Detailed study of the dual symmetry of dense quark matter within the effective action formalism*

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Abstract: The symmetry properties of the phase diagram of dense quark matter composed of u and d quarks with three colors have been investigated within the framework of massless (3+1)-dimensional Nambu–Jona-Lasinio (NJL) and QCD models. It turns out that, in the presence of baryon μ_B, isospin μ_I, chiral μ_5, and chiral isospin μ_{I5} chemical potentials, the Lagrangians of these models are invariant under the so-called dual transformation $\mathcal{D}$. In this paper, it has been shown that the path-integration measure of the corresponding partition functions is also invariant under $\mathcal{D}$. Consequently, the thermodynamic potentials of the NJL model (or QCD) are dually symmetric. In particular, this means that in the full $(\mu_B,\mu_I,\mu_5,\mu_{I5})$ phase portraits of these models, the chiral symmetry breaking (CSB) and charged pion condensation (PC) phases are arranged dually conjugately (or symmetrically) with respect to each other. Dual symmetry between the CSB and charged PC phases of quark matter is a fundamental property of massless dense QCD; it should be observed within the framework of any approximation to the QCD phase diagram, and it has important implications for studies of the phase structure.

Keywords: dense quark matter, Nambu-Jona-Lasinio (NJL) model, chiral symmetry breaking (CSB)

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I. INTRODUCTION

Dense quark matter described by Nambu–Jona-Lasinio (NJL) models with isospin and chiral imbalances exhibits a discrete duality $\mathcal{D}$ between chiral symmetry breaking (CSB) and charged pion condensation (PC) phenomena. Under a Pauli–Gursey–type rotation of the quark fields and the permutation of chemical potentials $\mu_I \leftrightarrow \mu_{I5}$, the two phases are mapped onto one another. This duality has been established in a series of works on $(1+1)$- and $(2+1)$-dimensional four-fermion models [1−10], on the realistic $(3+1)$-dimensional NJL model with two flavors and three colors [9, 11−13], and has been extended to the two-color case with diquark interactions [14, 15]; further dual structures of a three-color NJL model with color superconductivity were investigated in [16, 17]. Most recently, the duality was shown to hold at the level of the two- and three-color QCD Lagrangian itself [18, 19].

Although the invariance of the Lagrangian was established in those works, the invariance of the thermodynamic potential (TDP) was only implied. In the present paper, we close this gap. Within the effective-action formalism, we prove that the TDP of massless QCD and QCD-like models is invariant under $\mathcal{D}$, and we further show that the path-integral measure in the generating functional is also invariant under $\mathcal{D}$, so anomalies do not destroy the duality. Taken together, these results imply that the duality between CSB and charged PC is an intrinsic property of QCD and its effective models, holding beyond the mean-field or leading-N_c approximations.

The paper is organised as follows. Section II sets up the NJL model with the chemical potentials $\mu_B,\mu_I,\mu_{I5},\mu_5$ and recalls the mean-field duality. Section III formalises the dual transformation in the effective-action framework. Section IV extends the analysis to the NJL model with diquark channels, and Sec. V addresses two-flavor QCD itself. Appendices A–D collect technical material on the symmetries of the effective action and on the dual invariance of the path-integral measure.

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II. DUALITY BETWEEN CSB AND CHARGED PC IN THE MEAN-FIELD APPROXIMATION

We consider the three-color ($N_c = 3$), two-flavor, massless NJL Lagrangian extended by baryon, isospin, chiral-isospin, and chiral chemical potentials (cf. Refs. [12, 18, 19]):

$$L_{\rm NJL} = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi + \bar{\psi}\gamma^0\mathcal{M}\psi + \frac{G}{N_c}\left\{(\bar{\psi}\psi)^2 + (\mathrm{i}\bar{\psi}\gamma^5\vec{\tau}\psi)^2\right\}, \quad (1)$$

with the chemical-potential matrix

$$\mathcal{M} = \frac{\mu_B}{3} + \frac{\mu_I}{2}\tau_3 + \frac{\mu_{I5}}{2}\gamma^5\tau_3 + \mu_5\gamma^5, \quad (2)$$

where ψ is a flavor doublet and a color triplet (a color-doublet variant is treated separately in Sec. IV), and τ_k ($k = 1,2,3$) are the Pauli matrices acting in flavor space.

The Lagrangian (1) is invariant under the continuous group $U_B(1)\times U_{I_3}(1)\times U_{AI_3}(1)$, whose generators are given by [1)]

$$U_B(1):\ \psi\to\exp(\mathrm{i}\alpha/3)\psi;\ U_{I_3}(1):\ \psi\to\exp(\mathrm{i}\alpha\tau_3/2)\psi;$$
$$U_{AI_3}(1):\ \psi\to\exp(\mathrm{i}\alpha\gamma^5\tau_3/2)\psi. \quad (3)$$

The associated conserved densities are $n_B = \langle\bar{\psi}\gamma^0\psi\rangle/3$, $n_I = \langle\bar{\psi}\gamma^0\tau_3\psi\rangle/2$, and $n_{I5} = \langle\bar{\psi}\gamma^0\gamma^5\tau_3\psi\rangle/2$. By contrast, the chiral chemical potential μ_5 does not correspond to a conserved charge; it parametrizes matter with a fixed chiral density $n_5 = n_R - n_L$ on timescales over which chirality-changing processes have been exhausted [20–25]. The Lagrangian is also invariant under the electromagnetic $U_Q(1)$ symmetry with $Q = \mathrm{diag}(2/3, -1/3)$.

The motivation for retaining all four chemical potentials is astrophysical. Cold, dense, isospin-asymmetric quark matter is expected in neutron-star cores, and most existing analyses retain only μ_B and μ_I [26–54]. However, the strong magnetic fields characteristic of neutron stars induce nonzero n_5 and n_{I5} via the chiral magnetic [55] and chiral separation [56] effects (see also [13, 57]). Including all four potentials therefore provides a more realistic framework for magnetized dense quark matter.

A. Thermodynamic potential

Before studying the thermodynamics, we consider the partition function $\mathcal{Z}$ of the model (1) in the form of the following path integral:

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{N}^{-1}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left\{\mathrm{i}\int\mathrm{d}^4x L_{\rm NJL}\right\}, \quad (4)$$

where $\mathcal{N}$ is an infinite normalization constant and $\mathcal{D}\bar{\psi}\mathcal{D}\psi \equiv \mathcal{D}\bar{\psi}_u\mathcal{D}\bar{\psi}_d\mathcal{D}\psi_u\mathcal{D}\psi_d$. Note that if $\psi(x)\to U\psi(x)$ and $\bar{\psi}(x)\to\bar{\psi}(x)U^\dagger$, where U is a 2×2 flavor matrix, then the measures $\mathcal{D}\psi$ and $\mathcal{D}\bar{\psi}$ of the functional integral (4) transform according to the rule ((see, *e.g.*, Chapter 19 of Ref. [58] or Chapter 22 of Ref. [59]))

$$\mathcal{D}\psi\to\mathcal{D}(U\psi) = (\det U)^{-1}\mathcal{D}\psi,$$
$$\mathcal{D}\bar{\psi}\to\mathcal{D}(\bar{\psi}U^\dagger) = (\det U^\dagger)^{-1}\mathcal{D}\bar{\psi}. \quad (5)$$

Hence, if $U\in U_{I_3}(1)$, *i.e.*, $U = \cos\alpha + \mathrm{i}\tau_3\sin\alpha$ (see footnote 1), then $UU^\dagger = 1$, and the measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ in the path integral (4) remains unchanged. However, if $U\in U_{AI_3}(1)$, *i.e.*, $U = \cos\alpha + \mathrm{i}\gamma^5\tau_3\sin\alpha$ (see footnote 1), then $\psi_u\to\mathrm{e}^{\mathrm{i}\gamma^5\alpha}\psi_u$, $\psi_d\to\mathrm{e}^{-\mathrm{i}\gamma^5\alpha}\psi_d$, and $\bar{\psi}_u\to\bar{\psi}_u\mathrm{e}^{\mathrm{i}\gamma^5\alpha}$, $\bar{\psi}_d\to\bar{\psi}_d\mathrm{e}^{-\mathrm{i}\gamma^5\alpha}$. In this case, the path integral measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi \equiv \mathcal{D}\bar{\psi}_u\mathcal{D}\bar{\psi}_d\mathcal{D}\psi_u\mathcal{D}\psi_d$ transforms as follows:

$$\begin{aligned}&\mathcal{D}\bar{\psi}_u\mathcal{D}\bar{\psi}_d\mathcal{D}\psi_u\mathcal{D}\psi_d\to\\ &\mathcal{D}(\bar{\psi}_u\mathrm{e}^{\mathrm{i}\gamma^5\alpha})\mathcal{D}(\bar{\psi}_d\mathrm{e}^{-\mathrm{i}\gamma^5\alpha})\mathcal{D}(\mathrm{e}^{\mathrm{i}\gamma^5\alpha}\psi_u)\mathcal{D}(\mathrm{e}^{-\mathrm{i}\gamma^5\alpha}\psi_d)\\ &= \left[\det(\mathrm{e}^{2\mathrm{i}\gamma^5\alpha})\det(\mathrm{e}^{-2\mathrm{i}\gamma^5\alpha})\right]^{-1}\mathcal{D}\bar{\psi}_u\mathcal{D}\bar{\psi}_d\mathcal{D}\psi_u\mathcal{D}\psi_d\\ &= \mathcal{D}\bar{\psi}_u\mathcal{D}\bar{\psi}_d\mathcal{D}\psi_u\mathcal{D}\psi_d, \end{aligned} \quad (6)$$

i.e., it is invariant under transformations of the $U_{AI_3}(1)$ group. Taking into account the trivial fact that this measure is invariant under $U_B(1)$ transformations, we can conclude that not only the Lagrangian $L_{\rm NJL}$ (1), but also the integration measure of the functional integral (4) does not change under transformations (3). This means that the dynamics of dense quark matter, based on the Lagrangian (1), is indeed invariant under the $U_B(1)\times U_{I_3}(1)\times U_{AI_3}(1)$ group.

