Cosmological perturbation in New General Relativity: propagating mode from the violation of local Lorentz invariance*

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Abstract: We investigate the propagating modes of New General Relativity (NGR) in second-order linear perturbations in the Lagrangian density (first-order in field equations). Dirac-Bergmann analysis reveals a violation of local Lorentz invariance in NGR. We review the recent status of NGR, considering the results of its Dirac-Bergmann analysis. We then reconsider the vierbein perturbation framework and identify the origin of each perturbation field in the vierbein field components. This identification is necessary to adequately fix gauges while guaranteeing consistency with the invariance ensured by the Dirac-Bergmann analysis. We find that the spatially flat gauge is adequate for analyzing a theory with the violation of local Lorentz invariance. Based on the established vierbein perturbative framework, introducing a real scalar field as matter, we perform a second-order perturbative analysis of NGR with respect to tensor, scalar, pseudo-scalar, vector, and pseudo-vector modes. We reveal the possible propagating modes of each type of NGR. In particular, we find that Type 3 has five stable propagating modes, *i.e.*, tensor, scalar, and vector modes, compared to five non-linear degrees of freedom, which results in its Dirac-Bergmann analysis. The linear perturbation theory of Type 3 is preferable for applications to cosmology. Finally, we discuss our results in comparison to those of previous related work and conclude this paper.

Keywords: New General Relativity, Hamiltonian analysis, cosmological perturbation

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I. INTRODUCTION

New General Relativity (NGR) is an extension of Teleparallel Equivalent to General Relativity (TEGR) with three free parameters, in which torsion plays the primary role in describing gravity in a parity-preserving manner [1, 2]. NGR provides richer degrees of freedom (DOF) compared to TEGR [3], and this abundance has the potential to elucidate issues in cosmology such as dark energy [4−6], dark matter [6−8], and tensions in cosmological parameters [9−13]. One can check the present status of the tensions in Ref. [14]. To investigate these phenomenological issues, it is essential to clarify the nature of DOFs in NGR from the viewpoint of cosmological perturbations based on a constraint system. Recently, one of the authors revealed the constraint structure and counted the DOFs of NGR [15, 16]. However, while propagating modes around Minkowski spacetime have been well investigated in previous works [17−21], cosmological perturbations of NGR have not yet been sufficiently clarified [22].

Cosmological perturbations for NGR are special in the sense that NGR no longer satisfies the local Lorentz Invariance (LI) [15, 16]. In turn, this violation of local LI can produce new propagating modes that do not appear in conventional theories, such as GR and $f(R)$ gravity. Despite this, the standard perturbation theory typically includes only metric components [23], which correspond to the symmetric part of (co-)vierbein field components. To properly account for these additional propagating modes, it becomes necessary to incorporate the anti-symmetric part of the (co-)vierbein field into the perturbative framework, where the local Lorentz transformation operates.

Historically, $f(T)$-gravity has encountered the same

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issues as NGR. The first result on Dirac-Bergmann (DB) analysis [24–30] appeared in Ref. [31]. In this pioneering work, the authors implied the violation of local LI by stating that the first-class constraints corresponding to the local LI become second-class constraints. A closed algebra of first-class constraints in the Poisson bracket forms a gauge symmetry [32–37]. Thus, the result means that the LI is lost, at least as a local invariance. In this regard, also see Refs. [31, 38–43] for details. On the other hand, a vierbein perturbation theory has been established [44–47]. The authors of Refs. [44–46] incorporated the anti-symmetric part of vierbein components into the perturbations, thereby enabling us to consider a perturbation theory with the violation of local LI. The authors of Ref. [47] completed the perturbative framework by considering the inclusion of the pseudo-vector mode.

Recently, cosmological perturbation on NGR was performed using conformal transformations based on the results on Minkowski background spacetime [19, 22], where a conformal Newtonian gauge is imposed to investigate the propagation of each perturbative mode. However, there are still concerns in this analysis for the following reasons: 1) The conventional perturbative framework is applied [48], suggesting that the relationship between each perturbative variable and vierbein component remains unclear. In particular, in Ref. [48], the perturbations are introduced without explicit derivations, suggesting that we should identify each origin of the perturbation field. 2) The perturbative framework in this work does not take into account pioneering works [44–47], particularly Ref. [47]. The consistency with Refs. [19, 22, 48] should be investigated. 3) Refs. [19, 22, 48] do not reflect the result of the DB analysis in NGR [15, 16], meaning that there may remain a doubt that the gauge fixing is not appropriate; if this is the case, the propagating modes accounted for in this work could be insufficient. 4) The metric and torsion perturbations are calculated at first-order levels, implying that consideration of higher-order contributions in these variables may raise additional propagating modes.

To address these issues, we analyze the cosmological perturbation and reveal possible propagating modes in NGR. We perform perturbative expansion of the Lagrangian density of NGR up to the second order around the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. We introduce a scalar field as a matter field to realize the flat FLRW background; otherwise, the scale factor is constant, and our analysis is always reduced to existing analysis [19, 21] in the Minkowski background up to the scaling of the spatial coordinates. We investigate the propagation of perturbation modes and discuss the number of propagating modes according to the known classification of the three parameters in NGR. We summarize our results on the number of propagating modes in Table 2.

The remainder of this paper is organized as follows. In Sec. II, we review NGR in terms of DB analysis. In Sec. III, we reconstruct cosmological perturbations to clarify the origin of each perturbation in vierbein field components. In Sec. IV, we derive the background equations of NGR, incorporating a real scalar field as matter. In Sec. V, we investigate the propagating modes of tensor, (pseudo-)scalar, and (pseudo-)vector modes in each type of NGR. Finally, in Sec. VI, we summarize and conclude this paper.

Throughout this paper, we use the unit of $c^4/16\pi G = 1$. We denote spacetime indices by Greek letters, $\alpha,\beta,\gamma,\cdots,\mu,\nu,\rho,\cdots$; internal-space indices by capital Latin letters, $A,B,C,\cdots,I,J,K,\cdots$; and internal-space spatial indices and spacetime spatial indices by small Latin letters, $a,b,c,\cdots$ and $i,j,k,\cdots$, respectively. We express the covariant derivative with respect to the Levi-Civita connection as $\overset{\circ}{\nabla}$ and distinguish it from the covariant derivative with respect to an affine connection as ∇. For all calculations in this work, we utilized Cadabra [49], a free and excellent calculator.

II. NEW GENERAL RELATIVITY

A. Fundamental Ingredients

NGR consists of a (co-)vierbein field and a connection 1-form in the internal-space formulation. In this work, we consider NGR based on the Weitzenböck gauge in four-dimensional spacetime. This gauge restricts an affine connection to the Weitzenböck connection given as follows:[1)]

$$\overset{\mathrm{w}}{\Gamma}{}^{\rho}_{\mu\nu} = e_I{}^{\rho}\,\partial_\mu e^I{}_\nu\,, \tag{1}$$

where $e_I{}^{\mu}$ and $e^I{}_\mu$ are the vierbein and co-vierbein field (components), respectively. Here, these quantities are related by $e_I{}^{\mu}e^I{}_\nu = \delta^\mu{}_\nu$, or equivalently, $e_I{}^{\mu}e^J{}_\mu = \delta^J{}_I$. Thus, NGR requires only the (co-)vierbein to formulate it. Using this connection, the torsion tensor turns out to be

$$\overset{\mathrm{w}}{T}{}^{\rho}{}_{\mu\nu} = 2\overset{\mathrm{w}}{\Gamma}{}^{\rho}_{[\mu\nu]} = e_I{}^{\rho}\left(\partial_\mu e^I{}_\nu - \partial_\nu e^I{}_\mu\right). \tag{2}$$

Apparently, Eq. (1) does not satisfy the local LI. The same statement holds for Eq. (2). These properties are an

1) We note that in TEGR, the absence of spin connection gives rise to a boundary term when we perform a local Lorentz transformation on the action integral. This means that TEGR satisfies the local LI.

implication of the violation in NGR.[1)] In detail, see Sec. II-A in Ref. [15] and the original work Ref. [50].

To verify this violation in NGR, there are two methods: 1) Derive the field equation with respect to the tetrad field components and check the anti-symmetric part of the equation. If the anti-symmetric part of the equation identically holds, the theory satisfies the local LI. If this is not the case, extra DOFs would arise up to six; 2) Perform the DB analysis [24−30] and check the Poisson Bracket algebra (PB algebra) of constraints in the vierbein sector. If the algebra shows the Lie algebra of $SO(1,3)$, the given theory satisfies the local LI. If this is not the case, the theory violates the invariance and gives rise to extra DOFs up to six.[2)]

Constraint systems are classified into two types: regular and irregular systems [27, 29, 30, 51]. For regular systems, based on the DB analysis, we can calculate the nonlinear DOFs by the following formula: *nonlinear DOFs* $=$ (*phase space dimension* $-$ $2\times$# *of first-class constraint(s)* $-$ # *of second-class constraints*)/2, where the symbol # denotes "number." Here, the regularity of a system is defined by the first-order variation of every constraint being expressed by a linear combination of constraints existing in the theory. In the Hamiltonian constraint formulation, a closed PB algebra of first-class constraints provides a gauge symmetry of the given theory [32−37]. Thus, if a gauge symmetry, a closed PB algebra, is violated, # of first-class constraints decreases. This provides new second-class constraints and/or non-linear DOFs by the above definition. We remark that the nonlinear DOFs provide the upper bound of the possible number of propagating DOFs. Inheriting this property, the non-linear DOFs are sometimes called the full/total DOFs of a given theory. In the next subsection, we introduce NGR, in which the local LI is violated [15, 16].

For irregular systems, we must regularize the system to count the DOFs, although there is no generic method. Here, we call a theory irregular if it has a constraint that violates the property of regularity. Several examples can be verified in Refs. [16, 51]. In general, the regularized system does not change only in a local region of the constraint surface. That is, the method of regularization is different in each region of the constraint surface. This means that we cannot use the DOFs provided by the result of the DB analysis to set the upper bound of the perturbation theory. In the following, we omit the "w" on top of each quantity, such as the Weitzenböck connection, $\overset{\mathrm{w}}{\Gamma}{}^{\rho}{}_{\mu\nu}$, and torsion tensor, $\overset{\mathrm{w}}{T}{}^{\rho}{}_{\mu\nu}$, for simplicity.

