D^{(*)}\pi} \tag{9}$$

where $g = 0.59$, and $f_\pi = 132\,\text{MeV}$ is the π decay constant.

Owing to the $\eta-\pi^0$ mixing effect, we must consider the contribution from the isospin-conserving transition $\mathcal{D}^{(*)} \to \mathcal{D}^{(*)}\eta$ followed by $\eta \to \pi^0$ to violate the isospin symmetry. For the $\eta \to \pi^0$ process, we use the $\eta-\pi^0$ mixing angle $\theta_{\eta\pi^0}$, which is given by the leading order chiral expansion [52]

$$\tan(2\theta_{\eta\pi^0}) = \frac{\sqrt{3}}{2}\frac{m_d-m_u}{m_s-\hat{m}}. \tag{10}$$

where $\hat{m} = (m_u+m_d)/2$. The values of m_u, m_d, and m_s are taken from the PDG [50]. As $\theta_{\eta\pi^0}$ is very small, we consider

$$\theta_{\eta\pi^0} \simeq \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-\hat{m}}, \tag{11}$$

as broadly adopted in the literature.

B. Loop transition amplitudes of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$

Based on the effective Lagrangians given above, we can derive the corresponding loop amplitudes.

For the loop-level amplitudes without the $\eta-\pi^0$ mixing corresponding to Fig. 1, we denote the amplitude as $i\mathcal{M}_k(P_1,P_2,P_3)$, where $k=0,2,1$, and $1'$ correspond to $B_{c0}^{*+}, B_{c2}^{*+}, B_{c1}^{+}$, and $B_{c1}^{\prime +}$, respectively. Here, P_i represents the intermediate meson with momentum q_i, and $B_{\mu\nu}$ denotes the polarization tensor of the initial B_{c2}^{*+}. ε_μ^i denotes the polarization vector of the initial $B_{c1}^{(\prime)+}$, and ε_μ^f denotes the polarization vector of the final B_c^{*+}. Thus, the amplitudes corresponding to Fig. 1 can be expressed as

$$i\mathcal{M}_0(\mathcal{B},\mathcal{D},\mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c0}^*\mathcal{B}\mathcal{D}}g_{B_c\mathcal{B}\mathcal{D}^*}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}q_1^\mu\left(g_{\mu\nu}-\frac{q_{3\mu}q_{3\nu}}{m_3^2}\right)p_3^\nu}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)}\mathcal{F}(q_i^2), \tag{12}$$

$$i\mathcal{M}_0(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c0}^*\mathcal{B}^*\mathcal{D}^*}g_{B_c\mathcal{B}^*\mathcal{D}}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}\left(g^{\mu\nu}-\frac{q_1^\mu q_1^\nu}{m_1^2}\right)\left(g_{\mu\alpha}-\frac{q_{2\mu}q_{2\alpha}}{m_2^2}\right)q_{3\nu}p_3^\alpha}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)}\mathcal{F}(q_i^2), \tag{13}$$

$$\begin{aligned} i\mathcal{M}_0(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}^*) = &\int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c0}^*\mathcal{B}^*\mathcal{D}^*}g_{B_c\mathcal{B}^*\mathcal{D}^*}g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}}\left(g^{\mu\nu}-\frac{q_1^\mu q_1^\nu}{m_1^2}\right)g_{\mu\alpha}\left(g^{\alpha\beta}-\frac{q_2^\alpha q_2^\beta}{m_2^2}\right)}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \\ &\times\left(g^{\rho\sigma}-\frac{q_3^\rho q_3^\sigma}{m_3^2}\right)\varepsilon_{\nu\delta\rho\kappa}q_1^\delta q_3^\kappa \varepsilon_{\beta\lambda\sigma\tau}q_3^\lambda q_2^\tau \mathcal{F}(q_i^2), \end{aligned} \tag{14}$$

$$i\mathcal{M}_2(\mathcal{B},\mathcal{D},\mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c2}^*\mathcal{B}\mathcal{D}}g_{B_c\mathcal{B}\mathcal{D}^*}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}B_{\alpha\beta}q_1^\alpha q_2^\beta\left(g_{\mu\nu}-\frac{q_{3\mu}q_{3\nu}}{m_3^2}\right)q_1^\mu p_3^\nu}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)}\mathcal{F}(q_i^2), \tag{15}$$

$$i\mathcal{M}_2(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c2}^*\mathcal{B}^*\mathcal{D}^*}g_{B_c\mathcal{B}^*\mathcal{D}}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}B_{\mu\alpha}\left(g^{\mu\nu}-\frac{q_1^\mu q_1^\nu}{m_1^2}\right)\left(g^{\alpha\beta}-\frac{q_2^\alpha q_2^\beta}{m_2^2}\right)q_{3\nu}p_{3\beta}}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)}\mathcal{F}(q_i^2), \tag{16}$$

$$\begin{aligned} i\mathcal{M}_2(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}^*) = &\int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c2}^*\mathcal{B}^*\mathcal{D}^*}g_{B_c\mathcal{B}^*\mathcal{D}^*}g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}}B_{\mu\alpha}\left(g^{\mu\nu}-\frac{q_1^\mu q_1^\nu}{m_1^2}\right)\left(g^{\alpha\beta}-\frac{q_2^\alpha q_2^\beta}{m_2^2}\right)}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \\ &\times(g^{\rho\sigma}-\frac{q_3^\rho q_3^\sigma}{m_3^2})\varepsilon_{\nu\delta\rho\kappa}q_1^\delta q_3^\kappa \varepsilon_{\beta\lambda\sigma\tau}q_3^\lambda q_2^\tau \mathcal{F}(q_i^2), \end{aligned} \tag{17}$$

$$i\mathcal{M}_{1^{(\prime)}}(\mathcal{B},\mathcal{D}^*,\mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4}\frac{g_{B_{c1}^{(\prime)}\mathcal{B}\mathcal{D}^*}g_{B_c^*\mathcal{B}\mathcal{D}}g_{\mathcal{D}^*\mathcal{D}\mathcal{P}}\varepsilon_\mu^i\left(g^{\mu\nu}-\frac{q_2^\mu q_2^\nu}{m_2^2}\right)\varepsilon_\rho^f q_1^\rho p_{3\nu}}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)}\mathcal{F}(q_i^2), \tag{18}$$