Further, similarly to the Hubbard-Stratonovich transformation used in Refs [60–62], it is also possible in our case to introduce the auxiliary scalar fields $\sigma(x)$ and $\pi_{1,2,3}(x)$ and then rewrite the partition function (4) in the following form

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{N}'^{-1}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\mathcal{D}\pi_1\mathcal{D}\pi_2\mathcal{D}\pi_3\exp\left\{\mathrm{i}\int\mathrm{d}^4x\widetilde{L}_{\rm NJL}\right\}, \quad (7)$$

where

$$\begin{aligned}\widetilde{L}_{\rm NJL} &= \bar{\psi}\left[\gamma^\rho\mathrm{i}\partial_\rho - \sigma - \mathrm{i}\gamma^5\vec{\pi}\cdot\vec{\tau}\right]\psi + \bar{\psi}\gamma^0\mathcal{M}\psi - \frac{N_c}{4G}\left[\sigma^2 + \vec{\pi}^2\right]\\ &= L_{\rm NJL} - \frac{N_c}{4G}\left[\sigma + 2\frac{G}{N_c}(\bar{\psi}\psi)\right]^2\\ &\quad - \frac{N_c}{4G}\sum_{a=1}^{3}\left[\pi_a + 2\frac{G}{N_c}(\mathrm{i}\bar{\psi}\gamma^5\tau_a\psi)\right]^2. \end{aligned} \quad (8)$$

1) For the following, we recall that $\exp(\mathrm{i}\alpha\tau_3) = \cos\alpha + \mathrm{i}\tau_3\sin\alpha$, $\exp(\mathrm{i}\alpha\gamma^5\tau_3) = \cos\alpha + \mathrm{i}\gamma^5\tau_3\sin\alpha$.

After integrating Eq. (7) over the fields σ and $\vec{\pi}$, we obtain (4). It follows from Eq. (8) that $\widetilde{L}_{\rm NJL}$ differs from the original Lagrangian $L_{\rm NJL}$ by two terms; however, these terms have no effect on the dynamics of the model (1). It is clear from the fact that the Euler-Lagrange equations for auxiliary scalar fields are

$$\sigma(x) = -2\frac{G}{N_c}(\bar{\psi}\psi); \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{\psi}\mathrm{i}\gamma^5\tau_a\psi). \tag{9}$$

These relations do not contain time derivatives and therefore are not genuine equations of motion, but rather algebraic constraint equations. Consequently, when constructing the Hamiltonian of the system from the Lagrangian $\widetilde{L}_{\rm NJL}$, one must take these constraints (9) into account. As follows from Eq. (8), once the constraints are imposed, $\widetilde{L}_{\rm NJL}$ reduces exactly to the original Lagrangian $L_{\rm NJL}$. This means that the dynamics defined by the two Lagrangians are identical, although their corresponding Feynman rules differ.

The advantage of the bosonized Lagrangian $\widetilde{L}_{\rm NJL}$ lies in the presence of auxiliary scalar fields that describe collective quark states in different interaction channels. Owing to this feature, $\widetilde{L}_{\rm NJL}$, rather than the original $L_{\rm NJL}$, is more convenient for analyzing the properties of the ground state, *i.e*. the phase structure of dense quark matter. To this end, one proceeds from the partition function (7) to the generating functional $\mathcal{Z}_{\mathcal{M}}(J)$ of the Green's functions of the scalar fields

$$\mathcal{Z}_{\mathcal{M}}(J) = \exp^{\mathrm{i}N_c\mathcal{W}_{\mathcal{M}}(J)} = \mathcal{N}'^{-1}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\prod_{i=1}^{3}\mathcal{D}\pi_i \times\exp\left\{\mathrm{i}\int \mathrm{d}^4x\left[\widetilde{L}_{\rm NJL}+N_c(\sigma J_0+\vec{\pi}\cdot\vec{J})\right]\right\}, \tag{10}$$

where $\vec{\pi}\cdot\vec{J}\equiv\pi_1J_1+\pi_2J_2+\pi_3J_3$ and $J_0(x)$ and $J_{1,2,3}(x)$ are scalar sources.

Assume that the spinor fields transform according to one of the Abelian groups in (3). We determine how the scalar fields $\sigma(x)$ and $\pi_a(x)$ should transform so that the auxiliary Lagrangian (8) remains invariant. First, it is evident that if the spinor doublet ψ is transformed by the $U_B(1)$ group, then it is sufficient for the auxiliary scalar fields to remain unchanged. Second, if the spinor doublet is transformed by $U_{I_3}(1)$ or $U_{AI_3}(1)$, then, taking into account the footnote 1 and the explicit form of $\widetilde{L}_{\rm NJL}$ (8), one can show that this Lagrangian remains unchanged if the auxiliary scalar fields are transformed, respectively, according to the following rules

$$\begin{aligned}&U_{I_3}(1):\ \sigma\to\sigma;\ \ \pi_3\to\pi_3;\ \ \pi_1\to\cos(\alpha)\pi_1+\sin(\alpha)\pi_2;\\ &\pi_2\to\cos(\alpha)\pi_2-\sin(\alpha)\pi_1,\end{aligned} \tag{11}$$

$$\begin{aligned}&U_{AI_3}(1):\ \pi_1\to\pi_1;\ \ \pi_2\to\pi_2;\ \ \sigma\to\cos(\alpha)\sigma+\sin(\alpha)\pi_3;\\ &\pi_3\to\cos(\alpha)\pi_3-\sin(\alpha)\sigma.\end{aligned} \tag{12}$$

Note that the path integral (10) measure $\mathcal{D}\sigma\prod_{i=1}^{3}\mathcal{D}\pi_i$ is invariant under these scalar-field transformations. Whether the symmetry breakings in (11)-(12) occur can be determined from the thermodynamic potential (TDP). To obtain the TDP, we first derive the effective action $\Gamma(\sigma_{cl},\pi_{a_{cl}})$ by performing the Legendre transform of the functional $\mathcal{W}_{\mathcal{M}}(J)$ (10):

$$\Gamma(\sigma_{cl},\pi_{a_{cl}}) = \mathcal{W}_{\mathcal{M}}(J) - \int \mathrm{d}^4x\left\{\sigma_{cl}(x)J_0(x)+\sum_{a=1}^{3}\pi_{a_{cl}}(x)J_a(x)\right\}, \tag{13}$$

where $\sigma_{cl}(x)=\delta\mathcal{W}_{\mathcal{M}}(J)/\delta J_0(x)$ and $\pi_{a_{cl}}(x)=\delta\mathcal{W}_{\mathcal{M}}(J)/\delta J_a(x)$ denote the ground-state expectation values of the auxiliary scalar fields in the presence of external sources, *i.e.*, $\sigma_{cl}(x)=\langle\sigma(x)\rangle_J$ and $\pi_{a_{cl}}(x)=\langle\pi_a(x)\rangle_J$ (see Eq. (A2) in Appendix A). The classical fields $\sigma_{cl}(x)$ and $\pi_{a_{cl}}(x)$ satisfy the following equations

$$\frac{\delta\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})}{\delta\sigma_{cl}(x)} = -J_0(x), \quad \frac{\delta\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})}{\delta\pi_{k_{cl}}(x)} = -J_k(x). \tag{14}$$

From this, it is clear that the expectation values $\langle\sigma(x)\rangle$ and $\langle\pi_{1,2,3}(x)\rangle$ of the auxiliary scalar fields in the true ground state of the system must satisfy Eqs. (14) in the absence of external sources. In addition, we assume that our physical system is invariant under spacetime translations. This implies that $\langle\sigma(x)\rangle$ and $\langle\pi_{1,2,3}(x)\rangle$ are independent of the coordinates. Thus, these quantities are the components of the global minimum of the TDP $\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})$, which is defined by the relation

$$\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})\Big|_{\sigma_{cl},\pi_{a_{cl}}=const} = -\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})\int \mathrm{d}^4x. \tag{15}$$

The phase structure of the NJL model (1) is determined by the behaviour of $\langle\sigma(x)\rangle$ and $\langle\pi_{1,2,3}(x)\rangle$ as functions of the chemical potentials. The problem is significantly simplified because, as shown in Appendix A, the TDP $\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})$ depends only on two field combinations, $\Sigma\equiv\sqrt{\sigma_{cl}^2+\pi_{3_{cl}}^2}$ and $\Pi\equiv\sqrt{\pi_{1_{cl}}^2+\pi_{2_{cl}}^2}$, *i.e.*,

$$\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}}) \equiv \Omega\left(\Sigma,\Pi\right). \tag{16}$$

B. Dual symmetry in the mean-field approximation

For simplicity, both the phase structure of the NJL model (1) and its TDP (15) are usually studied in the so-called mean-field approximation or, equivalently, at leading order in the large-N_c expansion. In this case, using a general formula

$$\mathcal{N}'^{-1}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{\mathrm{i}\int \mathrm{d}^4x\bar{\psi}D\psi\right\} = [\det D]^{N_c}, \tag{17}$$