B. Violation of Local Lorentz Invariance in NGR and Extra Degrees of Freedom

The Lagrangian of NGR is given as follows [2]:

$$
\begin{aligned}
L_{\rm NGR} := \theta^{-1}\mathcal{L}_{\rm NGR} &= c_1\,T^{\mu\nu\rho}T_{\mu\nu\rho} + c_2\,T^{\mu\nu\rho}T_{\rho\mu\nu} + c_3\,T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\\
&= c_1\,g_{\mu\sigma}g^{\nu\lambda}g^{\rho\kappa}T^{\mu}{}_{\lambda\kappa}T^{\sigma}{}_{\nu\rho} + c_2\,g^{\nu\lambda}T^{\mu}{}_{\lambda\rho}T^{\rho}{}_{\mu\nu}\\
&\quad + c_3\,g^{\rho\nu}T^{\mu}{}_{\mu\rho}T^{\lambda}{}_{\lambda\nu}\,,
\end{aligned}
\tag{3}
$$

where θ is the determinant of the co-vierbein field components, and c_1, c_2, c_3 are three free parameters that range in real value. For example, in TEGR, which is equivalent to GR up to a boundary term in the action integral except for the geometry that describes spacetime, $c_1=-1/4$, $c_2=1/2$, $c_3=1$.

Applying the variational principle with respect to the (co-)vierbein field under the imposition of Dirichlet boundary conditions on these fields, we obtain the field equation

$$
\begin{aligned}
&\frac{1}{2}e_A{}^{\rho}\mathcal{T}_{\rho}{}^{\nu} = \theta^{-1}\partial_{\mu}(\theta e_A{}^{\rho}S_{\rho}{}^{\mu\nu}) + e_A{}^{\lambda}T^{\rho}{}_{\mu\lambda}S_{\rho}{}^{\mu\nu} - \frac{1}{2}e_A{}^{\nu}L_{\rm NGR}\,,\\
&S_{\rho}{}^{\mu\nu} := c_1\,T_{\rho}{}^{\mu\nu} + c_2\,T^{[\mu\nu]}{}_{\rho} + c_3\,\delta_{\rho}{}^{[\mu}T^{|\kappa|}{}_{\kappa}{}^{\nu]}\,.
\end{aligned}
\tag{4}
$$

Here, $e_A{}^{\rho}\mathcal{T}_{\rho}{}^{\nu}=\theta^{-1}\delta\mathcal{L}_{\rm matter}/\delta e^A{}_{\nu}$ is the push forward of the energy-momentum tensor by the vierbein field. Pulling back this equation from the internal space to spacetime and taking the anti-symmetric part of this field equation, we obtain

$$
\begin{aligned}
0 = &\,(-2c_2+c_3)\,\overset{\circ}{\nabla}_{\rho}T^{\rho}{}_{[\mu\nu]} - (2c_1+c_2)\,\overset{\circ}{\nabla}_{\rho}T_{[\mu\nu]}{}^{\rho}\\
&+\left(c_1-\frac{1}{2}c_2+\frac{1}{2}c_3\right)T^{\rho\sigma}{}_{[\mu}T_{\nu]\rho\sigma}\,,
\end{aligned}
\tag{5}
$$

where the circle "$\circ$" on top of ∇ denotes the covariant derivative with respect to the Levi-Civita connection. We remark that $\mathcal{T}_{[\mu\nu]}=0$. The number of independent components of Eq. (5) is six at most, and this number coincides with that of the generator of $SO(1,3)$-symmetry. In the TEGR case, Eq. (5) is, on the one hand, automatically satisfied, subject to automatic vanishment of each coefficient in Eq. (5) under $c_1=-1/4$, $c_2=1/2$, $c_3=1$. On the other hand, if Eq. (5) is not identically satisfied, the equation suggests the existence of propagating modes generated by the violation of the local LI.

1) In TEGR (Type 6 of NGR), a local Lorentz transformation of the Lagrangian provides a boundary term in the action integral. Thus, the theory guarantees the invariance only when the action contains boundary terms. In contrast, this is generally not the case in NGR except for Type 6 (TEGR).

2) In NGR, since the diffeomorphism invariance holds, the upper bound of non-linear DOFs is $(16\times2-8\times2)/2=8$. The two DOFs of eight are none other than the DOFs describing gravity (tensor modes). The remaining six DOFs are ascribed to the violation of the local Lorentz invariance. See also Refs. [15, 16, 21] in this point.

In the Hamiltonian formulation of NGR, we can classify the theory into nine independent types based on the $SO(3)$-irreducible decomposition of canonical momentum [50]. DB analysis on each type of NGR was performed in Refs. [15, 16] while satisfying the diffeomorphism invariance (hypersurface deformation algebra in terms of PB algebra) [29] in all types. The authors of Refs. [15, 16] clarified the constraint structure of each type, which is the expression of the internal local symmetry of each type in terms of PB algebra. The resulting PB algebra shows that all types do not have local LI except for Type 6 (TEGR). Based on this, the authors counted out the non-linear DOFs of each type. We summarize the results in Table 1.

All types violate the local LI and give rise to extra DOFs up to six, except for Type 6 (TEGR) [38, 39, 52, 53]. As all types satisfy the diffeomorphism invariance, the extra DOFs in each type should be ascribed to the violation of the local LI. This perspective is essential because it indicates that the propagating modes in NGR should be described in terms of the anti-symmetric part of the vierbein perturbation. In the next section, we revisit a complete formalism for describing linear perturbations of NGR around the flat FLRW spacetime with local LI.

III. COSMOLOGICAL PERTURBATION FROM THE PERSPECTIVE OF LOCAL LORENTZ INVARIANCE

We revisit the conventional framework of linear perturbations to clarify how each perturbation field is related to local Lorentz invariance. This step is important because, if one fixes a perturbation field associated with local Lorentz invariance in a theory where this invariance is violated, such gauge fixing may inadvertently remove a genuine propagating mode.

Cosmological perturbations are considered around the flat FLRW spacetime as follows:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{6}$$

where $a = a(t)$ is the scale factor, and i and j run from 1 to 3. In Refs. [45, 46], the authors decompose the co-vierbein field, $e^I{}_\mu$, into its symmetric part, $\bar{e}^I{}_\mu$, and anti-symmetric part, $\tilde{e}^I{}_\mu$. The symmetric and anti-symmetric parts describe the DOFs of the metric tensor and extra DOFs generated by the violation of the local LI, respectively. That is, we can split $e^I{}_\mu$ as follows:

$$e^I{}_\mu = \bar{e}^I{}_\mu + \tilde{e}^I{}_\mu\,. \tag{7}$$

According to Refs. [45, 46], we introduce perturbations in terms of the vierbein field with the condition $g_{\mu\nu} = e^I{}_\mu e^J{}_\nu\eta_{IJ} = \bar{e}^I{}_\mu\bar{e}^J{}_\nu\eta_{IJ}$. Here, $\eta_{IJ} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric.

In Ref. [47], the authors indicated that the perturbation theory formulated in Refs. [44–46] is incomplete because the anti-symmetric tensor introduced in Refs. [44–46] can be further decomposed into a pseudo-scalar and transverse pseudo-vector. The authors also redefined all perturbed fields and derived the gauge transformation of these fields. However, the literature confuses the symmetric and anti-symmetric parts of the co-vierbein decomposition and imposes gauge conditions that fix a part of the anti-symmetric components of the co-vierbein. Our aim is to investigate the propagation of the anti-symmetric parts of the co-vierbein, which represent the propagating modes generated by the violation of local LI. This property indicates that we should not fix the gauge cor-

Table 1. Conditions on parameters, c_1, c_2, c_3 and nonlinear DOFs of each type of NGR in the $SO(3)$-irreducible decomposition of canonical momentum. We remark that the sign of the parameter c_2 is opposite to that of the original work [50]. "Special" denotes the case that occurs only under the satisfaction of a set of specific conditions on Lagrange multipliers, whereas "Generic" denotes the case without any conditions. “Regularity” represents the closedness as a linear combination in the first-order variation of each constraint with respect to all constraints existing in a theory [27, 29, 30, 51]. For details, see Refs. [15, 16].

Theory	Conditions on parameter space (c_1, c_2, c_3)	Non-linear DOF	Regularity
Type 1	arbitrary	8	–
Type 2	$2c_1 - c_2 + c_3 = 0$	6	✓
Type 3	$2c_1 + c_2 = 0$	5	✓
Type 4	$2c_1 - c_2 = 0$	5	×
Type 5	$2c_1 - c_2 + 3c_3 = 0$	7	✓
Type 6 (TEGR)	$2c_1 - c_2 + c_3 = 0$ & $2c_1 + c_2 = 0$	2	✓
Type 7	$2c_1 + c_2 = 0$ & $2c_1 - c_2 = 0$	0 (Topological in bulk spacetime)	×
Type 8	$2c_1 + c_2 = 0$ & $2c_1 - c_2 + 3c_3 = 0$	6 (Generic) or 4 (Special)	✓
Type 9	$2c_1 - c_2 + c_3 = 0$ & $2c_1 - c_2 = 0$ & $2c_1 - c_2 + 3c_3 = 0$	3	✓

responding to the anti-symmetric part of the co-vierbein at least in advance. Thus, we must clarify the origin of each perturbation field in the pioneering work [47].

For our purpose, we reconsider the co-vierbein perturbation in Refs. [45−47]. Let $\bar{e}^I{}_\mu$ and $\tilde{e}^I{}_\mu$ decompose as follows:

$$\begin{aligned}
\bar{e}^0{}_\mu &= (1+\psi)\,\delta^0{}_\mu + a\,(\partial_i F + G_i)\,\delta^i{}_\mu\,,\\
\bar{e}^a{}_\mu &= a\,(1-\varphi)\,\delta^a{}_\mu + a\,\delta^{ak}\left(h_{jk}+\partial_j\partial_k B+\partial_j C_k+\partial_k C_j\right)\delta^j{}_\mu\,,
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
\tilde{e}^0{}_\mu &= a\,(\partial_i\alpha+\alpha_i)\,\delta^i{}_\mu\,,\\
\tilde{e}^a{}_\mu &= a\,\delta^{ai}\,\delta^{kl}\,\epsilon_{ijk}\left(\partial_l\tilde{\sigma}+\tilde{V}_l\right)\delta^j{}_\mu\,.
\end{aligned}\tag{9}$$

ψ, φ, B, and F are the scalar perturbations of the symmetric part of the co-vierbein field, $\bar{e}^I{}_\mu$. α and $\tilde{\sigma}$ are the scalar and pseudo-scalar perturbations of the anti-symmetric part of the co-vierbein field, $\tilde{e}^I{}_\mu$, respectively. C_i and G_i are the transverse vector perturbations of the symmetric part of the co-vierbein field, and α_i and $\tilde{V}_i$ are the transverse vector and pseudo-vector perturbations of the anti-symmetric part of the co-vierbein field, respectively. h_{ij} is the traceless and transverse spatial tensor perturbation, which corresponds to gravitational waves in GR. We note that the local Lorentz transformation acts on the first and second equations in Eq. (9) as a boost and rotation, respectively. Combining Eqs. (8) and (9) based on Eq. (7), we obtain

$$\begin{aligned}
e^0{}_\mu &= (1+\psi)\,\delta^0{}_\mu + a\,[\partial_i(F+\alpha)+(G_i+\alpha_i)]\,\delta^i{}_\mu\,,\\
e^a{}_\mu &= a\,(1-\varphi)\,\delta^a{}_\mu + a\,\delta^{ai}\left[h_{ji}+\partial_j\partial_i B+\partial_j C_i+\partial_i C_j\right.\\
&\quad\left.+\,\epsilon_{ijk}\,\delta^{kl}\left(\partial_l\tilde{\sigma}+\tilde{V}_l\right)\right]\delta^j{}_\mu\,.
\end{aligned}\tag{10}$$

We remark that we do not confuse the spatial indices of spacetime, $i,j,k,\cdots$, and the Lorentz indices, $a,b,c,\cdots$.