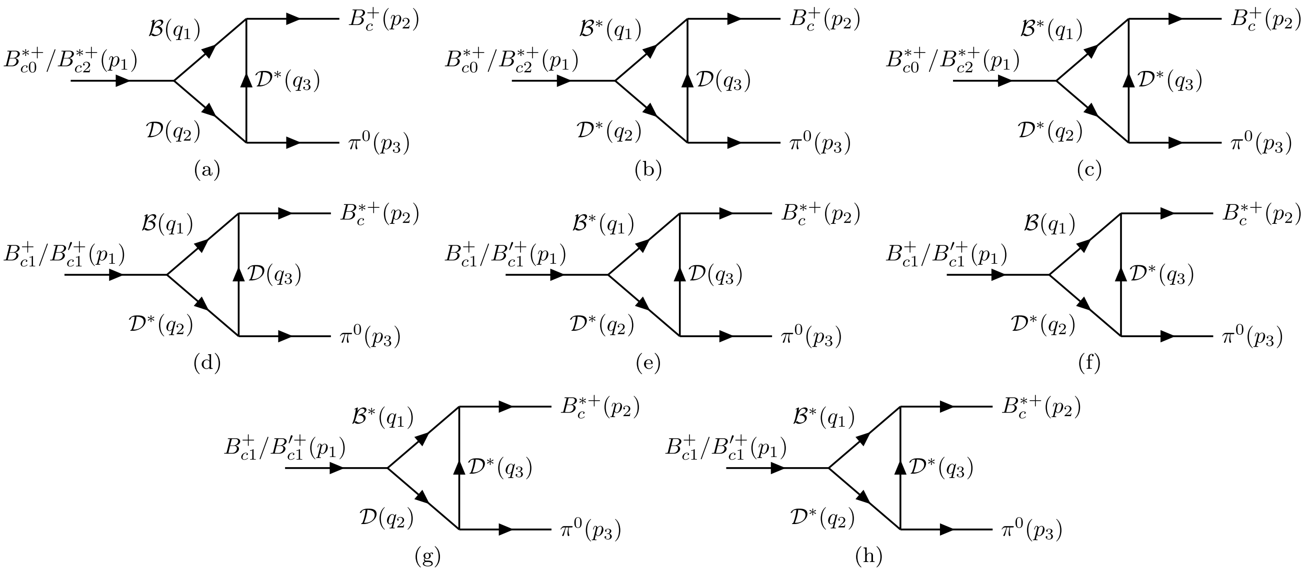

Fig. 1. Schematics of the decay $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ via intermediate meson loops without $\eta-\pi^0$ mixing, where $\mathcal{B} = (B^+, B^0)$ and $\mathcal{D} = (D^0, D^+)$. (a)–(h) illustrate different intermediate meson loops.

$$\begin{aligned} \mathrm{i}\mathcal{M}_{1^{(\prime)}}(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}) = &\int \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(\prime)}\mathcal{B}^*\mathcal{D}^*} g_{B_c^*\mathcal{B}^*\mathcal{D}} g_{\mathcal{D}^*\mathcal{D}\mathcal{P}} \varepsilon_{\mu\nu\rho\sigma} p_1^\mu \varepsilon_i^\nu \left(g^{\rho\alpha} - \dfrac{q_1^\rho q_1^\alpha}{m_1^2}\right)}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \\ &\times \left(g^{\sigma\beta} - \frac{q_2^\sigma q_2^\beta}{m_2^2}\right) \varepsilon_{\delta\lambda\kappa\alpha} p_2^\delta \varepsilon_f^\lambda q_1^\kappa p_{3\beta} \mathcal{F}(q_i^2), \end{aligned} \tag{19}$$

$$\mathrm{i}\mathcal{M}_{1^{(\prime)}}(\mathcal{B},\mathcal{D}^*,\mathcal{D}^*) = \int \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(\prime)}\mathcal{B}\mathcal{D}^*} g_{B_c^*\mathcal{B}\mathcal{D}^*} g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}} \varepsilon_\mu^i \left(g^{\mu\nu} - \dfrac{q_2^\mu q_2^\nu}{m_2^2}\right) \varepsilon_{\rho\sigma\alpha\beta} p_2^\rho \varepsilon_f^\sigma p_3^\alpha \left(g^{\beta\delta} - \dfrac{q_3^\beta q_3^\delta}{m_3^2}\right) \varepsilon_{\kappa\lambda\delta\nu} q_3^\kappa p_3^\lambda}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \mathcal{F}(q_i^2), \tag{20}$$

$$\mathrm{i}\mathcal{M}_{1^{(\prime)}}(\mathcal{B}^*,\mathcal{D},\mathcal{D}^*) = \int \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(\prime)}\mathcal{B}^*\mathcal{D}} g_{B_c^*\mathcal{B}^*\mathcal{D}^*} g_{\mathcal{D}^*\mathcal{D}\mathcal{P}} \varepsilon_\mu^i \left(g^{\mu\nu} - \dfrac{q_1^\mu q_1^\nu}{m_1^2}\right) (p_{2\nu}\varepsilon_\rho^f - p_{2\rho}\varepsilon_\nu^f) \left(g^{\rho\sigma} - \dfrac{q_3^\rho q_3^\sigma}{m_3^2}\right) p_{3\sigma}}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \mathcal{F}(q_i^2), \tag{21}$$

$$\begin{aligned} \mathrm{i}\mathcal{M}_{1^{(\prime)}}(\mathcal{B}^*,\mathcal{D}^*,\mathcal{D}^*) = &\int \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(\prime)}\mathcal{B}^*\mathcal{D}^*} g_{B_c\mathcal{B}^*\mathcal{D}^*} g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}} \varepsilon_{\mu\nu\rho\sigma} p_1^\mu \varepsilon_i^\nu \left(g^{\rho\alpha} - \dfrac{q_1^\rho q_1^\alpha}{m_1^2}\right)\left(g^{\sigma\beta} - \dfrac{q_2^\sigma q_2^\beta}{m_2^2}\right)}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \\ &\times (p_{2\alpha}\varepsilon_\delta^f - p_{2\delta}\varepsilon_\alpha^f)(g^{\delta\lambda} - \frac{q_3^\delta q_3^\lambda}{m_3^2}) \varepsilon_{\kappa\tau\beta\lambda} p_3^\kappa q_3^\tau \mathcal{F}(q_i^2). \end{aligned} \tag{22}$$

In the above equations, $\mathcal{F}(q_i^2)$ is a form factor adopted for cutting off the ultraviolet divergence in the loop integrals, and has the following form:

$$\mathcal{F}(q_i^2) = \prod_i \left(\frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 - q_i^2}\right), \tag{23}$$

where $\Lambda_i \equiv m_i + \alpha\Lambda_{\rm QCD}$, where m_i is the mass of the i-th internal particle, and the QCD energy scale $\Lambda_{\rm QCD} = 220$ MeV with $\alpha = 1 \sim 2$ as the cutoff parameter [29, 38, 53, 54].