One can then integrate over the spinor fields in Eq. (10)

$$\mathcal{Z}_{\mathcal{M}}(J) = \exp^{iN_c\mathcal{W}_{\mathcal{M}}(J)} = \mathcal{N}'^{-1}\int \mathcal{D}\sigma\prod_{k=1}^{3}\mathcal{D}\pi_k \times\exp\left\{\mathrm{i}N_c\left(S_{\mathrm{eff}}(\sigma,\pi_a)+\int \mathrm{d}^4x\left[\sigma J_0+\vec{\pi}\cdot\vec{J}\right]\right)\right\}, \tag{18}$$

where

$$\mathcal{S}_{\mathrm{eff}}(\sigma(x),\pi_a(x)) = -\int \mathrm{d}^4x\left[\frac{\sigma^2(x)+\pi_a^2(x)}{4G}\right]-\mathrm{iTr}_{sfx}\ln D,$$
$$D = \gamma^\nu \mathrm{i}\partial_\nu+\frac{\mu_B}{3}\gamma^0+\frac{\mu_I}{2}\tau_3\gamma^0+\frac{\mu_{I5}}{2}\tau_3\gamma^0\gamma^5+\mu_5\gamma^0\gamma^5 -\sigma(x)-\mathrm{i}\gamma^5\pi_a(x)\tau_a. \tag{19}$$

Here, the Tr-operation denotes the trace over spinor (s), flavor (f), and four-dimensional coordinate (x) spaces. As shown in Appendix B, starting from Eq. (18), one can readily obtain the effective action $\Gamma_{mf}(\sigma_{cl}(x),\pi_{a_{cl}}(x))$ of the model at leading order in the large-N_c expansion (or in the mean-field approximation)

$$\Gamma_{mf}(\sigma_{cl}(x),\pi_{a_{cl}}(x)) = \mathcal{S}_{\mathrm{eff}}(\sigma_{cl}(x),\pi_{a_{cl}}(x)). \tag{20}$$

It follows from Eq. (15) that the mean-field approximation to the TDP of the model is given by the relation

$$\Gamma_{mf}(\sigma_{cl}(x),\pi_{a_{cl}}(x))\Big|_{\sigma_{cl}(x),\pi_{a_{cl}}(x)=\mathrm{const}} = -\Omega_{mf}(\sigma_{cl},\pi_{a_{cl}})\int \mathrm{d}^4x. \tag{21}$$

The mean-field TDP (21) was investigated in a recent paper [12]. In particular, it was shown that $\Omega_{mf}(\sigma_{cl},\pi_{a_{cl}})$ is indeed a function only of the two variables Σ and Π (defined in the text preceding Eq. (16)), *i.e.*, $\Omega_{mf}(\sigma_{cl},\pi_{a_{cl}})\equiv\Omega_{mf}(\Sigma,\Pi)$, and that the mean-field TDP is invariant under the transformation

$$\widetilde{\mathcal{D}}:\ \Sigma\longleftrightarrow\Pi,\quad \mu_I\longleftrightarrow\mu_{I5}. \tag{22}$$

Because of this property, it was shown in Ref. [12] that there is a duality between the CSB and charged PC phenomena; accordingly, we refer to $\widetilde{\mathcal{D}}$ (22) as a dual transformation. This means that, at fixed μ_B and μ_5, the (μ_I,μ_{I5})-phase portrait (obtained within the mean-field approximation) of the NJL model (1) is symmetric under the simultaneous transformations CSB$\leftrightarrow$charged PC and $\mu_I\leftrightarrow\mu_{I5}$.

More specifically, suppose that, for certain fixed values of the chemical potentials μ_B, μ_5, $\mu_I=A$, and $\mu_{I5}=B$, the global minimum of the thermodynamic potential $\Omega_{mf}(\Sigma,\Pi)$ lies at the point $(\Sigma=\Sigma_0\neq 0,\Pi=0)$. In this case, for the chosen set of chemical potentials, the chiral $U_{AI_3}(1)$ symmetry (12) is spontaneously broken, and the model is realized in the chiral symmetry breaking (CSB) phase.

Then, owing to the invariance of the thermodynamic potential under the dual transformation $\widetilde{\mathcal{D}}$ (22), it immediately follows that, at the interchanged values of the chemical potentials, $\mu_I=B$ and $\mu_{I5}=A$, with μ_B and μ_5 unchanged, the global minimum of $\Omega_{mf}(\Sigma,\Pi)$ is located at the point $(\Sigma=0,\Pi=\Sigma_0)$. As follows from Eq. (11), in this region of the phase diagram the isospin $U_{I_3}(1)$ symmetry is spontaneously broken, and the system resides in the charged pion condensation (PC) phase.

Consequently, knowledge of the phase realized in the model (1) for a particular set of external parameters μ_B, μ_I, μ_{I5}, and μ_5 is sufficient to determine the corresponding *dually conjugate* phase realized after the interchange $\mu_I\leftrightarrow\mu_{I5}$, with μ_B and μ_5 kept fixed.

Moreover, various physical quantities, such as condensates, densities, and other observables, that characterize the two dually related phases are connected by the same dual transformation $\widetilde{\mathcal{D}}$. For example, the chiral condensate in the CSB phase at given μ_B, μ_I, μ_{I5}, and μ_5 equals the charged pion condensate in the corresponding dually conjugate PC phase, where the interchange $\mu_I\leftrightarrow\mu_{I5}$ is performed. Similarly, knowledge of the quark number density $n_q(\mu_I,\mu_{I5})$ in the initial CSB phase allows one to obtain the density in the conjugate PC phase simply by swapping the arguments, $n_q(\mu_I,\mu_{I5})\to n_q(\mu_{I5},\mu_I)$, and the same applies to all other thermodynamic quantities.

Thus, the symmetry of the thermodynamic potential of the massless NJL model (1) under the dual transformation (22) greatly simplifies the analysis of dense ($\mu_B\neq 0$) quark matter with simultaneous isospin ($\mu_I\neq 0$) and chiral isospin ($\mu_{I5}\neq 0$) asymmetries. However, it should be emphasized once again that the duality between the CSB and charged PC phenomena, established in Ref. [12], was demonstrated only at the level of the thermodynamic potential calculated within the mean-field (leading-order $1/N_c$) approximation. Whether this duality persists beyond mean-field, *i.e.*, in higher orders of the $1/N_c$ expansion, or holds for the full NJL model itself remains an

open and nontrivial question.

III. DUAL SYMMETRY OF THE SIMPLEST NJL LAGRANGIAN

In the present paper, we demonstrate the following: (i) there exists a discrete Pauli–Gursey-type transformation of the quark fields[1)] that converts the term in Lagrangian (1) responsible for the chiral symmetry breaking (CSB) interaction channel into the corresponding term describing the interaction channel in which charged pion condensation occurs, and vice versa.

Furthermore, (ii) under the same transformation, the terms in Lagrangian (1) corresponding to the isospin and chiral isospin charge densities are interchanged, whereas all other terms of the NJL Lagrangian remain unaffected.

Consequently, (iii) if the interchange $\mu_I \leftrightarrow \mu_{I5}$ is performed simultaneously with this Pauli–Gursey-type transformation, the entire Lagrangian (1) remains invariant. As a result, the dual symmetry (22) is a property of the full thermodynamic potential $\Omega(\Sigma,\Pi)$ of the model, and not merely of its mean-field approximation $\Omega_{mf}(\Sigma,\Pi)$. To investigate this issue, we define the following transformation of the flavor doublet $\psi(x)$ (as will be clear from the following discussion, it is indeed a Pauli–Gursey transformation)

$$\begin{aligned}&\text{PG}: \ \psi_R \to \psi_R, \ \ \psi_L \to \mathrm{i}\tau_1\psi_L,\\ &\text{or} \ \ \overline{\psi_R} \to \overline{\psi_R}, \ \ \overline{\psi_L} \to \overline{\psi_L}(-\mathrm{i})\tau_1,\end{aligned} \tag{23}$$

where $\psi = \psi_L + \psi_R$ and

$$\begin{aligned}&\psi_R \equiv \frac{1+\gamma^5}{2}\psi \equiv \Pi_+\psi, \ \ \psi_L \equiv \frac{1-\gamma^5}{2}\psi \equiv \Pi_-\psi,\\ &i.e. \ \ \overline{\psi_R} = \overline{\psi}\Pi_-, \ \ \overline{\psi_L} = \overline{\psi}\Pi_+.\end{aligned}$$

Alternatively, it can be expressed in the following form:

$$\begin{aligned}&\text{PG}: \ \psi \to \psi_R + \mathrm{i}\tau_1\psi_L, \ \ i.e.,\\ &\text{PG}\psi \equiv \text{PG}\begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix} = \begin{pmatrix}\psi_{uR} + \mathrm{i}\psi_{dL}\\ \psi_{dR} + \mathrm{i}\psi_{uL}\end{pmatrix}.\end{aligned} \tag{24}$$