However, this decomposition is not well-defined because the functional DOFs of each side of Eq. (10) do not match one another. That is, $e^0{}_\mu$ can encapsulate four DOFs, whereas the right-hand side has seven perturbation fields. In the second equation of Eq. (10), the co-vierbein has room to encapsulate these exceeded perturbation fields. Thus, to reconcile this inconsistency, we modify it as follows:

$$\begin{aligned}
e^0{}_\mu &= (1+\psi)\,\delta^0{}_\mu + a\,(\partial_i\alpha+\alpha_i)\,\delta^i{}_\mu\,,\\
e^a{}_\mu &= a\,(1-\varphi)\,\delta^a{}_\mu + \delta^{ai}\,(\partial_i F+G_i)\,\delta^0{}_\mu\\
&\quad + a\,\delta^{ai}\left[h_{ji}+\partial_j\partial_i B+\partial_j C_i+\partial_i C_j\right.\\
&\quad\left.+\,\epsilon_{ijk}\,\delta^{kl}\left(\partial_l\tilde{\sigma}+\tilde{V}_l\right)\right]\delta^j{}_\mu\,.
\end{aligned}\tag{11}$$

This perturbation theory can describe 10 (spacetime sector: h_{ij}, ψ, φ, F, B, G_i, C_i) + 6 (internal-space sector: α, $\tilde{\sigma}$, α_i, $\tilde{V}_i$) = 16 propagating DOFs at most. In particular, a violation of local LI causes possible propagating DOFs that are ascribed to the internal space. This decomposition is none other than that provided in Ref. [47] except for the difference in notation. Compared with Refs. [19, 22, 48], our parameterization coincides with theirs, except for the use of conformal time, the non-symmetrization of $\partial_i C_j$ terms, and the composition of variables that the local Lorentz transformation operates on.[1)]

Now, we consider the gauge transformation of these perturbed fields. For infinitesimally small coordinate transformations, $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$, the variation of the co-vierbein is given as follows:

$$\delta_\xi\, e^I{}_\mu = -\mathcal{L}_\xi\, e^I{}_\mu = -\xi^\nu\partial_\nu e^I{}_\mu - e^I{}_\nu\,\partial_\mu\xi^\nu\,.\tag{12}$$

Here, we denote by $\mathcal{L}_X Y$ the Lie derivative of Y with respect to X. Then, the gauge transformations of the perturbations are calculated as follows:

$$e^I{}_\mu \to e'^I{}_\mu = e^I{}_\mu + \delta_\xi e^I{}_\mu = e^I{}_\mu - \xi^\nu\partial_\nu e^I{}_\mu - e^I{}_\nu\,\partial_\mu\xi^\nu\,.\tag{13}$$

Decomposing the spatial component of ξ^μ, ξ^i, further into $\xi^i = \partial^i\xi + \xi^{(v)i}$, where $\xi^{(v)i}$ is a transverse vector, we can derive following transformation rules:

$$\begin{aligned}
&\psi' = \psi - \dot{\xi}^0\,, \quad \varphi' = \varphi + \frac{\dot{a}}{a}\xi^0\,,\\
&\alpha' = \alpha - \frac{1}{a}\xi^0\,, \quad B' = B - \frac{1}{a}\xi\,,\\
&F' = F - \left(\dot{\xi} - \frac{\dot{a}}{a}\xi\right), \quad \tilde{V}'_i = \tilde{V}_i - \frac{1}{a}\epsilon_{ijk}\delta^{jl}\delta^{km}\partial_l\xi^{(v)}_m\,,\\
&C'_i = C_i - \frac{1}{2a}\xi^{(v)}_i\,, \quad G'_i = G_i - \left(\dot{\xi}^{(v)}_i - \frac{\dot{a}}{a}\xi^{(v)}_i\right).
\end{aligned}\tag{14}$$

Here, the dot "$\cdot$" stands for the time derivative. Other perturbations, h_{ij}, α_i, and $\tilde{\sigma}$ do not change with respect to the infinitesimal gauge transformation given by Eq. (12).

1) Translating their notations into ours, it states that scalar $\alpha+F$, pseudo-scalar $\tilde{\sigma}$, vector α_i+G_i, and pseudo-vector $\tilde{V}_i$ correspond to local Lorentz rotation of the vierbein in Ref. [48]. Our current consideration does not follow from this reference for the following three reasons. a) α, F, $\tilde{\sigma}$, α_i, G_i, $\tilde{V}_i$ should be counted separately. Otherwise, the functional degrees of freedom of the vierbein components and the total number of perturbation fields do not match each other, implying that we implicitly induce constraints. This may accidentally change the original theory into a different one. b) local LI acts on $\tilde{V}_i+\partial_i\tilde{\sigma}$ for three rotations, but so does not for $\alpha_i+G_i+\partial_i(\alpha+F)$ for three boosts due to the incorporation of G_i and F. c) G_i and F correspond to the components of the metric perturbation, indicating that these fields do not contribute to internal-space symmetries.

This transformation coincides with that provided in the pioneering work [47] up to the difference in notation.[1)] Based on the above results, we can compose five set of gauge-invariant variables as follows:

$$\begin{aligned} &\beta=\psi+\varphi-a\dot{\alpha}\,, \qquad \gamma=\varphi+\dot{a}\alpha \qquad A=F-a\dot{B}\,,\\ &B_i=G_i-2a\dot{C}_i\,, \qquad \tilde{W}_i=\tilde{V}_i-2\epsilon_{ijk}\partial_j C_k\,. \end{aligned} \tag{15}$$

Taking h_{ij}, α_i, and $\tilde{\sigma}$ into account, our theory consists of twelve gauge-invariant variables in total; we can replace twelve out of sixteen perturbation variables in terms of these gauge-invariant variables.

Now, the origin of each perturbation field and its role are clear. We shall consider gauge-fixing conditions for our analysis. Fixing the gauge ξ^0, one of the three scalar perturbations, ψ, φ, or α, vanishes from the theory. For the gauge ξ, one of the two scalar perturbations, either B or F, vanishes. In the same way, fixing the gauge ${\xi^{(v)}}_i$, one of the pseudo-vector $\tilde{V}_i$, vector C_i, or vector G_i vanishes. If a given theory satisfies both diffeomorphism and local LI, we can simplify the perturbation theory as desired by fixing the gauge freely. In NGR, however, local LI is violated while preserving diffeomorphism invariance. Therefore, we should not fix the following perturbations that originated from the violation of local LI: α, α_i, $\tilde{\sigma}$, and $\tilde{V}_i$.

To formulate a perturbation theory for investigating the propagating DOFs in NGR, we consider an appropriate gauge choice. For the gauge ξ^0, we should fix it so that either of $\psi=0$ or $\varphi=0$ holds. In our work, we choose the gauge to realize $\varphi=0$. For the gauges ξ and ${\xi^{(v)}}_i$, in this work, we fix them for $B=0$ and $C_i=0$, respectively. Therefore, the possible propagating DOFs are h_{ij}, ψ, F, and G_i of the symmetric part of the co-vierbein field and α, α_i, $\tilde{\sigma}$, and $\tilde{V}_i$ of the anti-symetric part of the co-vierbein field. Explicitly, the co-vierbein field becomes

$$\begin{aligned} &{e^0}_0=1+\psi\,,\\ &{e^0}_i=a\left(\partial_i\alpha+\alpha_i\right),\\ &{e^a}_0=\delta^{ai}\left(\partial_i F+G_i\right),\\ &{e^a}_i=a\,{\delta^a}_i+a\,\delta^{al}\left[h_{li}+\epsilon_{lij}\delta^{jk}\left(\partial_k\tilde{\sigma}+\tilde{V}_k\right)\right]. \end{aligned} \tag{16}$$

The above gauge choice is well-known as the spatially flat gauge.

Finally, we note the following three points. 1) The authors in Ref. [47] indicates that in a parity-preserving theory the possible coupling of the pseudo-vector perturbation, $\tilde{V}_i$, with the vector perturbation, α_i, is $\epsilon_{ijk}(\partial_i\alpha_j)\tilde{V}_k$ only. Here, we modified their notation to ours. 2) In the linear perturbation theory, we can treat all modes separately except for the vector and pseudo-vector perturbations.[2)] 3) The background Lagrangian density of NGR becomes

$$\mathcal{L}^{(\mathrm{flat\,FLRW})}_{\mathrm{NGR}}=-3\left(2c_1-c_2+3c_3\right)a^3H^2 \tag{17}$$

where $H:=\dot{a}/a$ is the Hubble parameter. In particular, for Types 5, 8, and 9, the background Lagrangian density vanishes; that is, the existence of matter contributes to the time evolution of spacetime more than first order. We note that Types 2 and 3 are none other than Type 6 (TEGR) with a violation of local LI at least partially (for details, see Ref. [15]). In the following sections, we utilize this decomposition of the co-vierbein field, Eq. (16), to investigate propagating DOFs of each type of NGR up to second-order perturbations.

IV. BACKGROUND EQUATION WITH MATTER FIELD

We introduce a scalar field, Φ, as the matter source:

$$L_{\mathrm{matter}}=\theta^{-1}\mathcal{L}_{\mathrm{matter}}=-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-V(\Phi)\,. \tag{18}$$

Here, θ is the determinant of the co-frame field. Varying with respect to Φ, we of course obtain the field equation:

$$\theta^{-1}\partial_\mu\left(\theta g^{\mu\nu}\partial_\nu\Phi\right)-V'=0\,. \tag{19}$$

According to the pioneering work [47], we decompose Φ into the background and first-order perturbation part, Φ_0 and $\delta\Phi$, respectively, as follows:

$$\Phi=\Phi_0+\delta\Phi\,. \tag{20}$$

We can expand the potential term, V, up to second order with respect to $\delta\Phi$ as follows:

$$V(\Phi_0+\delta\Phi)=V_0+V_0'\delta\Phi+\frac{1}{2}V_0''(\delta\Phi)^2\,, \tag{21}$$

where the prime $'$ represents the derivative with respect to Φ. We write $V_0=V(\Phi_0)$, $V_0'=V'(\Phi_0)$, and $V_0''=V''(\Phi_0)$

1) In our notation, the sign of φ is opposite.