For the loop diagrams involving the $\eta-\pi^0$ mixing corresponding to Fig. 2, the difference from the loop diagrams of Fig. 1 lies in the coupling constants associated

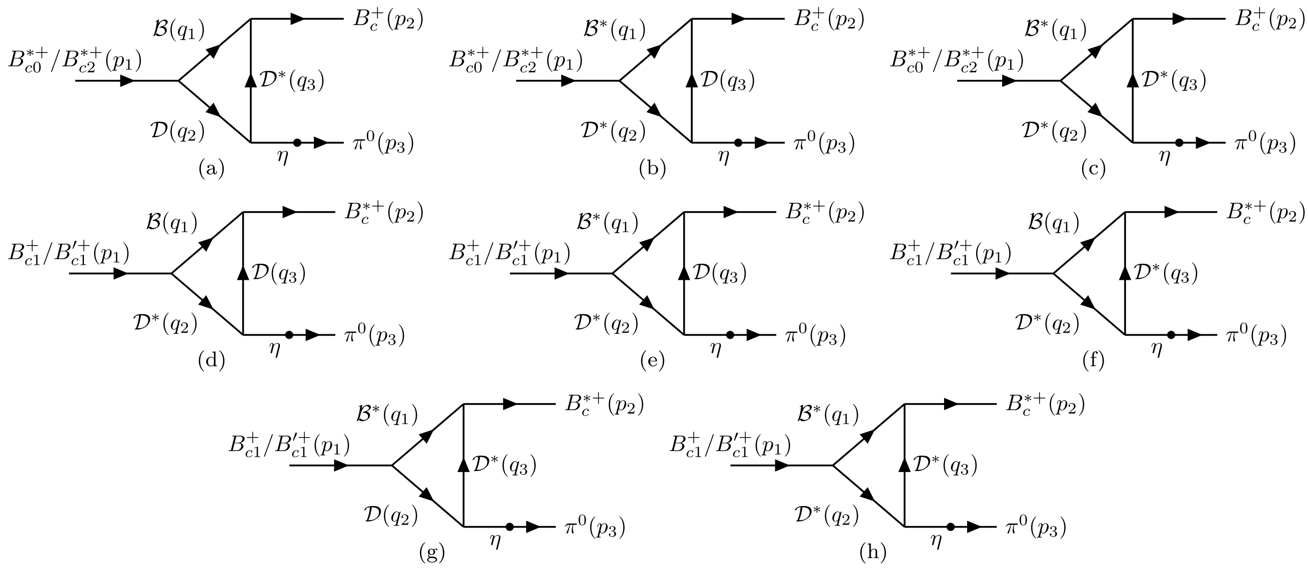

Fig. 2. Schematics of the decay $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ via intermediate meson loops and $\eta-\pi^0$ mixing, where $\mathcal{B}=(B^+,B^0,B_s^0)$ and $\mathcal{D}=(D^0,D^+,D_s^+)$. (a)–(h) illustrate different intermediate meson loops.

with the η and π^0 mesons. Specifically, for these two isospin-related channels, their amplitudes will interfere constructively in the $B_c(1P)^+ \to B_c^{(*)+}\eta$ transition, but destructively in the $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ transition. In addition, for $B_c(1P)^+ \to B_c^{(*)+}\eta$, we must consider the contributions from the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop diagrams. This is because η can be produced via its $s\bar{s}$ component. Therefore, it can be accessed by the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop transitions. In contrast, the loop transition for $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ contains the coupling vertex of $D_s^* \to D_s\pi^0$, which is isospin-violating and will be suppressed at the one-loop level. Thus, we only include the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop diagrams in those processes where the η meson is produced in the final state. They will then contribute to $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ via the $\eta-\pi^0$ mixing. We denote the amplitude as $\mathrm{i}\mathcal{M}_{0,2,1,1'}(P_1,P_2,P_3,\eta)$, and for each process in Fig. 2, their explicit expressions are given by

$$\begin{aligned}&\mathrm{i}\mathcal{M}_k(\mathcal{B}^{(*)},\mathcal{D}^{(*)},\mathcal{D}^{(*)},\eta)\\ &=\mathrm{i}\mathcal{M}_k(\mathcal{B}^{(*)},\mathcal{D}^{(*)},\mathcal{D}^{(*)})\frac{g_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\eta}\theta_{\eta\pi^0}}{g_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\pi}},\\ &k=0,2,1,1'.\end{aligned}\tag{24}$$

From the above amplitudes, we can obtain the total loop amplitude by summing up the contributions from the different loop diagrams. The decay $B_{c0}^{*+} \to B_c^+\pi^0$ is an SPP type decay process, the decay $B_{c2}^{*+} \to B_c^+\pi^0$ is a TPP type decay process, and the decay $B_{c1}^{(\prime)+} \to B_c^{*+}\pi^0$ is an AVP type decay process; hence, we can parametrize the total amplitude as

$$\begin{aligned}&\mathrm{i}\mathcal{M}_{B_{c0}^{*+}}=\mathrm{i}g_{B_{c0}^{*+}B_c^+\pi^0}m_{B_{c0}^*},\quad \mathrm{i}\mathcal{M}_{B_{c2}^{*+}}=\mathrm{i}\frac{g_{B_{c2}^{*+}B_c^+\pi^0}}{m_{B_{c2}^*}}B_{\mu\nu}p_2^\mu p_3^\nu,\\ &\mathrm{i}\mathcal{M}_{B_{c1}^{(\prime)+}}=\mathrm{i}g_{B_{c1}^{(\prime)+}B_c^{*+}\pi^0}m_{B_{c1}^{(\prime)+}}\varepsilon_\mu^i\varepsilon^{f\mu}.\end{aligned}\tag{25}$$