Using Eqs. (23)–(24), it is straightforward to determine how the simplest quark–antiquark structures in the Lagrangian (1) transform under this Pauli–Gursey-type transformation

$$\overline{\psi}\psi = \overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R \overset{PG}{\longrightarrow} \overline{\psi_R}\mathrm{i}\tau_1\psi_L - \overline{\psi_L}\mathrm{i}\tau_1\psi_R = -\mathrm{i}\overline{\psi}\gamma^5\tau_1\psi \tag{25}$$

$$\begin{aligned}\mathrm{i}\overline{\psi}\gamma^5\tau_1\psi = \mathrm{i}(-\overline{\psi_R}\tau_1\psi_L + \overline{\psi_L}\tau_1\psi_R) \overset{\text{PG}}{\longrightarrow}\\ -\mathrm{i}\overline{\psi_R}\tau_1\mathrm{i}\tau_1\psi_L - \mathrm{i}\overline{\psi_L}\mathrm{i}\tau_1\tau_1\psi_R = \overline{\psi}\psi.\end{aligned} \tag{26}$$

Similarly, it can be shown that

$$\mathrm{i}\overline{\psi}\gamma^5\tau_2\psi \overset{\text{PG}}{\longrightarrow} \mathrm{i}\overline{\psi}\gamma^5\tau_3\psi, \tag{27}$$

$$\mathrm{i}\overline{\psi}\gamma^5\tau_3\psi \overset{\text{PG}}{\longrightarrow} -\mathrm{i}\overline{\psi}\gamma^5\tau_2\psi. \tag{28}$$

This means that the four-fermion structures in Lagrangian (1) responsible for CSB and charged PC are transformed into one another by the PG transformation (23)–(24), *i.e.*

$$(\overline{\psi}\psi)^2 + (\mathrm{i}\overline{\psi}\gamma^5\tau_3\psi)^2 \overset{\text{PG}}{\longleftrightarrow} (\mathrm{i}\overline{\psi}\gamma^5\tau_1\psi)^2 + (\mathrm{i}\overline{\psi}\gamma^5\tau_2\psi)^2. \tag{29}$$

(Note that the original Pauli–Gursey transformation of Refs. [63, 64] connects the four-fermion structures of low-dimensional Lagrangians responsible for the CSB and superconducting channels (see, *e.g.*, Refs. [6−8, 65]). Therefore, to highlight this particular feature of the Pauli–Gursey transformation, in the present paper we refer to any transformation of spinor fields that connects the quark structures of different interaction channels as a Pauli–Gursey-type transformation.) Since the free term $\mathrm{i}\overline{\psi}\gamma^\mu\partial_\mu\psi$ of the Lagrangian (1) remains unchanged under the action of the PG transformation, the original NJL Lagrangian (1) is invariant under the Pauli–Gursey-type transformation (23)–(24) at zero chemical potentials. However, at nonzero chemical potentials, it is no longer a symmetry transformation of the NJL model. Indeed, under the action of the *PG* transformation (23)–(24), the quantities $\overline{\psi}\gamma^0\psi$ and $\overline{\psi}\gamma^0\gamma^5\psi$ remain unchanged, whereas

$$\begin{aligned}\overline{\psi}\gamma^0\gamma^5\tau_3\psi &= (\overline{\psi_R}\gamma^0\tau_3\psi_R - \overline{\psi_L}\gamma^0\tau_3\psi_L)\\ &\overset{\text{PG}}{\longleftrightarrow} (\overline{\psi_R}\gamma^0\tau_3\psi_R + \overline{\psi_L}\mathrm{i}\tau_1\gamma^0\tau_3\mathrm{i}\tau_1\psi_L)\\ &= (\overline{\psi_R}\gamma^0\tau_3\psi_R + \overline{\psi_L}\gamma^0\tau_3\psi_L) = \overline{\psi}\gamma^0\tau_3\psi.\end{aligned} \tag{30}$$

It follows that the densities of the isospin and chiral isospin charges, *i.e.*, the quantities $\overline{\psi}\gamma^0\tau_3\psi$ and $\overline{\psi}\gamma^0\gamma^5\tau_3\psi$ from Eq. (30), transform into each other. Consequently, when both $\mu_I \neq 0$ and $\mu_{I5} \neq 0$, the NJL Lagrangian (1) is

1) The most general form of the Pauli–Gursey transformation is given below Eq. (29).

no longer invariant under the Pauli–Gursey (PG) transformation. Thus, at nonvanishing μ_I and μ_{I5}, the discrete PG transformation ceases to be a true symmetry of the model.

Here, μ_I and μ_{I5} act as external classical backgrounds coupled to the isospin and chiral isospin currents $\overline{\psi}\gamma^0\tau_3\psi$ and $\overline{\psi}\gamma^0\gamma^5\tau_3\psi$, respectively. Since the PG rotation interchanges these two currents [Eq. (30)], interchanging their sources, $\mu_I \leftrightarrow \mu_{I5}$, restores the symmetry and maps the two source terms onto each other. This can also be seen by rewriting the source terms

$$\begin{aligned}&\frac{\mu_I}{2}\overline{\psi}\gamma^0\tau_3\psi+\frac{\mu_{I5}}{2}\overline{\psi}\gamma^0\gamma^5\tau_3\psi\\&=\frac{\mu_I+\mu_{I5}}{2}\overline{\psi_R}\gamma^0\tau_3\psi_R+\frac{\mu_I-\mu_{I5}}{2}\overline{\psi_L}\gamma^0\tau_3\psi_L.\end{aligned}$$

The PG rotation (23) acts only on the left-handed doublet, $\psi_L \to \mathrm{i}\tau_1\psi_L$, and reverses the sign of its bilinear, $\overline{\psi_L}\gamma^0\tau_3\psi_L \to -\overline{\psi_L}\gamma^0\tau_3\psi_L$ (since $\tau_1\tau_3\tau_1 = -\tau_3$), while leaving the right-handed doublet unchanged. Thus, it introduces a minus sign in the $(\mu_I - \mu_{I5})$ term and leaves the $(\mu_I + \mu_{I5})$ term unchanged; the interchange $\mu_I \leftrightarrow \mu_{I5}$ precisely compensates for this sign change.

Consequently, if the permutation of the chemical potentials $\mu_I \longleftrightarrow \mu_{I5}$ is performed simultaneously with the field transformations (23)–(24), the NJL Lagrangian remains invariant. Transformations under which not only the field variables of the Lagrangian but also some of the external parameters of the system (such as chemical potentials, coupling constants, etc.) are simultaneously transformed are hereafter referred to as *dual transformations*. Thus, the NJL Lagrangian (1) is invariant (or symmetric) under the discrete dual transformation $\mathcal{D}$ defined as

$$\mathcal{D}:\ \psi \to \psi_R + \mathrm{i}\tau_1\psi_L,\ \ \mu_I \longleftrightarrow \mu_{I5}. \tag{31}$$

Under this transformation, the quark bilinears corresponding to the CSB and charged PC interaction channels transform into one another (see, *e.g.*, Eq. (29)). Therefore, these channels are $\mathcal{D}$-dual to each other, or, equivalently, the CSB and charged PC phenomena are *dually* $\mathcal{D}$*-conjugated* within the framework of the NJL model (1).

The full thermodynamic potential (TDP) (15)–(16) of the NJL model is derived from the auxiliary bosonized Lagrangian $\widetilde{L}_{\rm NJL}$ (8). It is therefore natural to ask how the scalar fields $\sigma(x)$ and $\vec{\pi}(x)$ should transform under the dual transformation $\mathcal{D}$ (31) of the quark fields so that the auxiliary Lagrangian $\widetilde{L}_{\rm NJL}$ remains invariant. Given that the quark bilinears in $\widetilde{L}_{\rm NJL}$ transform according to Eqs. (25)–(28), it is straightforward to verify that the auxiliary scalar fields must transform as

$$\begin{aligned}\text{PG},\ \mathcal{D}:\ &\sigma(x) \to -\pi_1(x),\ \ \pi_1(x) \to \sigma(x),\\&\pi_2(x) \to \pi_3(x),\ \ \pi_3(x) \to -\pi_2(x),\end{aligned} \tag{32}$$

These transformations are required to leave $\widetilde{L}_{\rm NJL}$ invariant. The same transformation rules (32) can also be derived directly from the Euler–Lagrange equations (9) for the auxiliary fields by applying the dual transformation (31) to the quark fields in those equations.

Importantly, Appendix C provides a direct proof of the invariance of the functional measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ in the partition function (4) under the PG (and hence dual) transformation (31). Consequently, the full effective action $\Gamma_{\mathcal{M}}(\sigma_{cl},\vec{\pi}_{cl})$ and the full thermodynamic potential $\Omega_{\mathcal{M}}(\sigma_{cl},\vec{\pi}_{cl})$ (16) are invariant under the dual transformations (C11)–(C12) of the classical fields $\sigma_{cl}(x)$ and $\pi_{acl}(x)$, accompanied by the interchange of chemical potentials $\mu_I \leftrightarrow \mu_{I5}$ (see, for example, Eq. (C14) in Appendix C). This property—the $\mathcal{D}$-symmetry of the full TDP—is precisely the invariance of the mean-field TDP under the discrete dual transformation $\widetilde{\mathcal{D}}$ (22), which corresponds to the substitutions $\Sigma \leftrightarrow \Pi$ and $\mu_I \leftrightarrow \mu_{I5}$. Physically, this symmetry manifests itself as the duality between the CSB and charged PC phenomena.

Therefore, we conclude that the dual symmetry between the CSB and charged PC phases of dense quark matter with isospin and chiral isospin asymmetries (as described below Eq. (22))—originally observed in Ref. [12] within the mean-field approximation —is an intrinsic property of the massless NJL model (1) itself. Hence, the duality between the CSB and charged PC phases is realized throughout the full $(\mu_B,\mu_I,\mu_{I5},\mu_5)$ phase diagram of the model, rather than only within the mean-field (leading $1/N_c$) framework. This follows directly from the dual symmetry $\mathcal{D}$ (31) of both the microscopic NJL Lagrangian (1) and the functional integration measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ in the path integral.