2) Technically, the term $\partial_i\tilde{\sigma}$ behaves as a pseudo-vector perturbation, suggesting that $\tilde{\sigma}$ is independent of any of the other scalar perturbations in linear perturbations.

for simplicity. The isometry of the background flat-FLRW spacetime requires that the spatial derivative of the background scalar field Φ_0 vanishes:

$$\partial_i \Phi_0 = 0\,. \tag{22}$$

Based on this set-up, the field equations of NGR in Eq. (4) around the flat FLRW background spacetime are given as follows:

$$\begin{aligned} &-3(2c_1 - c_2 + 3c_3)H^2 = -\frac{1}{2}\dot{\Phi}_0\dot{\Phi}_0 + V_0\,,\\ &-(2c_1 - c_2 + 3c_3)\left(3H^2 + \dot{H}\right) = \frac{1}{2}V_0 - \frac{1}{4}\dot{\Phi}_0\dot{\Phi}_0\,. \end{aligned} \tag{23}$$

We can verify that these equations coincide with those in GR by setting $c_1 = -1/4$, $c_2 = 1/2$, $c_3 = 1$. Taking into account the result of its DB analysis [15, 16], the violation of local LI appears in the weight, $2c_1 - c_2 + 3c_3$, which couples matter with gravitation. In addition, combining them, we obtain an equation to describe the time evolution of the Hubble parameter:

$$-(2c_1 - c_2 + 3c_3)\dot{H} = \frac{1}{4}\dot{\Phi}_0\dot{\Phi}_0 - \frac{1}{2}V_0\,. \tag{24}$$

If $2c_1 - c_2 + 3c_3 = 0$ holds, that is, in Types 5, 8, and 9, the left-hand sides of two equations in Eq. (23) vanish. Therefore, the Hubble parameter is arbitrary, and in Types 5, 8, and 9, we cannot specify the background spacetime. This result can be interpreted as indicating that matter does not contribute to the time evolution of the background spacetime. Therefore, perturbative analyses in these types effectively correspond to studying the propagating DOFs just on maximally symmetric spacetime; these models are not suitable for cosmological applications. Nevertheless, the possibility of astrophysical applications, such as to black hole spacetime, still remains open.

Finally, the background field equation of the matter field, Eq. (19), around the background spacetime becomes

$$-\ddot{\Phi}_0 - 3H\dot{\Phi}_0 - V_0' = 0\,, \tag{25}$$

We use Eqs. (23) and (25) to eliminate the background variable, Φ_0, from the perturbed Lagrangian density in the subsequent sections. As a note, combining Eq. (24) with Eq. (25), we can solve the Hubble parameter and background matter field. Thus, if a second-order perturbed Lagrangian density contains a term that consists only of the background matter field and Hubble parameter, or equivalently, the scale factor, we can freely drop it without loss of generality under the satisfaction of the background equations. We also use this property in the subsequent sections.

V. PROPAGATING MODES

A. Tensor Perturbation

We calculate the tensor perturbation of NGR up to second order. The Lagrangian of the theory is given by Eq. (3). Focusing on the tensor terms of Eq. (16), we find that the co-vierbein and vierbein field components are derived as follows [47]:

$$\begin{aligned} &e^0{}_\mu\, \mathrm{d}x^\mu = \mathrm{d}t\,,\\ &e^a{}_\mu\, \mathrm{d}x^\mu = a\,\delta^{ai}\left(\delta_{ij} + h_{ij}\right)\mathrm{d}x^j\,,\\ &e_0{}^\mu\,\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}\,,\\ &e_a{}^\mu\,\frac{\partial}{\partial x^\mu} = a^{-1}\,\delta_{ai}\left(\delta^{ij} - h^{ij} + \delta_{kl}\,h^{ik}h^{jl}\right)\frac{\partial}{\partial x^j}\,. \end{aligned} \tag{26}$$

Using these formulae, the metric and its inverse tensor components are calculated as follows:

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu &= -\mathrm{d}t^2 + a^2\left(\delta_{ij} + 2\,h_{ij} + \delta^{kl}\,h_{ik}\,h_{jl}\right)\mathrm{d}x^i \mathrm{d}x^j\,,\\ g^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} &= -\frac{\partial}{\partial t}\frac{\partial}{\partial t} + a^{-2}\left(\delta^{ij} - 2\,h^{ij} + 3\,\delta_{kl}\,h^{ik}\,h^{jl}\right)\\ &\quad\times\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\,, \end{aligned} \tag{27}$$

where we used the relations: $g_{\mu\nu} = e^I{}_\mu\, e^J{}_\nu\, \eta_{IJ}$ and $g^{\mu\nu} = e_I{}^\mu\, e_J{}^\nu\, \eta^{IJ}$. The determinant of the co-vierbein field components with the traceless gauge is[1)]

$$\theta = a^3\left(1 - \frac{1}{2}\delta^{ik}\delta^{jl}\,h_{ij}\,h_{kl}\right)\,. \tag{28}$$

The tensor perturbation of Eq. (3) up to second order becomes

1) We used the following formula:

$$\det(1+\epsilon A) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\sum_{j=1}^{\infty}\epsilon^j\,\frac{(-1)^j}{j}\,\mathrm{tr}(A^j)\right)^k = 1 + \epsilon\,\mathrm{tr}(A) + \frac{1}{2}\epsilon^2\left[\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\right] + O(\epsilon^3)\,,$$

where ϵ is an infinitesimal parameter.

$$\begin{aligned}\mathcal{L}^{(\mathrm{TP})}_{\mathrm{NGR}} = & -3(2c_1-c_2+3c_3)a^3H^2 \\ & -a^3\,(2c_1-c_2)\,\delta^{ik}\,\delta^{jl}\,\dot{h}_{ij}\,\dot{h}_{kl} \\ & +2a^3\,H\,(2c_1-c_2+3c_3)\,\delta^{ik}\,\delta^{jl}\,\dot{h}_{ij}\,h_{kl} \\ & -\frac{1}{2}a^3\,H^2\,\delta^{ik}\,\delta^{jl}\,h_{ij}\,h_{kl} \\ & +a\,(2c_1+c_2)\,\delta^{il}\,\delta^{jm}\,\delta^{kn}\,\partial_i h_{jk}\,\partial_l h_{mn} \\ & -a\,(2c_1+c_2)\,\delta^{im}\,\delta^{jn}\,\delta^{kl}\,\partial_i h_{jk}\,\partial_l h_{mn}\,, \end{aligned} \tag{29}$$

We used the torsion tensor components given in Appendix A and applied the transverse and traceless gauge. In TEGR, where the parameters are chosen as $c_1=-1/4$, $c_2=1/2$, $c_3=1$, the overall sign of the kinetic term in Eq. (30) is $-(2c_1-c_2)=+1$. From Table 1, Types 4, 7, and 9 do not contain the kinetic term of the tensor mode, suggesting that these types are not simple extensions of GR. This classification coincides with that of the recent work in Ref. [21].

Substituting Eqs. (20), (21), (27), and (29) into Eq. (18), we rewrite the matter Lagrangian density as follows:

$$\begin{aligned}\mathcal{L}^{(\mathrm{TP})}_{\mathrm{matter}} = & \frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0 - a^3V_0 + a^3\dot{\Phi}_0\delta\dot{\Phi} - a^3V_0'\delta\Phi \\ & + \frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi} - \frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi - \frac{1}{2}a^3V_0''\delta\Phi\delta\Phi \end{aligned} \tag{30}$$

up to second order in terms of the perturbation fields. Combining the perturbed matter Lagrangian density with Eq. (30) and applying the background equations (Eqs. (23) and (25)), we obtain the total Lagrangian density as follows:

$$\begin{aligned}\mathcal{L}^{(\mathrm{TP;\,2nd\,order})}_{\mathrm{total}} = & \mathcal{L}^{(\mathrm{TP})}_{\mathrm{NGR}} + \mathcal{L}^{(\mathrm{TP})}_{\mathrm{matter}} \\ = & -a^3\,(2c_1-c_2)\,\delta^{ik}\,\delta^{jl}\,\dot{h}_{ij}\,\dot{h}_{kl} \\ & +2\,(2c_1-c_2+3c_3)\,a^3\,H\,\delta^{ik}\,\delta^{jl}\,\dot{h}_{ij}\,h_{kl} \\ & -\frac{1}{2}a^3\,H^2\,\delta^{ik}\,\delta^{jl}\,h_{ij}\,h_{kl} \\ & +a\,(2c_1+c_2)\,\delta^{il}\,\delta^{jm}\,\delta^{kn}\,\partial_i h_{jk}\,\partial_l h_{mn} \\ & -a\,(2c_1+c_2)\,\delta^{im}\,\delta^{jn}\,\delta^{kl}\,\partial_i h_{jk}\,\partial_l h_{mn} \\ & +\frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi} - \frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi \\ & -\frac{1}{2}a^3V_0''\delta\Phi\delta\Phi\,, \end{aligned} \tag{31}$$

where we have dropped the surface terms. If $2c_1-c_2\neq 0$, the first term in Eq. (32) remains. As a result, we conclude that Types 1, 2, 3, 5, 6 (TEGR), and 8 contain the propagating tensor mode, whereas Types 4, 7, and 9 do not. Moreover, the coefficient of the kinetic term of the tensor mode gives us a ghost-free condition:

$$2c_1-c_2<0\,, \tag{32}$$

and c_3 is arbitrary.