Then, the corresponding partial decay width is

$$\begin{aligned}&\Gamma(B_{c0}^{*+}\to B_c^+\pi^0)=\frac{g^2_{B_{c0}^{*+}B_c^+\pi^0}|\mathbf{p}_\pi|}{8\pi},\\ &\Gamma(B_{c2}^{*+}\to B_c^+\pi^0)=\frac{g^2_{B_{c2}^{*+}B_c^+\pi^0}|\mathbf{p}_\pi|^5}{60\pi m^4_{B_{c2}^{*+}}},\\ &\Gamma(B_{c1}^{(\prime)+}\to B_c^{*+}\pi^0)=\left(3+\frac{|\mathbf{p}_\pi|^2}{m^2_{B_c^{*+}}}\right)\frac{g^2_{B_{c1}^{(\prime)+}B_c^{*+}\pi^0}|\mathbf{p}_\pi|}{24\pi},\end{aligned}\tag{26}$$

where $|\mathbf{p}_\pi|$ is the momentum of the π^0 in the rest frame of the initial particle.

C. Coupling constants of $B_{c(0,1,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ vertices

We calculate the amplitudes for the processes $B_{c(0,1,2)}^{(*)} \to \mathcal{B}^{(*)}\mathcal{D}^{(*)}$ using the ${}^3P_0$ model, and match them with those obtained from the effective Lagrangian approach. Thus, the coupling constants in the effective Lagrangian can be determined.

The corresponding transition operator in the ${}^3P_0$ model is given by [32–34, 47, 55]

$$\begin{aligned}T=&-3\gamma\sum_m\langle 1,m;1,-m\,|\,0,0\rangle\int \mathrm{d}^3\mathbf{p}_3 d^3\mathbf{p}_4\,\delta^3(\mathbf{p}_3+\mathbf{p}_4)\\ &\times\mathcal{Y}_1^m\left(\frac{\mathbf{p}_3-\mathbf{p}_4}{2}\right)\chi^{34}_{1,-m}\phi_0^{34}\omega_0^{34}b^\dagger_{3i}(\mathbf{p}_3)d^\dagger_{4j}(\mathbf{p}_4),\end{aligned}\tag{27}$$

where i and j are color indices, $\chi^{34}_{1,-m}$ is the spin wave function, ϕ_0^{34} is the flavor singlet wave function, $\omega_0^{34} = \delta_{ij}/\sqrt{3}$ is the color singlet wave function, and $\mathcal{Y}_1^m(\mathbf{p}) = |\mathbf{p}|Y_1^m(\theta,\phi)$.

The meson wave function is given by [56]

$$|M(\mathbf{P},J,J_z)\rangle = \sum_{S_z,L_z,c_i} \langle L,L_z;S,S_z \mid J,J_z\rangle \times \int \mathrm{d}^3\mathbf{p}_1 d^3\mathbf{p}_2\, \delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{P})\psi_{N,L,L_z}(\mathbf{p}_1,\mathbf{p}_2) \times \frac{\delta_{c_1c_2}}{\sqrt{3}}\phi_{f_1,f_2}\chi^{S,S_z}_{s_1,s_2}\, b^\dagger_{c_1,f_1,s_1,\mathbf{p}_1}\, d^\dagger_{c_2,f_2,s_2,\mathbf{p}_2}|0\rangle, \tag{28}$$

where c_j, s_j, and f_j $(j=1,2)$ denote the color, spin, and flavor indices of the quark, respectively. $b^\dagger$ and $d^\dagger$ are the creation operators for quarks and antiquarks, respectively, and ψ_{N,L,L_z} is the spatial wave function in the harmonic oscillator basis.

The spatial wave functions for S-wave and P-wave mesons are given by

$$\psi_{0,0,0}(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{\pi^{3/4}R^{3/2}}\exp\left(-\frac{(\mathbf{p}_1-\mathbf{p}_2)^2}{8R^2}\right), \quad \psi_{0,1,m}(\mathbf{p}_1,\mathbf{p}_2) = \sqrt{\frac{2}{3}}\frac{|\mathbf{p}_1-\mathbf{p}_2|}{\pi^{1/4}R^{5/2}}Y_1^m(\theta,\phi)\exp\left(-\frac{(\mathbf{p}_1-\mathbf{p}_2)^2}{8R^2}\right), \tag{29}$$

where Y_1^m is the spherical harmonic function, and R is the parameter of the meson wave function.

The $B_c(1P)^+$ multiplet contains four physical states: $B_{c0}^{*+}(1^3P_0)$, $B_{c2}^{*+}(1^3P_2)$, $B_{c1}^+(1P_1)$, and $B_{c1}^{\prime+}(1P_1^\prime)$. The B_{c1}^+ and $B_{c1}^{\prime+}$ states are mixtures of the 1^1P_1 and 1^3P_1 configurations, with the mixing defined as

$$\begin{pmatrix} B_{c1}^{\prime+} \\ B_{c1}^+ \end{pmatrix} = \begin{pmatrix} \cos\theta_{1P} & \sin\theta_{1P} \\ -\sin\theta_{1P} & \cos\theta_{1P} \end{pmatrix} \begin{pmatrix} 1^1P_1 \\ 1^3P_1 \end{pmatrix}, \tag{30}$$

where $\theta_{1P} = 22.4°$. Therefore, for $B_{c1}^{(\prime)+}$, the mixing of the wave functions must be considered.

The transition amplitude for $A\to BC$ in the $^3P_0$ model is given by

$$\mathcal{M}_q = \langle BC|T|A\rangle, \tag{31}$$

and the amplitude obtained from the effective Lagrangian approach is denoted as $\mathcal{M}_h$. The relation between the two amplitudes is $\mathcal{M}_h = 8\pi^{3/2}\sqrt{m_A m_B m_C}\mathcal{M}_q$, from which the coupling constants in the effective Lagrangian can be determined. By substituting the wave functions from Eq. (32) into Eq. (35), and comparing the resulting amplitude with that obtained from the effective Lagrangian in Eq. (5), the coupling constants for the $B_c^{(*)}$ meson vertices can be extracted. The explicit expressions for the coupling constants are given in Appendix A.