IV. DUAL SYMMETRY OF NJL LAGRANGIAN WITH COLOR SUPERCONDUCTIVITY

Let us consider a more general type of NJL model. Recall that in the recent paper [17], the phase structure of the four-fermion model whose Lagrangian $L_{\rm CSC}$,

$$L_{\rm CSC} = L_{\rm NJL} + H\sum_{k=1,2,3}[\overline{\psi^c}\gamma^5\tau_2\epsilon_k\psi][\overline{\psi}\gamma^5\tau_2\epsilon_k\psi^c], \tag{33}$$

which differs from Eq. (1) by an additional diquark-interaction term, was discussed in the mean-field approximation. In Eq. (33), $\psi^c = C\overline{\psi}^T$, $\overline{\psi^c} = \psi^T C$ are charge-conjugated spinors, where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge-conjugation matrix (the symbol T denotes transposition), and ϵ_k $(k = 1,2,3)$ are 3×3 matrices acting in the three-dimen-

sional fundamental representation of the color $SU_c(3)$ group, with matrix elements $(\epsilon_k)_{ij}=\epsilon_{ijk}$ (the latter quantity is the totally antisymmetric tensor in the indices $i,j,k=1,2,3$). The remaining notation is the same as that for the Lagrangian $L_{\rm NJL}$ (see the text below Eq. (1)). As for the simpler NJL model discussed in Section II, when analyzing the phase structure of the model (33), it is also convenient to work with its auxiliary Lagrangian $\widetilde{L}_{\rm CSC}$,

$$\widetilde{L}_{\rm CSC}=\widetilde{L}_{\rm NJL}-\frac{1}{4H}\Delta_k^*\Delta_k-\frac{\Delta_k^*}{2}[\overline{\psi^c}\gamma^5\tau_2\epsilon_k\psi]-\frac{\Delta_k}{2}[\overline{\psi}\gamma^5\tau_2\epsilon_k\psi^c], \tag{34}$$

where $\widetilde{L}_{\rm NJL}$ is the auxiliary semibosonized Lagrangian in Eq. (8). In addition to the spinor fields $\psi(x)$ and auxiliary bosonic fields $\sigma(x),\vec{\pi}(x)$ (see Eq. (9)), $\widetilde{L}_{\rm CSC}$ contains auxiliary scalar diquark fields $\Delta_k(x)$ and $\Delta^*_{k'}(x)$, with $k,k'=1,2,3$, which transform in the color $SU_c(3)$ antitriplet and triplet representations, respectively. The Euler–Lagrange equations of motion for these fields are given by

$$\Delta_k(x)=-2H[\overline{\psi^c}\gamma^5\tau_2\epsilon_k\psi],\ \ \Delta_k^*(x)=-2H[\overline{\psi}\gamma^5\tau_2\epsilon_k\psi^c]. \tag{35}$$

Their ground-state expectation values are the order parameters of the color superconducting phase.[1)]

Using the microscopic auxiliary Lagrangian (34), it is much easier to construct the formal effective action $\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})$. First, one can write down the generating functional $\mathcal{Z}_{\rm CSC}(J,\overline{\eta},\eta)$ of the model (33)

$$\begin{aligned}\mathcal{Z}_{\rm CSC}(J,\overline{\eta},\eta)=\mathcal{N}'^{-1}\int\mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}\sigma\prod_{i=1}^{3}\mathcal{D}\pi_i\mathcal{D}\Delta_i\mathcal{D}\Delta_i^*\\ \times\exp\left\{\mathrm{i}\int\mathrm{d}^4x\left[\widetilde{L}_{\rm CSC}+N_c(\sigma J_0+\vec{\pi}\cdot\vec{J}\right.\right.\\ \left.\left.+\overline{\eta}_k\Delta_k+\eta_k\Delta_k^*)\right]\right\}.\end{aligned} \tag{36}$$

The effective action can be constructed using the generating functional $\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta):\mathcal{Z}_{\rm CSC}(J,\overline{\eta},\eta)=\exp^{\mathrm{i}N_c\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta)}$.

$$\begin{aligned}\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})=\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta)\\ -\int\mathrm{d}^4x\{\sigma_{cl}(x)J_0(x)+\pi_{a_{cl}}(x)J_a(x)\\ +\overline{\eta}_a(x)\Delta_{a_{cl}}(x)+\eta_a(x)\Delta^*_{a_{cl}}(x)\},\end{aligned}$$

where $N_c=3$, and summation over the repeated indices $a=1,2,3$ and $k=1,2,3$ in Eq. (36) is implied. The arguments of the effective action, namely the classical scalar fields $\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}}$, are defined by the following equations:

$$\begin{aligned}\sigma_{cl}(x)=\frac{\delta\mathcal{W}_{\rm CSC}}{\delta J_0(x)},\quad \pi_{a_{cl}}(x)=\frac{\delta\mathcal{W}_{\rm CSC}}{\delta J_a(x)},\\ \Delta_{k_{cl}}(x)=\frac{\delta\mathcal{W}_{\rm CSC}}{\delta\overline{\eta}_k(x)},\quad \Delta^*_{k_{cl}}(x)=\frac{\delta\mathcal{W}_{\rm CSC}}{\delta\eta_k(x)},\end{aligned} \tag{37}$$

from which it follows (see also Appendix A) that the classical scalar fields are the ground-state expectation values of the auxiliary scalar fields in the presence of external sources, *i.e.*, $\sigma_{cl}(x)=\langle\sigma(x)\rangle_J$, $\Delta_{k_{cl}}(x)=\langle\Delta_k(x)\rangle_\eta,\ldots$.

It is straightforward to show that the fields $\sigma_{cl}(x)$, $\pi_{a_{cl}}(x)$, $\Delta_{k_{cl}}(x)$, and $\Delta^*_{k'_{cl}}(x)$ satisfy the following equations:

$$\begin{aligned}\frac{\delta\Gamma_{\rm CSC}}{\delta\sigma_{cl}(x)}=-J_0(x),\quad \frac{\delta\Gamma_{\rm CSC}}{\delta\pi_{a_{cl}}(x)}=-J_a(x),\\ \frac{\delta\Gamma_{\rm CSC}}{\delta\Delta_{k_{cl}}(x)}=-\overline{\eta}_k(x),\quad \frac{\delta\Gamma_{\rm CSC}}{\delta\Delta^*_{k_{cl}}(x)}=-\eta_k(x).\end{aligned} \tag{38}$$

Similarly to the discussion in Section II above, the expectation values $\langle\sigma(x)\rangle$, $\langle\pi_{1,2,3}(x)\rangle$, $\langle\Delta_{1,2,3}(x)\rangle$ and $\langle\Delta^*_{1,2,3}(x)\rangle$ of the auxiliary scalar fields in the true ground state of the system must satisfy equations (38) with vanishing external sources, $J_0=J_a=\overline{\eta}_k=\eta_l=0$. Furthermore, the physical system is assumed to be invariant under spacetime translations; therefore, $\langle\sigma(x)\rangle$, $\langle\pi_{1,2,3}(x)\rangle$, $\langle\Delta_k(x)\rangle$ and $\langle\Delta^*_l(x)\rangle$ should not depend on coordinates. Thus, the TDP $\Omega_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})$ is defined by the relation

$$\begin{aligned}\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})\Big|_{\ldots,\Delta_{k_{cl}},\Delta^*_{k'_{cl}}=\rm const}\\ =-\Omega_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})\int\mathrm{d}^4x,\end{aligned} \tag{39}$$

which is indeed a function of the constant scalar fields $\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}}$, one should note that these quantities are the components of the global minimum of the TDP.

It is straightforward to see that, due to the invariance of the Lagrangian (34) and the path-integral measure with respect to the Abelian $U_{I_3}(1)$ and $U_{AI_3}(1)$ symmetries (3), the total TDP $\Omega_{\rm CSC}$ as a function of σ_{cl} and $\pi_{a_{cl}}$ depends only on two combinations of these quantities, Σ and Π, which are defined around Eq. (16) [2)]. To discuss the symmetry properties of the TDP $\Omega_{\rm CSC}$ with respect to the color $SU_c(3)$ group and its dependence on the scalar color

1) The details of the construction of the auxiliary semi-bosonic Lagrangian eLCSC are described both in Ref. [17] and in earlier reviews devoted to color superconductivity [66–71].