B. Scalar Perturbation

We calculate the scalar perturbation of NGR up to second order. Focusing on the scalar terms of Eq. (16), the co-vierbein, vierbein, metric, inverse metric components, and determinant of the co-vierbein are respectively derived as follows:

$$\begin{aligned} e^0{}_\mu\,\mathrm{d}x^\mu = & (1+\psi)\mathrm{d}t + a\,\partial_i\alpha\,\mathrm{d}x^i\,, \\ e^a{}_\mu\,\mathrm{d}x^\mu = & \delta^{ai}\,\partial_iF\,\mathrm{d}t + a\,\delta^{aj}\,\delta_{ij}\,\mathrm{d}x^i\,, \\ e_0{}^\mu\,\frac{\partial}{\partial x^\mu} = & \left[1-\psi+\psi^2+\delta^{ij}\partial_iF\partial_j\alpha\right]\frac{\partial}{\partial t} + a^{-1}\left[(1-\psi)\delta^{ij}\,\partial_jF\right]\frac{\partial}{\partial x^i}\,, \\ e_a{}^\mu\,\frac{\partial}{\partial x^\mu} = & \left[-(1-\psi)\,\delta_a{}^i\,\partial_i\alpha\right]\frac{\partial}{\partial t} + a^{-1}\left[\delta_a{}^i + \delta_a{}^j\,\partial_j\alpha\,\delta^{ik}\partial_kF\right]\frac{\partial}{\partial x^i}\,, \\ g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = & \left[-(1+2\psi+\psi^2)+\delta^{ij}\,\partial_iF\,\partial_jF\right]\mathrm{d}t\mathrm{d}t + 2a\left[\partial_iF-(1+\psi)\,\partial_i\alpha\right]\mathrm{d}t\mathrm{d}x^i + a^2\left[\delta_{ij}-\partial_i\alpha\,\partial_j\alpha\right]\mathrm{d}x^i\mathrm{d}x^j\,, \\ g^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = & \left[-(1-2\psi+3\psi^2)-2\delta^{ij}\,\partial_iF\,\partial_j\alpha+\delta^{ij}\,\partial_i\alpha\,\partial_j\alpha\right]\frac{\partial}{\partial t}\frac{\partial}{\partial t} + 2\,a^{-1}\left[(1-\psi)\delta^{ij}\,\partial_jF-(1-2\psi)\delta^{ij}\,\partial_j\alpha\right]\frac{\partial}{\partial t}\frac{\partial}{\partial x^i} \\ & + a^{-2}\left[\delta^{ij}+\delta^{ik}\,\delta^{jl}\,\partial_kF\partial_lF+2\,\delta^{ik}\partial_kF\,\delta^{jl}\partial_l\alpha\right]\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\,, \\ \theta = & a^3(1+\psi-\delta^{ij}\partial_iF\partial_j\alpha)\,. \end{aligned} \tag{33}$$

The torsion tensor components are given in Appendix B.

Differing from the tensor perturbation, we must apply not only the background equations but also constraints to decouple propagating modes. To show the latter usage, we display an intermediate step. After deriving the perturbed Lagrangian and adopting the background equations, we obtain

$$\begin{aligned}\mathcal{L}^{(\mathrm{SP;2nd\,order})}_{\mathrm{total}} &= \mathcal{L}^{(\mathrm{SP})}_{\mathrm{NGR}}+\mathcal{L}^{(\mathrm{SP})}_{\mathrm{matter}} = (2c_1-c_2+c_3)a^3\delta^{ij}\partial_i\dot{\alpha}\partial_j\dot{\alpha}+4(2c_1-c_2+3c_3)a^3H\delta^{ij}\partial_i\dot{\alpha}\partial_j\alpha-3(c_2-3c_3)a^3H^2\delta^{ij}\partial_i\alpha\partial_j\alpha\\
&+\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\delta^{ij}\partial_i\alpha\partial_j\alpha+2a^2\dot{\Phi}_0\delta^{ij}\partial_i\alpha\partial_j\delta\Phi\\
&+\psi\Big[2(2c_1-c_2+c_3)a^2\delta^{ij}\partial_i\partial_j\dot{\alpha}+4(2c_1-c_2+3c_3)a^2H\delta^{ij}\partial_i\partial_j\alpha-6(2c_1-c_2+3c_3)a^2\delta^{ij}\partial_i\partial_jF\\
&-2(2c_1-c_2+c_3)a^2H\delta^{ij}\partial_i\partial_jF-a^3\dot{\Phi}_0\delta\dot{\Phi}-a^3V_0'\delta\Phi\Big]\\
&+\psi\Big[-(2c_1-c_2+c_3)a\delta^{ij}\partial_i\partial_j\psi-3(2c_1-c_2+3c_3)a^3H^2\psi+\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\psi\Big]\\
&+F\Big[6(2c_1-c_2+3c_3)a^3\delta^{ij}\partial_i\partial_j\dot{\alpha}+(2c_1-7c_2+21c_3)a^3H^2\delta^{ij}\partial_i\partial_j\alpha+2a^2\dot{\Phi}_0\delta^{ij}\partial_i\partial_j\delta\Phi-V_0a^3\delta^{ij}\partial_i\partial_j\alpha\\
&-\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\partial_i\partial_j\alpha\Big]+F\Big[10c_1a^3H^2\delta^{ij}\partial_i\partial_jF-(2c_1-c_2+2c_3)a\delta^{ij}\delta^{kl}\partial_i\partial_j\partial_k\partial_lF\Big]\\
&+\frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi}-\frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi-\frac{1}{2}a^3V_0''\delta\Phi\delta\Phi\,, \end{aligned}\tag{34}$$

where we integrated by parts and neglected surface terms.

Varying with respect to ψ and F, we derive the following constraint equations:

$$\begin{aligned}0 &= 2(2c_1-c_2+c_3)a^2\delta^{ij}\partial_i\partial_j\dot{\alpha}+4(2c_1-c_2+3c_3)a^2H\delta^{ij}\partial_i\partial_j\alpha-6(2c_1-c_2+3c_3)a^2\delta^{ij}\partial_i\partial_jF-2(2c_1-c_2+c_3)a^2H\delta^{ij}\partial_i\partial_jF\\
&-a^3\dot{\Phi}_0\delta\dot{\Phi}-a^3V_0'\delta\Phi-2(2c_1-c_2+c_3)a\delta^{ij}\partial_i\partial_j\psi-6(2c_1-c_2+3c_3)a^3H^2\psi+a^3\dot{\Phi}_0\dot{\Phi}_0\psi\end{aligned}\tag{35}$$

and

$$\begin{aligned}0 &= 6(2c_1-c_2+3c_3)a^3\delta^{ij}\partial_i\partial_j\dot{\alpha}+(2c_1-7c_2+21c_3)a^3H^2\delta^{ij}\partial_i\partial_j\alpha+2a^2\dot{\Phi}_0\delta^{ij}\partial_i\partial_j\delta\Phi-V_0a^3\delta^{ij}\partial_i\partial_j\alpha\\
&-\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\partial_i\partial_j\alpha+20c_1a^3H^2\delta^{ij}\partial_i\partial_jF-2(2c_1-c_2+2c_3)a\delta^{ij}\delta^{kl}\partial_i\partial_j\partial_k\partial_lF\,.\end{aligned}\tag{36}$$

Substituting these constraints into Eq. (35), we obtain the second-order perturbed Lagrangian density as follows:

$$\begin{aligned}\mathcal{L}^{(\mathrm{SP;2nd\,order})}_{\mathrm{total}} &= \mathcal{L}^{(\mathrm{SP})}_{\mathrm{NGR}}+\mathcal{L}^{(\mathrm{SP})}_{\mathrm{matter}} = (2c_1-c_2+c_3)a^3\delta^{ij}\partial_i\dot{\alpha}\partial_j\dot{\alpha}+4(2c_1-c_2+3c_3)a^3H\delta^{ij}\partial_i\dot{\alpha}\partial_j\alpha-3(c_2-3c_3)a^3H^2\delta^{ij}\partial_i\alpha\partial_j\alpha\\
&+\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\delta^{ij}\partial_i\alpha\partial_j\alpha+2a^2\dot{\Phi}_0\delta^{ij}\partial_i\alpha\partial_j\delta\Phi\\
&+\psi\Big[(2c_1-c_2+c_3)a\delta^{ij}\partial_i\partial_j\psi+3(2c_1-c_2+3c_3)a^3H^2\psi-\frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\psi\Big]\\
&+F\Big[-10c_1a^3H^2\delta^{ij}\partial_i\partial_jF+(2c_1-c_2+2c_3)a\delta^{ij}\delta^{kl}\partial_i\partial_j\partial_k\partial_lF\Big]\\
&+\frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi}-\frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi-\frac{1}{2}a^3V_0''\delta\Phi\delta\Phi\,.\end{aligned}\tag{37}$$

We find that all perturbation fields, α, ψ, and F, decouple from each other, and the scalar mode α can propagate. The ghost-free condition for the scalar mode α is given as

$$2c_1-c_2+c_3>0\,.\tag{38}$$

For regular systems, the scalar mode α does not propagate in Types 2, 6 (TEGR), and 9, whereas it does in all other types. For irregular systems, this mode propagates in Types 4 and 7. We remark that, as mentioned in Sec. II.A, in irregular systems, the DOFs based on the DB analysis cannot provide the upper bound of the perturbation. Thus, in Type 7, the propagating mode α can exist.

C. Pseudo-scalar Perturbation

We calculate the pseudo-scalar perturbation of NGR up to second order. Focusing on the pseudo-scalar terms of Eq. (16), the co-vierbein, vierbein, metric, inverse metric components, and determinant of the co-vierbein are respectively derived as follows:

$$\begin{aligned}
e^0{}_\mu \, \mathrm{d}x^\mu &= \mathrm{d}t\,, \\
e^a{}_\mu \, \mathrm{d}x^\mu &= a\,\delta^{aj} \left(\delta_{ij} - \epsilon_{ijk}\,\delta^{kl}\,\partial_l \tilde{\sigma}\right) \mathrm{d}x^i\,, \\
e_0{}^\mu \, \frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial t}\,, \\
e_a{}^\mu \, \frac{\partial}{\partial x^\mu} &= a^{-1} \left[\delta_a{}^i - \delta_{ak}\,\epsilon^{ijk}\,\partial_j \tilde{\sigma} \right.\\
&\quad \left. + \delta_{an}\,\delta_{lm}\,\epsilon^{lik}\,\epsilon^{mjn}\,\partial_j\tilde{\sigma}\,\partial_k\tilde{\sigma}\right] \frac{\partial}{\partial x^i}\,, \\
g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu &= -\mathrm{d}t\mathrm{d}t + a^2 \left[\delta_{ij} \right.\\
&\quad \left. + \epsilon_{mik}\,\epsilon_{ljn}\,\delta^{ml}\,\delta^{kp}\,\delta^{nq}\,\partial_p\tilde{\sigma}\,\partial_q\tilde{\sigma}\right] \mathrm{d}x^i \mathrm{d}x^j\,, \\
g^{\mu\nu}\,\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} &= -\frac{\partial}{\partial t}\frac{\partial}{\partial t} + a^{-2}\left[\delta^{ij} \right.\\
&\quad \left. - \delta_{ml}\,\epsilon^{mik}\,\epsilon^{ljn}\,\partial_k\tilde{\sigma}\,\partial_n\tilde{\sigma}\right] \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\,, \\
\theta &= a^3 \left(1 + \delta^{ij}\partial_i\tilde{\sigma}\partial_j\tilde{\sigma}\right)\,,
\end{aligned} \tag{39}$$

where ϵ^{ijk} is the Levi-Civita symbol.[1)] The torsion tensor components given in Appendix C.