III. NUMERICAL RESULTS AND DISCUSSIONS

Proceeding to the numerical calculations of the loop contributions for $B_c(1P)^+ \to B_c^{(*)+}\pi^0$, the relevant coupling constants can be calculated using Eqs. (A1), (A4) and (9). The parameters used in the calculations are summarized in Table 1 and the $^3P_0$ model coupling constant $\gamma_{c\bar{b}} = 2.145$ [57] 1). The masses of the relevant particles are taken from the PDG [50], the masses of B_{c0}^{*+} and B_{c2}^{*+} are given by [27, 28]

$$m_{B_{c0}^{*+}} = 6704.8 \pm 5.5 \pm 2.8 \pm 0.3\ \text{MeV}, \quad m_{B_{c2}^{*+}} = 6752.4 \pm 9.5 \pm 3.1 \pm 0.3\ \text{MeV}, \tag{32}$$

and the masses of B_{c1}^+, $B_{c1}^{\prime+}$, and B_c^{*+} are adopted from Ref. [13]: $m_{B_{c1}^+} = 6741$ MeV, $m_{B_{c1}^{\prime+}} = 6750$ MeV, and $m_{B_c^{*+}} = 6338$ MeV.

The explicit values of the coupling constants are given in Tables 2−6.

With the above coupling constants, we can calculate the partial decay widths of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$. In Table 7, we list the calculation results of the partial decay width of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ with five typical cutoff parameter values $\alpha = 1.0$, 1.35, 1.5, 1.65, and 2.0. For the typical value $\alpha = 1.5 \pm 0.15$ [29, 53, 54], we present the predicted partial decay widths of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ in Table 8. The partial width $\Gamma(B_{c0}^{*+} \to B_c^+\pi^0)$ is nearly three orders of magnitude larger than $\Gamma(B_{c2}^{*+} \to B_c^+\pi^0)$. This indicates that the $B_c^+\pi^0$ decay channel can be used to distinguish B_{c0}^{*+} and B_{c2}^{*+} from each other given the sufficient data samples from experiments. Meanwhile, if B_c^+ and B_c^{*+} can be distinguished by the measurement of $B_c^{*+} \to B_c^+\gamma$, it seems that these four states will then be separated. It is likely that, in the $B_c^+\pi^0$ channel, one will only observe one state, i.e., B_{c0}^{*+}, whereas in the $B_c^{*+}\pi^0$ channel, one may observe two peaks for B_{c1}^+ and $B_{c1}^{\prime+}$. However, their structures as the mixture of the 1^1P_1 and 1^3P_1 configurations would rely on the detailed mixing dynamics. Note that the isospin-violating decay process has indeed been significantly suppressed. Such a measurement would be extremely challenging. However, for the future establishment of these four states, such an effort should still be pursued.

To observe more clearly the role played by the intermediate loop transitions, we plot the dependence of the decay width of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ on the cut-off paramet-

1) Our value of γ is higher than that used in [57] by a factor of $\sqrt{24\pi}$ due to the different operator convention and the different normalization of the wave function.

Table 1. Harmonic oscillator strengths for meson wavefunctions involved in the $B^{(*)}_{c(0,1,2)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ couplings adopted from potential quark model studies [12, 13, 58−61].

HO Strength	R_D	R_{D^*}	R_{D_s}	$R_{D_s^*}$	R_B	R_{B^*}	R_{B_s}	$R_{B_s^*}$	R_{B_c}	$R_{B^*_{c0}}$	$R_{B^*_{c2}}$	$R_{B^1_{c1}(^1P_1)}$	$R_{B^3_{c1}(^3P_1)}$
Value/MeV	601	516	651	562	580	542	636	595	1010	760	670	700	710

Table 2. Values of the $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}$ coupling constants.

Coupling constant	$g_{D^{*0}D^{*0}\pi^0}$	$g_{D^{*+}D^{*+}\pi^0}$	$g_{D^0D^{*0}\pi^0}$	$g_{D^+D^{*+}\pi^0}$	$g_{D^{*0}D^{*0}\eta}$
Value	6.32 GeV^{-1}	−6.32 GeV^{-1}	12.23	−12.23	6.16 GeV^{-1}
Coupling constant	$g_{D^{*+}D^{*+}\eta}$	$g_{D^0D^{*0}\eta}$	$g_{D^+D^{*+}\eta}$	$g_{D_s^{*+}D_s^{*+}\eta}$	$g_{D_s^{*+}D_s^+\eta}$
Value	6.16 GeV^{-1}	11.94	11.94	−1.49 GeV^{-1}	−3.03

Table 3. Values of the $B_c^{*+}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_c^{*+}B^+D^0}$	$g_{B_c^{*+}B^{*+}D^0}$	$g_{B_c^{*+}B^+D^{*0}}$	$g_{B_c^{*+}B^{*+}D^{*0}}$	$g_{B_c^{*+}B_s^0D_s^+}$	$g_{B_c^{*+}B_s^{*0}D_s^+}$	$g_{B_c^{*+}B_s^0D_s^{*+}}$	$g_{B_c^{*+}B_s^{*0}D_s^{*+}}$
Value	27.1	4.31 GeV^{-1}	4.49 GeV^{-1}	14.5	27.4	4.37 GeV^{-1}	4.54 GeV^{-1}	14.7

Table 4. Values of the $B^{(*)+}_{c(0,2)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_c^+B^+D^{*0}}$	$g_{B_c^+B^{*+}D^0}$	$g_{B_c^+B^{*+}D^{*0}}$	$g_{B_c^+B_s^0D_s^{*+}}$	$g_{B_c^+B_s^{*0}D_s^+}$	$g_{B_c^+B_s^{*0}D_s^{*+}}$	$g_{B_{c0}^{*+}B^+D^0}$
Value	24.1	23.18	1.82 GeV^{-1}	24.4	23.6	1.85 GeV^{-1}	7.51 GeV
Coupling constant	$g_{B_{c0}^{*+}B^{*+}D^{*0}}$	$g_{B_{c0}^{*+}B_s^0D_s^+}$	$g_{B_{c0}^{*+}B_s^{*0}D_s^{*+}}$	$g_{B_{c2}^{*+}B^+D^0}$	$g_{B_{c2}^{*+}B^{*+}D^{*0}}$	$g_{B_{c2}^{*+}B_s^0D_s^+}$	$g_{B_{c2}^{*+}B_s^{*0}D_s^{*+}}$
Value	4.50 GeV	8.64 GeV	5.23 GeV	3.58 GeV^{-1}	0.365 GeV^{-1}	3.53 GeV^{-1}	0.398 GeV^{-1}