2) A similar procedure was carried out in Sec. II.A within the framework of the simpler NJL model (1), and it can be readily repeated in the more general model (33).

diquarks $\Delta_{k_{cl}}, \Delta^*_{k'_{cl}}$, it is necessary to take into account the following fact. Although the quantized fields from the auxiliary Lagrangian (34) transform according to $\psi \to U\psi$, $\Delta \to U^*\Delta$ and $\Delta^* \to U\Delta^*$ (where $U \in SU_c(3)$), the classical diquark fields Δ_{cl} and Δ^*_{cl}, which are included in the set of arguments of the thermodynamic potential $\Omega_{\rm CSC}$, transform as follows: $\Delta_{cl} \to \Delta'_{cl} = (U^*)^{-1}\Delta_{cl} = U^T\Delta_{cl}$, $\Delta^*_{cl} \to \Delta'^*_{cl} = U^{-1}\Delta^*_{cl}$ (for details, see Appendix A). In terms of the color components of the scalar diquark fields, this means that

$$\Delta'_{a_{cl}} = \Delta_{b_{cl}} U_{ba}, \quad \Delta'^*_{a_{cl}} = \left(U^{-1}\right)_{ab} \Delta^*_{b_{cl}}. \tag{40}$$

Because both the $L_{\rm CSC}$ (33)−(34) and the path integration measures $\mathcal{D}\overline{\psi}\mathcal{D}\psi$, $\prod_{i=1}^3 \mathcal{D}\Delta_i \mathcal{D}\Delta_i^*$, and $\mathcal{D}\sigma\prod_{i=1}^3 \mathcal{D}\pi_i$ in the partition functional are invariant under the $SU_c(3)$ group, it can be shown that the effective action $\Gamma_{\rm CSC}(\sigma_{cl}, \pi_{a_{cl}}, \Delta_{cl}, \Delta^*_{cl})$ is invariant with respect to $SU_c(3)$. Consequently, the TDP $\Omega_{\rm CSC}$ of the model is also invariant under the $SU_c(3)$ group, *i.e.*

$$\Omega_{\rm CSC}(\sigma_{cl}, \pi_{a_{cl}}, \Delta'_{cl}, \Delta'^*_{cl}) = \Omega_{\rm CSC}(\sigma_{cl}, \pi_{a_{cl}}, \Delta_{cl}, \Delta^*_{cl}).$$

Thus, instead of six color arguments, the TDP $\Omega_{\rm CSC}$ must depend only on the single color-field combination, $\Delta \equiv \left(\Delta_{1_{cl}}\Delta^*_{1_{cl}} + \Delta_{2_{cl}}\Delta^*_{2_{cl}} + \Delta_{3_{cl}}\Delta^*_{3_{cl}}\right)^{1/2}$. As is clear from Eq. (40), the quantity Δ is invariant under $SU_c(3)$ transformations. (Some details of this conclusion can be clarified on the basis of the material presented in Appendix A.)

Hence, the total TDP $\Omega_{\rm CSC}$ of the NJL model (33)−(34) depends only on three field combinations, *i.e.*

$$\Omega_{\rm CSC}(\sigma_{cl}, \pi_{a_{cl}}, \Delta_{k_{cl}}, \Delta^*_{k'_{cl}}) \equiv \Omega_{\rm CSC}(\Sigma, \Pi, \Delta). \tag{41}$$

Thus, if at the global minimum point of the TDP $\Omega_{\rm CSC}(\Sigma, \Pi, \Delta)$ the quantity Δ is nonzero, then at least one of the quantities $\Delta_{k_{cl}}$ is also nonzero, and the color $SU_c(3)$ symmetry is spontaneously broken. In this case, diquark pairing occurs, and the so-called color superconducting phase is realized in the NJL model (33) and related models.

Recall that in Section III, we proved that the Lagrangian (1) of the simplest massless NJL model is invariant under the dual transformation $\mathcal{D}$ (31), thereby leading to the duality between the CSB and charged PC phases. We now prove that the same property holds for the NJL model (33) with an additional diquark interaction channel.

To verify that the Lagrangian $L_{\rm CSC}$ is invariant under the dual transformation $\mathcal{D}$, it is sufficient to check the invariance of only the second term on the right-hand side of Eq. (33). This can be shown by the following simple transformations (see also Eqs. (23)−(24))

$$\begin{aligned}\Delta_k(x) \sim \overline{\psi^c}\gamma^5\tau_2\epsilon_k\psi &= \psi^T C\gamma^5\tau_2\epsilon_k\psi = \psi_R^T C\tau_2\epsilon_k\psi_R - \psi_L^T C\tau_2\epsilon_k\psi_L \\ &\xrightarrow{\mathcal{D}} \psi_R^T C\tau_2\epsilon_k\psi_R - \psi_L^T \mathrm{i}\tau_1 C\tau_2\epsilon_k \mathrm{i}\tau_1\psi_L \\ &= \psi_R^T C\tau_2\epsilon_k\psi_R - \psi_L^T C\tau_2\epsilon_k\psi_L = \overline{\psi^c}\gamma^5\tau_2\epsilon_k\psi.\end{aligned} \tag{42}$$

In addition, the validity of a similar relation can be readily established

$$\Delta^*_k(x) \sim \overline{\psi}\gamma^5\tau_2\epsilon_k\psi^c \xrightarrow{\mathcal{D}} \overline{\psi}\gamma^5\tau_2\epsilon_k\psi^c. \tag{43}$$

Hence, $L_{\rm CSC}$ as a whole is invariant under the duality transformation $\mathcal{D}$ (31). The auxiliary Lagrangian $\widetilde{L}_{\rm CSC}$ (34) of the model is also invariant under this duality transformation, provided that the corresponding transformations of the spinor fields (31) are supplemented by the transformations (32) of the auxiliary meson fields $\sigma(x)$ and $\vec{\pi}_a(x)$, and that the auxiliary diquark fields are taken to remain unchanged (see Eqs. (42)−(43)).

It is important to note that Appendices C and D prove the invariance of the measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ of the generating functional (36) under the duality transformation (31). The invariance of the measure $\mathcal{D}\sigma\prod_{i=1}^3 \mathcal{D}\pi_i \prod_{k=1}^3 \mathcal{D}\Delta_k \mathcal{D}\Delta_k^*$ under the PG transformations (32) and the unchanged scalar diquark fields Δ_k and Δ_k^* is rather evident, as discussed in Appendix C. Appendix D considers in detail whether the regularization procedure, which is inevitable in effective models, can break the duality of the model. Consequently (see Appendix C), not only the auxiliary Lagrangian $\widetilde{L}_{\rm CSC}$ but also the full effective action $\Gamma_{\rm CSC}(\sigma_{cl}, \vec{\pi}_{cl}, \Delta_{k_{cl}}, \Delta^*_{k'_{cl}})$ and the full TDP $\Omega_{\rm CSC}(\Sigma, \Pi, \Delta)$ (41) of the model are invariant under the PG transformations (C11)−(C12) of the classical fields $\sigma_{cl}(x)$ and $\pi_{a_{cl}}(x)$, with unchanged classical diquark fields $\Delta_{k_{cl}}$ and $\Delta^*_{k'_{cl}}$, whereas $\mu_I \leftrightarrow \mu_{I5}$. We refer to this entire procedure as the dual transformation $\mathcal{D}$ of the TDP. This last property, *i.e.*, the $\mathcal{D}$ symmetry of the full TDP, is precisely its invariance under the dual transformation $\widetilde{\mathcal{D}}$ (22) (when $\Sigma \leftrightarrow \Pi$, $\mu_I \leftrightarrow \mu_{I5}$, and Δ is unchanged) of the TDP found in the mean-field approximation [17]. Therefore, the $\mathcal{D}$ symmetry of the TDP should manifest itself in the full phase portrait of the model as a duality between the CSB and charged PC phenomena.

As a result, we see that the dual symmetry between the CSB and charged PC phenomena in a dense quark medium with isospin and chiral isospin asymmetries (see the description below Eq. (22)), predicted earlier in different NJL models [12, 17] within the framework of the mean-field approximation to the phase structure, is in fact a genuine property of dense quark matter described by the massless NJL models (1) and (33). Thus, the duality

between the CSB and charged PC phenomena is realized in the full $(\mu_B,\mu_I,\mu_{I5},\mu_5)$-phase diagrams of these models, and not only within the scope of the mean-field approximation, since it is a consequence of the dual symmetry $\mathcal{D}$ (31) of both the microscopic NJL Lagrangians (1) and (33) and the measures $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ and $\mathcal{D}\sigma\prod_{i=1}^{3}\mathcal{D}\pi_i\prod_{k=1}^{3}\mathcal{D}\Delta_k\mathcal{D}\Delta_k^*$ of path integration.

V. DUAL SYMMETRY OF DENSE QCD

In Sections III and IV, we established an important result: two formally distinct massless NJL models, (1) and (33), which effectively describe the low-energy and moderate-density regimes of massless QCD, both possess the dual symmetry $\mathcal{D}$ (31) between the CSB and charged PC phenomena. This result naturally suggests that the Lagrangian of massless dense QCD itself should also be invariant under the same dual transformation $\mathcal{D}$. In this section, we demonstrate that this is indeed the case and discuss several related properties of QCD that confirm the existence of a dual symmetry between the CSB and charged PC phases in fundamental QCD.

As a starting point, we consider the Lagrangian for the quark sector of massless QCD, extended by baryon, isospin, chiral, and chiral isospin chemical potentials:

$$\begin{aligned}L_{QCD} = {}& \mathrm{i}\bar{\psi}\gamma^\nu\nabla_\nu\psi + \frac{\mu_B}{3}\bar{\psi}\gamma^0\psi + \mu_5\bar{\psi}\gamma^0\gamma^5\psi \\ & + \frac{\mu_I}{2}\bar{\psi}\gamma^0\tau_3\psi + \frac{\mu_{I5}}{2}\bar{\psi}\gamma^0\gamma^5\tau_3\psi,\end{aligned} \tag{44}$$

where $\psi(x)$ is the two-flavor quark doublet. The covariant derivative $\nabla_\nu = \partial_\nu - \mathrm{i}g\lambda_a A_\nu^a(x)$ acts in color space, where λ_a $(a=1,\ldots,8)$ are the 3×3 Gell–Mann matrices and $A_\nu^a(x)$ are the $SU_c(3)$ gauge fields. The Lagrangian (44) therefore describes dense quark matter in the presence of both isospin and chiral asymmetries.