Repeating the same procedure as that of the tensor and scalar perturbation, we obtain

$$\begin{aligned}
\mathcal{L}^{(\text{pseud SP};\,\text{2nd order})}_{\text{total}} &= \mathcal{L}^{(\text{pseud SP})}_{\text{NGR}} + \mathcal{L}^{(\text{pseud SP})}_{\text{matter}} = -2(2c_1+c_2)\delta^{ij}\partial_i\dot{\tilde{\sigma}}\partial_j\dot{\tilde{\sigma}} - 4(2c_1-c_2+3c_3)Ha^3\delta^{ij}\partial_i\tilde{\sigma}\partial_i\dot{\tilde{\sigma}} + (2c_1-c_2)a\delta^{ij}\delta^{kl}\partial_i\partial_k\tilde{\sigma}\partial_j\partial_l\tilde{\sigma} \\
&\quad - 3(2c_1-c_2+3c_3)H^2a^3\delta^{ij}\partial_i\tilde{\sigma}\partial_j\tilde{\sigma} + 2(2c_1+3c_2)a\delta^{ij}\delta^{kl}\partial_i\partial_j\tilde{\sigma}\partial_k\partial_l\tilde{\sigma} - a^3V_0\delta^{ij}\partial_i\tilde{\sigma}\partial_j\tilde{\sigma} + \frac{1}{2}a^3\dot{\Phi}_0\dot{\Phi}_0\delta^{ij}\partial_i\tilde{\sigma}\partial_j\tilde{\sigma} \\
&\quad + \frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi} - \frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi - \frac{1}{2}a^3V_0''\delta\Phi\delta\Phi\,.
\end{aligned} \tag{40}$$

The ghost-free condition for the pseudo-scalar mode $\tilde{\sigma}$ is

$$2c_1 + c_2 < 0\,, \tag{41}$$

and c_3 is arbitrary.

We find that the perturbation field $\tilde{\sigma}$ can propagate in NGR. For regular systems, the pseudo-scalar mode does not propagate in Types 3, 6 (TEGR), and 8, whereas it does in Types 1, 2, 5, and 9. For irregular systems, Type 7 does not contain the pseudo-scalar mode, while Type 4 does.

D. Vector and Pseudo-vector Perturbation

We calculate the vector and pseudo-vector perturbations of NGR up to second order. We note that a coupling term between α_i and $\tilde{V}_i$ appears, indicating that we cannot separate these perturbations. Focusing on the vector and pseudo-vector terms of Eq. (16), we obtain the respective co-vierbein, vierbein, metric, and inverse metric components as follows:

$$\begin{aligned}
e^0{}_\mu \, \mathrm{d}x^\mu &= \mathrm{d}t + a\,\alpha_i\,\mathrm{d}x^i\,, \\
e^a{}_\mu \, \mathrm{d}x^\mu &= \delta^{ai}\,G_i\,\mathrm{d}t + a\,\delta^{aj}\left(\delta_{ij} - \epsilon_{ijk}\,\delta^{kl}\,\tilde{V}_l\right)\mathrm{d}x^i\,, \\
e_0{}^\mu \, \frac{\partial}{\partial x^\mu} &= \left[1 + \delta^{ij}\alpha_i G_j\right]\frac{\partial}{\partial t} + a^{-1}\left[-\delta^{ij}G_j - \epsilon^{ijk}G_j\tilde{V}_k\right]\frac{\partial}{\partial x^i}\,, \\
e_a{}^\mu \, \frac{\partial}{\partial x^\mu} &= \left[-\delta_a{}^i\alpha_i + \delta_{aj}\epsilon^{ijk}\alpha_i\tilde{V}_k\right]\frac{\partial}{\partial t} + a^{-1}\left[\delta_a{}^i + \epsilon^{ijk}\delta_{aj}\tilde{V}_k + \delta^{ij}\delta_a{}^k G_j\alpha_k - \delta_{kl}\delta_{an}\epsilon^{kij}\epsilon^{lnm}\tilde{V}_j\tilde{V}_m\right]\frac{\partial}{\partial x^i}\,,
\end{aligned}$$

1) Expanding the third term of the right-hand side of the fourth equation in Eq. (40), we get

$$e_a{}^\mu\,\frac{\partial}{\partial x^\mu} = a^{-1}\left[\delta_a{}^n - \delta_{ai}\,\epsilon^{ijn}\,\partial_j\tilde{\sigma} + \delta_{ai}\,\delta^{ij}\,\partial_j\tilde{\sigma}\,\partial_n\tilde{\sigma} - \delta_{an}\,\delta^{ij}\,\partial_i\tilde{\sigma}\,\partial_j\tilde{\sigma}\right]\frac{\partial}{\partial x^n}\,.$$

This result coincides exactly with the pioneering work Ref. [47]. However, we do not expand the Levi-Civita symbol at this stage for convenience in calculation.

$$
\begin{aligned}
g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu &= \left[-1+\delta^{ij}G_iG_j\right]\mathrm{d}t\mathrm{d}t + a\left[-2\alpha_i + 2G_i - 2\delta^{jl}\delta^{km}\epsilon_{ijk}G_l\tilde{V}_m\right]\mathrm{d}t\mathrm{d}x^i + a^2\left[\delta_{ij} - \alpha_i\alpha_j + \delta^{pn}\delta^{kl}\delta^{qr}\epsilon_{pik}\epsilon_{njq}\tilde{V}_l\tilde{V}_r\right]\mathrm{d}x^i\mathrm{d}x^j\,,\\
g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} &= \left[-1-2\delta^{ij}\alpha_iG_j + \delta^{ij}\alpha_i\alpha_j\right]\frac{\partial}{\partial t}\frac{\partial}{\partial t} + a^{-1}\left[2\delta^{ij}G_j - 2\delta^{ij}\alpha_j + 2\epsilon^{ijk}G_j\tilde{V}_k\right]\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\\
&\quad + a^{-2}\left[\delta^{ij} - \delta^{ik}\delta^{jl}G_kG_l + 2\delta^{ik}\delta^{jl}\alpha_kG_l - \delta_{lm}\epsilon^{lik}\epsilon^{mjn}\tilde{V}_k\tilde{V}_n\right]\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\,,\\
\theta &= a^3\left(1+\delta^{ij}\tilde{V}_i\tilde{V}_j - \delta^{ij}\alpha_iG_j\right)\,,
\end{aligned}
\tag{42}
$$

The torsion tensor components are given in Appendix D.

Repeating the same procedure, we obtain

$$
\begin{aligned}
\mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP\,;\,2nd\,order})}_{\mathrm{total}} &= \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{NGR}} + \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{matter}} = (\in\rfloor_\infty - \rfloor_\in + \rfloor_\ni)\dashv^\ni\delta^{\rangle|}\dot{\alpha}_\rangle\dot{\alpha}_| + \triangle(\in\rfloor_\infty - \rfloor_\in + \ni\rfloor_\ni)\dashv^\ni\mathcal{H}\delta^{\rangle|}\dot{\alpha}_\rangle\alpha_|\\
&\quad - 2c_1a\delta^{ij}\delta^{kl}\partial_i\alpha_k\partial_j\alpha_l - 3(c_2-3c_3)a^3H^2\delta^{ij}\alpha_i\alpha_j - 2(2c_1+c_2)a^3\delta^{ij}\dot{\tilde{V}}_i\dot{\tilde{V}}_j\\
&\quad - 4(2c_1-c_2+3c_3)a^3H\delta^{ij}\dot{\tilde{V}}_i\tilde{V}_j - a^3V_0\delta^{ij}\tilde{V}_i\tilde{V}_j + (2c_1-c_2+c_3)a\delta^{ij}\delta^{kl}\partial_i\tilde{V}_k\partial_j\tilde{V}_l\\
&\quad - 3(2c_1-c_2+3c_3)a^3H^2\delta^{ij}\tilde{V}_i\tilde{V}_j + 2(2c_2-c_3)a^2\epsilon^{ijk}\partial_i\dot{\alpha}_j\tilde{V}_k - 4(2c_1-c_2+3c_3)a^2H\epsilon^{ijk}\alpha_i\partial_j\tilde{V}_k\\
&\quad + G_i\left[-2c_1a\delta^{ij}\delta^{kl}\partial_k\partial_lG_j + 6c_1a^3H^2\delta^{ij}G_j\right] + \frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi} - \frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi - \frac{1}{2}V_0''a^3\delta\Phi\delta\Phi\,.
\end{aligned}
\tag{43}
$$

We notice that the perturbation field G_i decouples from other fields, but α_i and $\tilde{V}_i$ are coupled with each other, as in the sixth line of Eq. (45). The ghost-free conditions for the vector and pseudo-vector modes, α_i and $\tilde{V}_i$, are given as

$$
2c_1 - c_2 + c_3 > 0\,, \tag{44}
$$

and

$$
2c_1 + c_2 < 0 \qquad \text{and} \qquad c_3 \text{ is arbitrary} \tag{45}
$$

respectively.

First, we consider regular systems (see Table 1). In the case $2c_1 - c_2 + c_3 = 0$ and $2c_1 + c_2 = 0$, that is, in Type 6 (TEGR), we find that α and $\tilde{V}_i$ do not propagate, as desired.

In the case of solely $2c_1 - c_2 + c_3 = 0$, that is, in Types 2 and 9, the perturbed Lagrangian density becomes as follows:

$$
\begin{aligned}
\mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{total}} &= \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{NGR}} + \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{matter}} = -2(2c_1+c_2)a^3\delta^{ij}\dot{\tilde{V}}_i\dot{\tilde{V}}_j - 4(2c_1-c_2+3c_3)a^3H\delta^{ij}\dot{\tilde{V}}_i\tilde{V}_j - a^3V_0\delta^{ij}\tilde{V}_i\tilde{V}_j\\
&\quad - 3(2c_1-c_2+3c_3)a^3H^2\delta^{ij}\tilde{V}_i\tilde{V}_j + \alpha_i\Big[4(2c_1-c_2+3c_3)a^3H\delta^{ij}\dot{\alpha}_j\\
&\quad + 12(2c_1-c_2+3c_3)a^3H^2\delta^{ij}\alpha_j + 4(2c_1-c_2+3c_3)a^3\dot{H}\delta^{ij}\alpha_j - 2c_1a\delta^{ij}\delta^{kl}\partial_k\partial_l\alpha_j + 3(c_2-3c_3)a^3H^2\delta^{ij}\alpha_j\Big]\\
&\quad + G_i\left[-2c_1a\delta^{ij}\delta^{kl}\partial_k\partial_lG_j + 6c_1a^3H^2\delta^{ij}G_j\right] + \frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi} - \frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi - \frac{1}{2}V_0''a^3\delta\Phi\delta\Phi\,,
\end{aligned}
\tag{46}
$$

where we used the constraint with respect to α_i. We find that all the perturbation modes decouple from each other, and the pseudo-vector mode $\tilde{V}_i$ propagates in Types 2 and 9.