Table 5. Values of the $B_{c1}^+\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_{c1}^+B^+D^{*0}}$	$g_{B_{c1}^+B^{*+}D^{*0}}$	$g_{B_{c1}^+B^{*+}D^0}$	$g_{B_{c1}^+B_s^0D_s^{*+}}$	$g_{B_{c1}^+B_s^{*0}D_s^{*+}}$	$g_{B_{c1}^+B_s^{*0}D_s^+}$
Value	9.13GeV	1.33	9.18 GeV	10.1 GeV	1.49	10.1 GeV

Table 6. Values of the $B_{c1}'^+\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_{c1}'^+B^+D^{*0}}$	$g_{B_{c1}'^+B^{*+}D^{*0}}$	$g_{B_{c1}'^+B^{*+}D^0}$	$g_{B_{c1}'^+B_s^0D_s^{*+}}$	$g_{B_{c1}'^+B_s^{*0}D_s^{*+}}$	$g_{B_{c1}'^+B_s^{*0}D_s^+}$
Value	8.50 GeV	1.24	8.19 GeV	9.39 GeV	1.38	9.40 GeV

Table 7. Partial decay widths of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ with $\alpha = 1.0$, 1.35, 1.5, 1.65, and 2.0 in units of eV.

α	1.0	1.35	1.5	1.65	2.0
$\Gamma(B_{c0}^{*+} \to B_c^+\pi^0)$	0.40	1.42	2.16	3.12	6.35
$\Gamma(B_{c2}^{*+} \to B_c^+\pi^0)(\times 10^{-3})$	0.95	3.79	6.06	9.17	20.7
$\Gamma(B_{c1}^+ \to B_c^{*+}\pi^0)$	0.64	2.35	3.62	5.29	11.1
$\Gamma(B_{c1}'^+ \to B_c^{*+}\pi^0)$	0.60	2.17	3.34	4.88	10.2

Table 8. Predicted partial decay widths of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ for $\alpha = 1.5 \pm 0.15$.

Decay channel	$B_{c0}^{*+} \to B_c^+\pi^0$	$B_{c2}^{*+} \to B_c^+\pi^0$	$B_{c1}^+ \to B_c^{*+}\pi^0$	$B_{c1}'^+ \to B_c^{*+}\pi^0$
Partial decay width	$2.2^{+1.0}_{-0.7}$eV	$(6.1^{+3.1}_{-2.3})\times 10^{-3}$eV	$3.6^{+1.7}_{-1.3}$eV	$3.3^{+1.5}_{-1.2}$eV

er α in Fig. 3. The behavior of the decay widths as a function of the cutoff parameter α is observed to be consistent with $D_s^{*+} \to D_s^+\pi^0$ [29]. To further clarify the contributions from different loop diagrams in the processes $B_c(1P)^+ \to B_c^{(*)+}\pi^0$, we present in Fig. 4 the dependence of the different loop contributions to $g_{B_c(1P)^+B_c^{(*)+}\pi^0}$ on the cutoff parameter α. Note that, for all four processes of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$, the loop diagrams involving the exchange of a $\mathcal{D}^*$ meson and containing a $\mathcal{D}$ meson as an internal line provide the dominant contribution, which is

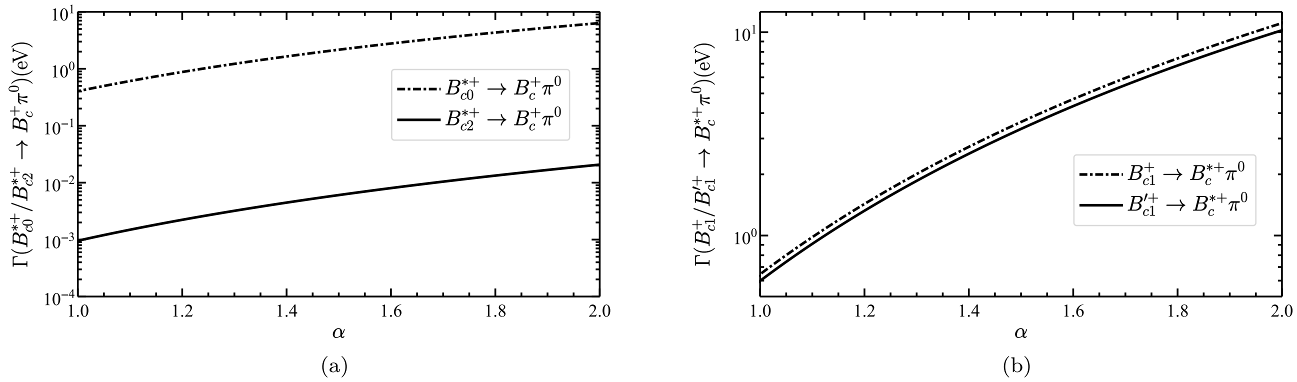

Fig. 3. Partial decay width of $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ as a function of the cutoff parameter α.