We now show that the dual symmetry $\mathcal{D}$, defined in Eq. (31)—which, in the NJL framework, relates the CSB and charged PC channels—is also realized in the fundamental QCD Lagrangian (44). It follows from the discussion in the previous sections that the sum of the chemical-potential terms in Eq. (44) is invariant under $\mathcal{D}$. The kinetic term is invariant as well, since

$$\begin{aligned}\mathrm{i}\bar{\psi}\gamma^\nu\nabla_\nu\psi &= \mathrm{i}\bar{\psi}_R\gamma^\nu\nabla_\nu\psi_R + \mathrm{i}\bar{\psi}_L\gamma^\nu\nabla_\nu\psi_L \xrightarrow{\mathcal{D}} \mathrm{i}\bar{\psi}_R\gamma^\nu\nabla_\nu\psi_R \\ &\quad + \mathrm{i}\bar{\psi}_L(-\mathrm{i}\tau_1)\gamma^\nu\nabla_\nu\mathrm{i}\tau_1\psi_L \\ &= \mathrm{i}\bar{\psi}_R\gamma^\nu\nabla_\nu\psi_R + \mathrm{i}\bar{\psi}_L\gamma^\nu\nabla_\nu\psi_L = \mathrm{i}\bar{\psi}\gamma^\nu\nabla_\nu\psi,\end{aligned} \tag{45}$$

Thus, the full QCD Lagrangian (44) is symmetric under the dual transformation $\mathcal{D}$ (31).

It is well known in quantum field theory that the invariance of a classical Lagrangian under a given group of transformations does not necessarily imply the invariance of the corresponding quantum theory under the same transformations (see, for example, Section 9.6 of Ref. [58]). Such discrepancies, known as anomalies, arise when a symmetry of the classical action is not preserved after quantization. This situation typically occurs for axial transformations of spinor fields. To verify that a symmetry group G is non-anomalous, one must explicitly demonstrate the invariance of the functional integration measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ in the partition function of the model under transformations of G.

For illustration, we refer the reader to Appendix A, where the invariance of the $U_{I_3}(1)$ symmetry (11) is demonstrated for both the auxiliary NJL Lagrangian $\widetilde{L}_{\rm NJL}$ of Eq. (1) and its effective action $\Gamma(\sigma_{cl},\pi_{acl})$. There, it is shown that the key step in establishing this invariance is to prove that the integration measure is invariant under the $U_{I_3}(1)$ transformation.

As shown above, the QCD Lagrangian (44) is symmetric under the dual transformation $\mathcal{D}$ (31). Because this transformation involves the spinor matrix γ^5, one might suspect that $\mathcal{D}$ could be anomalous. If this were the case, neither the full effective action nor the thermodynamic potential would remain dually symmetric, and the dual correspondence between the CSB and charged PC phases of dense quark matter would be lost. However, as proven in Appendix C, this suspicion is unfounded: the functional integration measure in the QCD generating functional is invariant under the dual transformation $\mathcal{D}$.

Specifically, Appendix C shows that the measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ is invariant under both vector transformations $U(\vec{\theta})\in SU_V(2)$ and axial transformations $U_A(\vec{\omega})\in SU_A(2)$ (see Eqs. (C2) and (C3), respectively). Consequently, it is invariant under their product $U(\vec{\theta})U_A(\vec{\omega})$. By choosing the specific parameters $\vec{\theta}=-\vec{\omega}=(\pi/4,0,0)$, this combined transformation coincides exactly with the dual (Pauli–Gursey) transformation $\mathcal{D}$ (31) of the spinor doublet $\psi(x)$. Therefore, the path-integral measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ in massless QCD is invariant under $\mathcal{D}$, and this transformation is *non-anomalous*.

It follows that both the effective action and the thermodynamic potential of QCD are symmetric under $\mathcal{D}$. Hence, in the full $(\mu_B,\mu_I,\mu_{I5},\mu_5)$ phase diagram, the CSB phase is dually conjugate to the charged PC phase. This dual symmetry should be regarded as a fundamental property of massless QCD and is expected to manifest itself in any consistent approximation to the QCD phase structure.

VI. SUMMARY AND CONCLUSIONS

In this work, we have demonstrated that the Lagrangians of the massless NJL models (1) and (33), formulated for three-color u and d quarks and including the baryon, isospin, chiral, and chiral isospin chemical potentials μ_B,

μ_I, μ_{I5}, and μ_5, possess a discrete dual symmetry $\mathcal{D}$ (31) that relates the chiral symmetry breaking (CSB) and charged pion condensation (PC) interaction channels (see Sections II–IV). Since the path-integral measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ in the generating functionals of these models is also invariant under the $\mathcal{D}$ transformation (as shown in Appendix C), it follows that the corresponding total thermodynamic potentials, viewed as functions of the chemical potentials and the order parameters Σ and Π associated with the CSB and charged PC phases, are invariant under the dual transformation (22).

Consequently, in the *complete* phase diagram of each model, the CSB and charged PC phases appear as dual, mirror-symmetric counterparts. In particular, at fixed μ_B and μ_5, the (μ_I,μ_{I5}) phase portrait exhibits mirror symmetry with respect to the line $\mu_I = \mu_{I5}$. Thus, we have shown that the dual symmetry between the CSB and charged PC phenomena is an intrinsic property of the full massless NJL models (1) and (33), rather than merely a feature of their mean-field (leading $1/N_c$) approximation, as previously established in Refs. [11, 12, 17].

Furthermore, in Section V, we proved that the same dual symmetry $\mathcal{D}$ (31) is also realized in the fundamental massless QCD Lagrangian (44) for three-color u and d quarks. This result implies that the dual correspondence between CSB and charged PC phenomena is not an artifact of effective theories but a *fundamental property of dense massless QCD*. Hence, it should manifest itself within any consistent approximation to the QCD phase diagram. Moreover, in the case of two-color QCD, additional dual correspondences arise, enriching the symmetry structure of the phase diagram [18, 19].

Establishing this dual correspondence directly from first principles, *i.e.*, within QCD itself, opens the possibility of a broad class of applications of dual symmetries in studies of the QCD phase structure. The concept of duality is no longer confined to the chiral dynamics of quarks: in principle, it can encompass other QCD phenomena, including gluonic degrees of freedom, transitions between confined and deconfined phases, and topological properties. Moreover, the dual symmetry is not restricted to the range of validity of effective models; it should hold at any energy scale and for both real and imaginary values of the chemical potentials, potentially offering new insights into the analytic continuation of QCD thermodynamics.

Finally, we note that the dual symmetry $\mathcal{D}$ has been rigorously established here for massless quarks. In realistic situations, where quarks are massive, remnants of this symmetry should persist to some extent. A physical current mass breaks the symmetry (under the Pauli–Gursey rotation (23), the mass term $m_0\bar{\psi}\psi$ is transformed into the charged-pion channel Eqs. (25)–(26)); however, this breaking is mild, being set by the small mass $m_0 \simeq 5.5$ MeV compared with the much larger scales of dense matter. Indeed, the mean-field analysis of the NJL model at the physical point (with nonzero current masses) [13] demonstrates that the CSB–PC duality becomes approximate but remains clearly visible and provides an excellent approximation at chemical potentials larger than half the pion mass. (Indeed, once the chemical potentials exceed a reasonable value, the CSB and charged PC regions appear practically as in the chiral limit.) Thus, there is no indication that this approximation should not be valid more generally, beyond the mean-field level. For simplicity, the present study has been restricted to zero temperature ($T = 0$), but we see no compelling reason why the duality $\mathcal{D}$ in the phase structure of dense quark matter should disappear at finite temperature. We therefore expect the $\mathcal{D}$-duality to remain a robust and general feature of QCD matter under a wide range of conditions.

APPENDIX A: SYMMETRY PROPERTIES OF EFFECTIVE ACTION AND TDP

• We first consider the symmetry properties of the effective action

$$\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}}) \equiv \Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{3_{cl}};\pi_{1_{cl}},\pi_{2_{cl}})$$
$$= \mathcal{W}_{\mathcal{M}}(J) - \int \mathrm{d}^4x \left\{ \sigma_{cl}(x)J_0(x) + \sum_{a=1}^{3} \pi_{a_{cl}}(x)J_a(x) \right\}, \quad \text{(A1)}$$

where

$$\sigma_{cl}(x) \equiv \frac{\delta \mathcal{W}_{\mathcal{M}}(J)}{\delta J_0(x)} = \frac{1}{\mathcal{Z}_{\mathcal{M}}(J)} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \prod_{a=1}^{3} \mathcal{D}\pi_a \sigma(x)$$
$$\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4y \left[\widetilde{L}_{\mathrm{NJL}} + N_c(\sigma J_0 + \pi_a J_a) \right] \right\} = \langle \sigma(x) \rangle_J,$$
$$\pi_{k_{cl}}(x) \equiv \frac{\delta \mathcal{W}_{\mathcal{M}}(J)}{\delta J_k(x)} = \frac{1}{\mathcal{Z}_{\mathcal{M}}(J)} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \prod_{a=1}^{3} \mathcal{D}\pi_a \pi_k(x)$$
$$\times \exp\left\{ \mathrm{i} \int \mathrm{d}^4y \left[\widetilde{L}_{\mathrm{NJL}} + N_c(\sigma J_0 + \pi_a J_a) \right] \right\}$$
$$= \langle \pi_k(x) \rangle_J, \quad \text{(A2)}$$

which follow from the invariance of the auxiliary Lagrangian $\widetilde{L}_{\mathrm{NJL}}$ (8) under the transformations (3), (11), and (12).