In the case of solely $2c_1 + c_2 = 0$, that is, in Types 3 and 8, the perturbed Lagrangian density is

$$
\begin{aligned}
\mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{total}} &= \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{NGR}} + \mathcal{L}^{(\mathrm{VP\,and\,pseudo\,VP})}_{\mathrm{matter}} = (\in\rfloor_\infty - \rfloor_\in + \rfloor_\ni)\dashv^\ni\delta^{\rangle|}\dot{\alpha}_\rangle\dot{\alpha}_| + \triangle(\in\rfloor_\infty - \rfloor_\in + \ni\rfloor_\ni)\dashv^\ni\mathcal{H}\delta^{\rangle|}\dot{\alpha}_\rangle\alpha_|\\
&\quad - 2c_1a\delta^{ij}\delta^{kl}\partial_i\alpha_k\partial_j\alpha_l - 3(c_2-3c_3)a^3H^2\delta^{ij}\alpha_i\alpha_j + \tilde{V}_i\Big[-4(2c_1-c_2+3c_3)a^3H\delta^{ij}\dot{\tilde{V}}_j\\
&\quad - 12(2c_1-c_2+3c_3)a^3H^2\tilde{V}_j - 4(2c_1-c_2+3c_3)a^3\dot{H}\delta^{ij}\tilde{V}_j\\
&\quad + a^3V_0\delta^{ij}\tilde{V}_j + (2c_1-c_2+c_3)a\delta^{ij}\delta^{kl}\partial_k\partial_l\tilde{V}_j + 3(2c_1-c_2+3c_3)a^3H^2\delta^{ij}\tilde{V}_j\Big]
\end{aligned}
$$

$$+G_i\left[-2c_1 a\delta^{ij}\delta^{kl}\partial_k\partial_l G_j+6c_1 a^3 H^2\delta^{ij}G_j\right]+\frac{1}{2}a^3\delta\dot{\Phi}\delta\dot{\Phi}-\frac{1}{2}a\delta^{ij}\partial_i\delta\Phi\partial_j\delta\Phi-\frac{1}{2}V_0'' a^3\delta\Phi\delta\Phi\,, \tag{47}$$

where we used the constraint with respect to $\tilde{V}_i$. We find that all perturbation modes decouple from each other, and the vector mode α_i propagates in Types 3 and 8.

In Type 1, the parameters c_1, c_2, and c_3 can be freely chosen. This flexibility allows us to select values for c_1, c_2, and c_3 such that they satisfy both $2c_2-c_3=0$ and $2c_1-c_2+3c_3=0$, which is valid for decoupling α and $\tilde{V}_i$ modes. Then, all perturbation modes decouple from each other, and the vector mode α_i and pseudo-vector mode $\tilde{V}_i$ propagate. We conclude that Type 1 can contain at most the vector and pseudo-vector modes. In Type 5, however, we cannot make the perturbation modes α_i and $\tilde{V}_i$ decouple due to the existence of the first term in the sixth line of Eq. (45). Taking into account the result of the DB analysis [15], which suggests that the upper bound of DOFs in Type 5 is seven, we can conclude that either α_i or $\tilde{V}_i$ propagates.

Second, we consider irregular systems (see Table 1).

Type 7 contains the vector mode α_i because this type satisfies a common condition with Type 3. In Type 4, unfortunately, we cannot decouple any modes in Eq. (45). In Ref. [16], it is shown that the constraint surface, denoted by Γ_0, of Type 4 contains that of its regularized system, denoted by Γ_1, which implies that a part of the perturbation modes could be ascribed to the DOFs of the outside region of the regularized system $\Gamma_0\cap\neg\Gamma_1$. If this is the case, Type 4 contains either the vector mode α_i or pseudo-vector mode $\tilde{V}_i$.

VI. CONCLUSIONS

In this work, we investigated the propagating mode in each type of NGR up to second order. After summarizing recent progress on the Hamiltonian analysis of NGR, we reconsidered the vierbein perturbation framework and clarified the correspondence between each perturbation field and vierbein component. Throughout this, we addressed issues 1) and 2) in Sec. I. We revealed that the spatially flat gauge is an adequate gauge choice in a theory with the violation of local LI, which addresses issue 3) in Sec. I. To consider cosmological perturbations, we introduced a scalar field as matter and derived the background-field equations of NGR. Finally, we performed the perturbative analysis of NGR up to second order to reveal the propagating modes in each type of NGR. The results are summarized in Table 2. The emergence of modes is consistent with the consideration in Sec. III-F of Ref. [15] and Sec. III-F of Ref. [16]. We found new propagating modes in the second-order perturbation theory of NGR, which addresses issue 4) in Sec. I.

We compare our result with the previous works [19, 21, 22]. The perturbative analysis around the Minkowski background has been performed by several groups in Refs. [19, 21]. In Ref. [21], Types 1, 2, 3, 5, 6 (TEGR), and 8 are considered gravitational theories including tensorial propagating modes. The results of this work coincide with those of Ref. [19] except for Type 8; it concluded that the tensor mode does not exist. In Ref. [22], applying the conformal transformation to the result in Ref. [19], cosmological perturbative analysis is carried out. In all cases, no new propagating modes appear; pure gauge degrees of freedom are converted into constrained variables. Our result differs from Ref. [22], except for Types 1 and 6 (TEGR). We shall enumerate the differences as follows:[1)]

- Type 1

There is almost no difference from [22] except for the vector and pseudo-vector modes, α_i and $\tilde{V}_i$, that decouple only in a specific case. A ghost-free parameter space exists, and our result differs from that in Ref. [21].

- Type 2

The pseudo-vector mode $\tilde{V}_i$ propagates in our case, in addition to the propagating modes in Ref. [22]. A ghost-free parameter space exists, and our result coincides with that in Ref. [21].

- Type 3

The vector mode α_i propagates in our case, in addition to the propagating modes in Ref. [22]. A ghost-free parameter space exists, and our result coincides with that in Ref. [21].

- Type 4

In our case, the scalar mode α propagates. Either the vector or pseudo-vector modes, α_i or $\tilde{V}_i$, can propagate in a specific case, whereas only half of α_i always propagates in Ref. [22]. The pseudo-scalar mode $\tilde{\sigma}$ propagates as in Ref. [22]. A ghost-free parameter space exists.

1) For the reader's convenience, we list the correspondence of perturbation fields between our work and Ref. [19] as follows (the left variables are ours, the right variables are those in Ref. [19]):

$$\varphi\leftrightarrow\psi,\quad \psi\leftrightarrow\phi,\quad B\leftrightarrow\sigma,\quad F\leftrightarrow\zeta,\quad \alpha\leftrightarrow\beta,\quad \tilde{\sigma}\leftrightarrow s,\quad C_i\leftrightarrow c_i,\quad G_i\leftrightarrow v_i,\quad \alpha_i\leftrightarrow u_i,\quad \tilde{V}_i\leftrightarrow\chi_i\,.$$

The tensor mode is represented using the standard notation. In our formulation, C_i is introduced as $2\partial_{(i}C_{j)}$ according to the convention, by contrast, they introduce it just as c_i. Moreover, in our analysis, the perturbation field G_i does not propagate, which is a constrained variable. Thus, we can regard the perturbation field α_i in our notation as both $\mathcal{M}_i=(u_i-v_i)/2$ and $\mathcal{L}_i=(u_i+v_i)/2$ in their notation.

Table 2. Summary of our work on the linear perturbations of NGR around the flat FLRW background spacetime. Perturbations that are listed on the left-hand side of ";" always propagate. Perturbations given on the right-hand side of ";" can propagate under the imposition of a specific condition on the parameters. “XXX - YYY” denotes the range from XXX to YYY. Type 6 is TEGR, which is equivalent to GR. We remark that for irregular systems, the non-linear DOF cannot restrict the number of perturbation modes. See Sec. II A or Refs. [16, 51] for details.

Theory	Regularity	# of Non-linear DOF (DB analysis)	Propagating Modes (Perturbative analysis)	Ghost-free conditions
Type 1	–	8	6 - 8: ($h_{ij},\alpha,\tilde{\sigma},\alpha_i$(or $\tilde{V}_i$); $\tilde{V}_i$(or α_i))	$2c_1-c_2<0$ & $2c_1-c_2+c_3>0$ & $2c_1+c_2<0$
Type 2	✓	6	5: ($h_{ij},\tilde{\sigma},\tilde{V}_i$)	$2c_1-c_2<0$ & $2c_1+c_2<0$
Type 3	✓	5	5: (h_{ij},α,α_i)	$2c_1-c_2<0$ & $2c_1-c_2+c_3>0$
Type 4	×	5	2 - 4: ($\alpha,\tilde{\sigma}$; either α_i or $\tilde{V}_i$)	$2c_1-c_2+c_3>0$ & $2c_1+c_2<0$
Type 5	✓	7	6: ($h_{ij},\alpha,\tilde{\sigma}$, either α_i or $\tilde{V}_i$)	$2c_1-c_2<0$ & $2c_1-c_2+c_3>0$ & $2c_1+c_2<0$
Type 6	✓	2	2: (h_{ij})	$2c_1-c_2<0$
Type 7	×	0 (Topological)	3: (α,α_i)	$2c_1-c_2+c_3>0$
Type 8	✓	6 or 4 (Bifurcate)	5: (h_{ij},α,α_i)	$2c_1-c_2<0$ & $2c_1-c_2+c_3>0$
Type 9	✓	3	3: ($\tilde{\sigma},\tilde{V}_i$)	$2c_1+c_2<0$

● Type 5

The scalar mode α propagates in our case. However, the vector mode α_i propagates in a specific case, and if this mode propagates, the pseudo-vector mode $\tilde{V}_i$ cannot propagate, and vice versa. Both the scalar and vector modes propagate in Ref. [22]. The propagation of the tensor and pseudo-scalar modes, h_{ij} and $\tilde{\sigma}$, is the same as in Ref. [22]. A ghost-free parameter space exists, and our result differs from that in Ref. [21].

● Type 6 (TEGR)

There is no difference from Ref. [22]. A ghost-free parameter space exists, and our result coincides with that in Ref. [21].

● Type 7

The scalar and vector modes, α and α_i, propagate, but the tensor mode h_{ij} does not in our analysis. On the contrary, only the tensor mode h_{ij} propagates in Ref. [22]. A ghost-free parameter space exists.

● Type 8

The scalar, vector, and tensor modes, α, α_i, and h_{ij}, propagate in our case, whereas no propagating mode exists in Ref. [22]. A ghost-free parameter space exists, and our result coincides with that in Ref. [21].

● Type 9

The pseudo-vector mode $\tilde{V}_i$ propagates in our case, in addition to the propagating modes in Ref. [22]. A ghost-free parameter space exists.