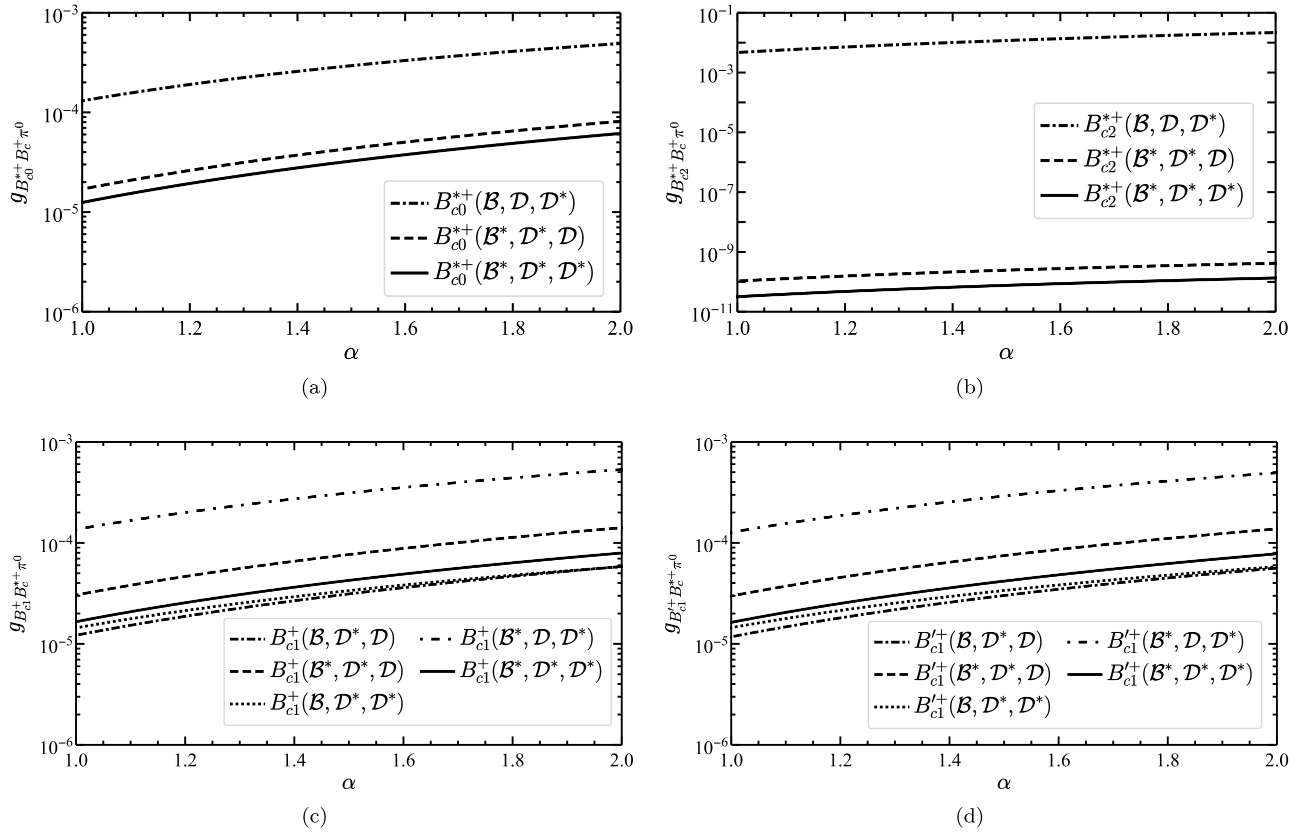

Fig. 4. Dependence of the contributions from different loop diagrams to $g_{B_c(1P)^+B_c^{(*)+}\pi^0}$ on the cutoff parameter α. (a)–(d) correspond to different $B_c(1P)^+$ states.

more than one order of magnitude larger than the other loop contributions. This is particularly pronounced in the case of $B_{c2}^{*+} \to B_c^+\pi^0$.

IV. SUMMARY

In this study, we calculate the partial decay widths for the four isospin-violating decay channels $B_c(1P)^+ \to B_c^{(*)+}\pi^0$ using the effective Lagrangian approach. We assume that the isospin-violating π^0 production via the $U(1)$ anomaly term is dual to the intermediate meson loops involving $\mathcal{B}^{(*)}$ and $\mathcal{D}^{(*)}$ rescatterings. The isospin-violating transition is a consequence of the cancellations between the two kinds of loops involving the intermediate heavy mesons containing $u\bar{u}$ or $d\bar{d}$. In particular, we observe that the loop diagrams with $\mathcal{B}^{(*)}\mathcal{D}$ rescatterings by exchanging a $\mathcal{D}^*$ provide the dominant contributions. Our results show that the partial decay width of $B_{c0}^{*+} \to B_c^+\pi^0$ is approximately three orders of magnitude larger than that for $B_{c2}^{*+} \to B_c^+\pi^0$. This significant difference can be exploited experimentally to distinguish between the B_{c0}^{*+} and B_{c2}^{*+} states in the final state of $B_c^+\pi^0$. Meanwhile, the two axial-vector states $B_{c1}^+/B_{c1}^{\prime+}$ can be possibly identified in $B_{c1}^+/B_{c1}^{\prime+} \to B_c^{*+}\pi^0$ though their structures as the mixture of the 1^1P_1 and 1^3P_1 states will rely on the detailed mixing dynamics. These measurements would be challenging for the present and even future experiments. However, we emphasize that they are crucial for finally establishing these states in experiments.

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APPENDIX A: COUPLING CONSTANTS OF BOTTOM-CHARM MESONS

Explicit expressions for the $B_{c(0,1,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ couplings, which are calculated in the $^3P_0$ model, are listed below. Namely, for the B_c^+ couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$, we have

$$\begin{aligned}
g_{B_c\mathcal{B}^*\mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{B_c}m_{\mathcal{B}^*}m_{\mathcal{D}}}\gamma_{c\bar{b}}(R_{B_c}R_{\mathcal{B}^*}R_{\mathcal{D}})^{3/2}(R_{B_c}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}}^2)+2R_{\mathcal{B}^*}^2R_{\mathcal{D}}^2)}{\sqrt{3}(R_{B_c}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}}^2)^{5/2}},\\
g_{B_c\mathcal{B}\mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{B_c}m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}(R_{B_c}R_{\mathcal{B}}R_{\mathcal{D}^*})^{3/2}(R_{B_c}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}^*}^2)+2R_{\mathcal{B}}^2R_{\mathcal{D}^*}^2)}{\sqrt{3}(R_{B_c}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}}^2R_{\mathcal{D}^*}^2)^{5/2}},\\
g_{B_c\mathcal{B}^*\mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}(R_{B_c}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{3/2}(R_{B_c}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+2R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)}{\sqrt{3m_{B_c}}(R_{B_c}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)^{5/2}}.
\end{aligned}\tag{A1}$$