The additional propagation modes in Types 2, 3, and 9 can be attributed to the higher-order perturbative terms included in our analysis. By contrast, the differences in the propagating modes of Types 4, 5, 7, and 8 between Ref. [22] and our work may stem from different gauge choices. It should be noted that these analyses were not conducted in terms of gauge-invariant variables. Here, we note again that a violation of symmetry limits the proper choice of gauge. A perturbation field originating from a broken symmetry should not be fixed, and such variables should not be confused with other perturbation fields that respect symmetry. For instance, one should not impose a gauge such as $\tilde{V}_i=0$ or $\alpha=F$, as F obeys diffeomorphism invariance, which holds in NGR, whereas $\tilde{V}_i$ and α are related to local Lorentz invariance, which is not preserved in NGR. Otherwise, such discrepancies may be signal issues in the perturbative framework. A more detailed investigation of this point is left for future work.

In Table 2, we observe discrepancies between the columns # of Non-linear DOF and Propagating mode. There are two possible scenarios that explain these discrepancies. The first possibility is that the linearization process accidentally restores part of the symmetry. In this case, additional first-class constraints appear, and as a result, the total number of DOFs is reduced. The second possibility is that strong coupling occurs. In this case, the kinetic term responsible for the propagation of a mode may arise only at higher order in perturbation theory, thereby resolving the apparent absence of the propagating mode at the linear level. Clarifying which of these possibilities explains the discrepancies in each type is an important issue for future work.

Most importantly, in cosmological applications, it has been shown that Type 3 still preserves $SO(3)$ invariance. (for instance, see Refs. [15, 16]). The preservation of $SO(3)$ invariance in Type 3 implies that the modes corresponding to this symmetry never propagate at the non-linear level. Our current results based on the perturbative approach are consistent with this picture and also with existing works [19, 21, 22], independent of any concerns regarding an improper gauge choice or potential issues in the perturbative framework. Furthermore, in Ref. [21], Type 3 allows parameter ranges for c_1, c_2, and c_3 that render the theory stable in the Minkowski background spacetime. The same property can be expected in cosmological perturbations, since, according to Refs. [19, 22], a proper conformal transformation connects the results of perturbative analysis around the Minkowski background to those of the flat FLRW background spacetime. In our analysis, Type 3 has a ghost-free region in the parameter space; thus, our perspectives align with each other at all points.

Given that the number of DOFs in the DB and perturbative analyses coincide and that Type 3 contains the propagating tensor modes, if Type 3 does not suffer from strong couplings, a healthy MAG theory can be obtained for cosmological applications. If this is the case, the theory will provide a new perspective on the large-scale structure formation, including dark matter issues due to the violation of boost invariance, whereas it retains the properties of isotropy in the cosmic microwave background by virtue of the $SO(3)$ invariance of the theory. However, if this is not the case, we should investigate the possibility of implementing the screening mechanism [54, 55] to remedy the strong couplings, with the aim of applying it to astrophysics. For example, violations of the local LI could leave observable imprints in gravitational wave signals [56]. These issues must be considered in future further investigations.

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APPENDIX A: TORSION TENSOR IN TENSOR PERTURBATION

We calculate the torsion tensor components up to second order as follows:

$$
\begin{aligned}
&T^0{}_{0i} = 0\,,\\
&T^0{}_{ij} = 0\,,\\
&T^i{}_{0j} = H\delta^i{}_j + \delta^{ik}\dot{h}_{kj} - h^{ik}\dot{h}_{kj}\,,\\
&T^i{}_{jk} = -\delta^{il}\partial_k h_{lj} + \delta^{il}\partial_j h_{lk} + h^{il}\partial_k h_{lj} - h^{il}\partial_j h_{lk}\,,\\
&T^\mu{}_{0\mu} = 3H + \delta^{ij}\dot{h}_{ij} - h^{ij}\dot{h}_{ij}\,,\\
&T^\mu{}_{i\mu} = -\delta^{jk}\partial_j h_{ik} + \delta^{jk}\partial_i h_{jk} + h^{jk}\partial_j h_{ik} - h^{jk}\partial_i h_{jk}\,,
\end{aligned}
\tag{A1}
$$

where h is the trace of h_{ij}. We set $h^i{}_j = h_j{}^i = \delta^{ik}h_{kj}$ and $h^i{}_j = h_j{}^i = \delta_{jk}h^{ik}$.

APPENDIX B: TORSION TENSOR IN SCALAR PERTURBATION

We calculate the torsion tensor components up to second order as follows:

$$
\begin{aligned}
&T^0{}_{0i} = a(\partial_i\dot{\alpha} - \psi\partial_i\dot{\alpha}) - \partial_i\psi + \delta^{jk}\partial_j\alpha\partial_k\partial_i F + \psi\partial_i\psi\,,\\
&T^0{}_{ij} = 0\,,\\
&T^i{}_{0j} = H\delta^i{}_j - a^{-1}\delta^{ij}\left(\partial_i\partial_j F + \partial_i F\partial_j\psi\right)\\
&\qquad + 2H\delta^{ij}\partial_i F\partial_j\alpha + \delta^{ij}\partial_i F\partial_j\dot{\alpha}\,,\\
&T^i{}_{jk} = 0\,,\\
&T^\mu{}_{0\mu} = 3H - a^{-1}\delta^{ij}\partial_i\partial_j F - a^{-1}\delta^{ij}\partial_i F\partial_j\psi\\
&\qquad + 2H\delta^{ij}\partial_i F\partial_j\alpha + \delta^{ij}\partial_i F\partial_j\dot{\alpha}\,,\\
&T^\mu{}_{i\mu} = \partial_i\psi - \delta^{ij}\partial_i\alpha\partial_j F - \psi\partial_i\psi + a(-\partial_i\dot{\alpha} + \psi\partial_i\dot{\alpha})\,.
\end{aligned}
\tag{B1}
$$

APPENDIX C: TORSION TENSOR IN PSEUDO-SCALAR PERTURBATION

We calculate the torsion tensor components up to second order as follows:

$$
\begin{aligned}
&T^0{}_{0i} = 0\,,\\
&T^0{}_{ij} = 0\,,\\
&T^i{}_{0j} = H\delta^i_j - \delta^{ik}\delta^{lm}\epsilon_{jkl}\partial_m\dot{\tilde{\sigma}} - \delta^{ik}\partial_j\tilde{\sigma}\partial_k\dot{\tilde{\sigma}} + \delta^i{}_j\delta^{kl}\partial_k\tilde{\sigma}\partial_l\dot{\tilde{\sigma}}\,,\\
&T^i{}_{jk} = \delta^{il}\delta^{mn}\epsilon_{jlm}\partial_n\partial_k\tilde{\sigma} - \delta^{il}\delta^{mn}\epsilon_{klm}\partial_n\partial_j\tilde{\sigma} + \delta^{il}\partial_j\tilde{\sigma}\partial_l\partial_k\tilde{\sigma}\\
&\qquad - \delta^{il}\partial_k\tilde{\sigma}\partial_l\partial_j\tilde{\sigma} - \delta^i{}_j\delta^{lm}\partial_l\tilde{\sigma}\partial_m\partial_k\tilde{\sigma} + \delta^i{}_k\delta^{lm}\partial_l\tilde{\sigma}\partial_m\partial_j\tilde{\sigma}\,,\\
&T^\mu{}_{0\mu} = 3H + 2\delta^{ij}\partial_i\tilde{\sigma}\partial_j\dot{\tilde{\sigma}}\,,\\
&T^\mu{}_{i\mu} = \delta^{jl}\delta^{km}\epsilon_{ijk}\partial_l\partial_m\tilde{\sigma} + \delta^{kl}\partial_i\tilde{\sigma}\partial_k\partial_l\tilde{\sigma} + \delta^{jk}\partial_j\tilde{\sigma}\partial_k\partial_i\tilde{\sigma}\,,
\end{aligned}
\tag{C1}
$$

where ϵ_{ijk} is the Levi-Civita symbol.

APPENDIX D: TORSION TENSOR IN VECTOR AND PSEUDO-VECTOR PERTURBATION

We calculate the torsion tensor components up to second order as follows:

$$
\begin{aligned}
T^{0}{}_{0i} &= a\dot{\alpha}_i + \delta^{jk}\alpha_j\partial_i G_k + a\delta^{jk}\delta^{lm}\epsilon_{ijl}\alpha_k\dot{\tilde{V}}_m\,, \\
T^{0}{}_{ij} &= a\left(\partial_i\alpha_j - \partial_j\alpha_i\right) + a\left(\delta^{kl}\delta^{mn}\epsilon_{jkm}\alpha_l\partial_i\tilde{V}_n - \delta^{kl}\delta^{mn}\epsilon_{ikm}\alpha_l\partial_j\tilde{V}_n\right)\,, \\
T^{i}{}_{0j} &= H\delta^{i}{}_{j} - a^{-1}\delta^{ik}\partial_j G_k + \delta^{il}\delta^{km}\epsilon_{ljk}\dot{\tilde{V}}_m + a^{-1}\delta^{ik}\delta^{lm}\delta^{pq}\epsilon_{klp}\tilde{V}_q\partial_j G_m - \delta^{ik}G_k\dot{\alpha}_j + \delta^{kl}\delta^{im}\delta^{pq}\delta^{rs}\epsilon_{jkp}\epsilon_{mlr}\dot{\tilde{V}}_q\tilde{V}_s\,, \\
T^{i}{}_{jk} &= \delta^{li}\delta^{mn}\epsilon_{jlm}\partial_k\tilde{V}_n - \delta^{li}\delta^{mn}\epsilon_{klm}\partial_j\tilde{V}_n + \delta^{il}G_l\partial_k\alpha_j - \delta^{il}G_l\partial_j\alpha_k - \delta^{il}\delta^{mn}\delta^{pq}\delta^{rs}\epsilon_{lnp}\epsilon_{jmr}\tilde{V}_q\partial_k\tilde{V}_s + \delta^{il}\delta^{mn}\delta^{pq}\delta^{rs}\epsilon_{lnp}\epsilon_{kmr}\tilde{V}_q\partial_j\tilde{V}_s\,, \\
T^{\mu}{}_{0\mu} &= 3H - a^{-1}\delta^{ij}\partial_i G_j - a^{-1}\delta^{ij}\delta^{kl}\delta^{mn}\epsilon_{ikm}\partial_j G_n\tilde{V}_l - \delta^{ij}G_i\dot{\alpha}_j + 2\delta^{ij}\dot{\tilde{V}}_i\tilde{V}_j\,, \\
T^{\mu}{}_{i\mu} &= -a\dot{\alpha}_i + \delta^{jk}\delta^{lm}\epsilon_{ijl}\partial_k\tilde{V}_m - a\delta^{jk}\delta^{lm}\epsilon_{ijl}\alpha_k\dot{\tilde{V}}_m + \delta^{jk}G_j\partial_k\alpha_i - \delta^{jk}G_j\partial_i\alpha_k - \delta^{jk}\alpha_k\partial_i G_j + \delta^{jk}\tilde{V}_i\partial_j\tilde{V}_k + \delta^{jk}\tilde{V}_j\partial_i\tilde{V}_k\,.
\end{aligned}
$$

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