For the B_c^{*+} couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$, we have

$$\begin{aligned}
g_{B_c^*\mathcal{B}\mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{B_c^*}m_{\mathcal{B}}m_{\mathcal{D}}}\gamma_{c\bar{b}}(R_{B_c^*}R_{\mathcal{B}}R_{\mathcal{D}})^{3/2}(R_{B_c^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)+2R_{\mathcal{B}}^2R_{\mathcal{D}}^2)}{\sqrt{3}(R_{B_c^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)+R_{\mathcal{B}}^2R_{\mathcal{D}}^2)^{5/2}},\\
g_{B_c^*\mathcal{B}^*\mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}}m_{\mathcal{D}}}\gamma_{c\bar{b}}(R_{B_c^*}R_{\mathcal{B}^*}R_{\mathcal{D}})^{3/2}(R_{B_c^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}}^2)+2R_{\mathcal{B}^*}^2R_{\mathcal{D}}^2)}{\sqrt{3m_{B_c^*}}(R_{B_c^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}}^2)^{5/2}},\\
g_{B_c^*\mathcal{B}\mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}(R_{B_c^*}R_{\mathcal{B}}R_{\mathcal{D}^*})^{3/2}(R_{B_c^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}^*}^2)+2R_{\mathcal{B}}^2R_{\mathcal{D}^*}^2)}{\sqrt{3m_{B_c^*}}(R_{B_c^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}}^2R_{\mathcal{D}^*}^2)^{5/2}},\\
g_{B_c^*\mathcal{B}^*\mathcal{D}^*} &= \frac{2\sqrt{2}\pi^{\frac{1}{4}}\sqrt{m_{B_c^*}m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}(R_{B_c^*}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{3/2}(R_{B_c^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+2R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)}{\sqrt{3}(R_{B_c^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)^{5/2}}.
\end{aligned}\tag{A2}$$

For the B_{c0}^{*+} couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$, we have

$$\begin{aligned}
g_{B_{c0}^*\mathcal{B}\mathcal{D}} &= \frac{8\pi^{\frac{1}{4}}\sqrt{m_{B_{c0}^*}m_{\mathcal{B}}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{\mathcal{B}}R_{\mathcal{D}}(R_{B_{c0}^*}R_{\mathcal{B}}R_{\mathcal{D}})^{5/2}}{(R_{B_{c0}^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)+R_{\mathcal{B}}^2R_{\mathcal{D}}^2)^{\frac{5}{2}}},\\
g_{B_{c0}^*\mathcal{B}^*\mathcal{D}^*} &= \frac{16\pi^{\frac{1}{4}}\sqrt{m_{B_{c0}^*}m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B_{c0}^*}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{5/2}}{3(R_{B_{c0}^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)^{\frac{5}{2}}}.
\end{aligned}\tag{A3}$$

For the B_{c2}^{*+} couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$, we have

$$\begin{aligned}
g_{B_{c2}^*\mathcal{B}\mathcal{D}} &= \frac{\pi^{\frac{1}{4}}\sqrt{m_{B_{c2}^*}m_{\mathcal{B}}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{B_{c2}^*}^{5/2}(R_{\mathcal{B}}R_{\mathcal{D}})^{\frac{3}{2}}(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)(R_{B_{c2}^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)+2R_{\mathcal{B}}^2R_{\mathcal{D}}^2)}{\sqrt{3}(R_{B_{c2}^*}^2(R_{\mathcal{B}}^2+R_{\mathcal{D}}^2)+R_{\mathcal{B}}^2R_{\mathcal{D}}^2)^{7/2}},\\
g_{B_{c2}^*\mathcal{B}^*\mathcal{D}^*} &= \frac{16\pi^{\frac{1}{4}}\sqrt{m_{B_{c2}^*}m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B_{c2}}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{5/2}}{\sqrt{3}(R_{B_{c2}^*}^2(R_{\mathcal{B}^*}^2+R_{\mathcal{D}^*}^2)+R_{\mathcal{B}^*}^2R_{\mathcal{D}^*}^2)^{\frac{5}{2}}}.
\end{aligned}\tag{A4}$$

For the $B_{c1}^{(\prime)+}$ couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$, consisdering the mixing between 1^1P_1 and 1^3P_1, we have

$$
\begin{aligned}
g_{B'_{c1}\mathcal{B}\mathcal{D}^*} &= \cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B'_{c1}}m_{\mathcal{B}}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}}R_{\mathcal{D}^*}(R_{B^1_{c1}}R_{\mathcal{B}}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}}R^2_{\mathcal{D}^*})^{5/2}} + \sqrt{2}\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B'_{c1}}m_{\mathcal{B}}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}}R_{\mathcal{D}^*}(R_{B^3_{c1}}R_{\mathcal{B}}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}}R^2_{\mathcal{D}^*})^{5/2}}, \\
g_{B_{c1}\mathcal{B}\mathcal{D}^*} &= -\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B_{c1}}m_{\mathcal{B}}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}}R_{\mathcal{D}^*}(R_{B^1_{c1}}R_{\mathcal{B}}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}}R^2_{\mathcal{D}^*})^{5/2}} + \sqrt{2}\cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B_{c1}}m_{\mathcal{B}}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}}R_{\mathcal{D}^*}(R_{B^3_{c1}}R_{\mathcal{B}}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}}R^2_{\mathcal{D}^*})^{5/2}},
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
g_{B'_{c1}\mathcal{B}^*\mathcal{D}} &= \cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B'_{c1}}m_{\mathcal{B}^*}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}}(R_{B^1_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}})^{5/2}} + \sqrt{2}\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B'_{c1}}m_{\mathcal{B}^*}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}}(R_{B^3_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}})^{5/2}}, \\
g_{B_{c1}\mathcal{B}^*\mathcal{D}} &= -\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B_{c1}}m_{\mathcal{B}^*}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}}(R_{B^1_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}})^{5/2}} + \sqrt{2}\cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{B_{c1}}m_{\mathcal{B}^*}m_{\mathcal{D}}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}}(R_{B^3_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}})^{5/2}},
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
g_{B'_{c1}\mathcal{B}^*\mathcal{D}^*} &= \cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B^1_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B'_{c1}}}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}^*})^{5/2}} + \sqrt{2}\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B^3_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B'_{c1}}}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}^*})^{5/2}}, \\
g_{B_{c1}\mathcal{B}^*\mathcal{D}^*} &= -\sin\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B^1_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B_{c1}}}(R^2_{B^1_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}^*})^{5/2}} + \sqrt{2}\cos\theta_{1P}\frac{8\pi^{\frac{1}{4}}\sqrt{m_{\mathcal{B}^*}m_{\mathcal{D}^*}}\gamma_{c\bar{b}}R_{\mathcal{B}^*}R_{\mathcal{D}^*}(R_{B^3_{c1}}R_{\mathcal{B}^*}R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B_{c1}}}(R^2_{B^3_{c1}}(R^2_{\mathcal{B}^*}+R^2_{\mathcal{D}^*})+R^2_{\mathcal{B}^*}R^2_{\mathcal{D}^*})^{5/2}}.
\end{aligned}
\tag{A7}
$$

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