We now examine how the invariance of the auxiliary Lagrangian (8) under these transformations is realized in terms of the functional $\mathcal{W}_{\mathcal{M}}(J)$ (10). For simplicity, we consider in detail only the subgroup $U_{I_3}(1)$. To this end, we change the integration variables in this functional integral as follows: $\psi(x) \to \psi'(x) = \exp(\mathrm{i}\alpha\tau_3/2)\psi(x)$, $\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \to \begin{pmatrix} \pi'_1 \\ \pi'_2 \end{pmatrix} = O(\alpha) \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$, where $O(\alpha)$ is a

2×2 matrix of the form $O(\alpha)=\begin{pmatrix}\cos\alpha,\sin\alpha\\-\sin\alpha,\cos\alpha\end{pmatrix}$ and scalar fields $\sigma(x),\pi_3(x)$ remain unchanged. This change of variables is simply a transformation of the fermionic fields under the action of $U_{I_3}(1)$ (3), and of the scalar fields according to the rule (11). One must take into account that, under this transformation, neither the integration measure nor the Lagrangian changes. The part of the source terms that involves only the scalar fields π_1 and π_2 is transformed as follows (summation over repeated indices from 1 to 2 is implied):

$$\pi_a J_a \to \pi'_a J_a = O(\alpha)_{ai}\pi_i J_a = \pi_i O^T(\alpha)_{ia} J_a \equiv \pi_i J'_i, \tag{A3}$$

where $J'_i = O^T(\alpha)_{ia}J_a$; that is, $J'_1=\cos\alpha J_1-\sin\alpha J_2$ and $J'_2=\sin\alpha J_1+\cos\alpha J_2$. Thus, it follows that

$$\mathcal{W}_{\mathcal{M}}(J)\equiv\mathcal{W}_{\mathcal{M}}(J_0,J_3|J_1,J_2)=\mathcal{W}_{\mathcal{M}}(J_0,J_3|J'_1,J'_2). \tag{A4}$$

This means that the functional $\mathcal{W}_{\mathcal{M}}(J_0,J_3|J_1,J_2)$ is invariant under the $SO(2)$ group acting on the two sources J_1 and J_2; that is, it remains unchanged under the transformation $J_1,J_2\to J'_1,J'_2$. We now perform the Legendre transform of the functional $\mathcal{W}_{\mathcal{M}}(J_0,J_3|J'_1,J'_2)$:

$$\begin{aligned}&\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{3_{cl}};\pi'_{1_{cl}},\pi'_{2_{cl}})=\mathcal{W}_{\mathcal{M}}(J_0,J_3|J'_1,J'_2)\\&\quad-\int \mathrm{d}^4x\left\{\sigma_{cl}(x)J_0(x)+\pi_{3_{cl}}(x)J_3(x)+\sum_{a=1}^{2}\pi'_{a_{cl}}(x)J'_a(x)\right\},\end{aligned} \tag{A5}$$

where

$$\begin{aligned}\pi'_{a_{cl}}(x)&=\frac{\delta\mathcal{W}_{\mathcal{M}}(J_0,J_3|J'_1,J'_2)}{\delta J'_a(x)}\\&=\int \mathrm{d}^4z\frac{\delta\mathcal{W}_{\mathcal{M}}(J_0,J_3|J_1,J_2)}{\delta J_i(z)}\frac{\delta J_i(z)}{\delta J'_a(x)}\\&=\pi_{i_{cl}}(x)O(\alpha)_{ia}.\end{aligned} \tag{A6}$$

To obtain the final expression in Eq. (A6), we use the relations $J_i(z)=O(\alpha)_{ia}J'_a(z)$ and $\delta J_i(z)/\delta J'_a(x)=O(\alpha)_{ia}\delta(x-z)$. In addition, the equality (A4) for the functional $\mathcal{W}_{\mathcal{M}}(J_0,J_3|J'_1,J'_2)$ and the definition (A2) of $\pi_{i_{cl}}(x)$ are taken into account. It follows from Eqs. (A6) and (A3) that $\pi'_{a_{cl}}(x)J'_a(x)=\pi_{a_{cl}}(x)J_a(x)$. Combining these results, we find that

$$\begin{aligned}&\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{3_{cl}};\pi'_{1_{cl}},\pi'_{2_{cl}})\\&=\mathcal{W}_{\mathcal{M}}(J)-\int \mathrm{d}^4x\left\{\sigma_{cl}(x)J_0(x)+\sum_{a=1}^{3}\pi_{a_{cl}}(x)J_a(x)\right\}\\&\equiv\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{3_{cl}};\pi_{1_{cl}},\pi_{2_{cl}}),\end{aligned} \tag{A7}$$

That is, the effective action (A1) or (13) remains invariant when $\pi_{1_{cl}}(x)$ and $\pi_{2_{cl}}(x)$ transform as an $SO(2)$ doublet. Similarly, one can show that the effective action $\Gamma(\sigma_{cl},\pi_{3_{cl}};\pi_{1_{cl}},\pi_{2_{cl}})$ is also invariant when $\sigma_{cl}(x)$ and $\pi_{3_{cl}}(x)$ transform as an $SO(2)$ doublet. Hence,

$$\Gamma_{\mathcal{M}}(\sigma'_{cl},\pi'_{3_{cl}};\pi'_{1_{cl}},\pi'_{2_{cl}})=\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{3_{cl}};\pi_{1_{cl}},\pi_{2_{cl}}), \tag{A8}$$

where $\begin{pmatrix}\pi'_{1_{cl}}\\\pi'_{2_{cl}}\end{pmatrix}=\begin{pmatrix}\cos\alpha,-\sin\alpha\\\sin\alpha,\cos\alpha\end{pmatrix}\begin{pmatrix}\pi_{1_{cl}}\\\pi_{2_{cl}}\end{pmatrix}$ and $\begin{pmatrix}\sigma'_{cl}\\\pi'_{3_{cl}}\end{pmatrix}=\begin{pmatrix}\cos\beta,-\sin\beta\\\sin\beta,\cos\beta\end{pmatrix}\begin{pmatrix}\sigma_{cl}\\\pi_{3_{cl}}\end{pmatrix}$. Then, if the scalar fields σ_{cl} and $\pi_{a_{cl}}$ do not depend on spacetime coordinates, it follows that the thermodynamic potential $\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})$ depends only on two variables, $\Sigma=\sqrt{\sigma_{cl}^2+\pi_{3_{cl}}^2}$ and $\Pi=\sqrt{\pi_{1_{cl}}^2+\pi_{2_{cl}}^2}$.

• Next, we similarly discuss the transformation properties of the effective action $\Gamma_{\mathrm{CSC}}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})$ of the NJL model (33)−(34) (defined in Section IV) with respect to the color $SU_c(3)$ group. [1] To begin, let us recall that the auxiliary scalar diquark fields $\Delta_a(x)$ and $\Delta^*_b(x)$ in $\widetilde{L}_{\mathrm{CSC}}$ transform under the color $SU_c(3)$ group as follows

$$\Delta_a\to\Delta'_a=U^*_{ab}\Delta_b,\qquad \Delta^*_a\to\Delta'^*_a=U_{ab}\Delta^*_b, \tag{A9}$$

where $U\in SU_c(3)$ is a 3×3 matrix. This fact can be readily established using the equations of motion (35) for the auxiliary diquark fields, for which one must perform the $SU_c(3)$ transformation of the spinor fields, $\psi(x)\to U\psi(x)$. It follows from Eq. (A9) that the diquark fields $\Delta_a(x)$ form an antitriplet, whereas the set of $\Delta^*_b(x)$ $(a,b=1,2,3)$ forms a triplet of the color $SU_c(3)$ group. Moreover, the integration measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi\prod_{i=1}^{3}\mathcal{D}\Delta_i\mathcal{D}\Delta^*_i$ in the generating functional $\mathcal{Z}_{\mathrm{CSC}}(J,\overline{\eta},\eta)$ (36) is invariant under $SU_c(3)$ transformations.

Now, let us determine how the scalar diquark sources $\overline{\eta}_a$ and η_b, which appear in the generating functional (36), should be transformed so that this functional remains invariant under $SU_c(3)$ transformations. Because both $\widetilde{L}_{\mathrm{CSC}}$ and the integration measure of this functional are $SU_c(3)$

1) Its properties with respect to the transformations (3), (11) and (12) are the same as those of the effective action (A1) of the NJL model (1).

invariant, we can change variables in the functional integral, $\Delta_a \to \Delta'_a$ and $\Delta^*_a \to \Delta'^*_a$, as in Eq. (A9), and then define the $SU_c(3)$ transformed sources by the following relations (summation over repeated indices is implied)

$$\begin{aligned}&\overline{\eta}_a\Delta'_a = \overline{\eta}_a U^*_{ab}\Delta_b = \overline{\eta}'_b\Delta_b \Longrightarrow \overline{\eta}'_b = U^{\dagger}_{ba}\overline{\eta}_a,\\ &\eta_a\Delta'^*_a = \eta_a U_{ab}\Delta^*_b = \eta'_b\Delta^*_b \Longrightarrow \eta'_b = U^T_{ba}\eta_a.\end{aligned} \tag{A10}$$

It follows that $\mathcal{Z}_{\rm CSC}(J,\overline{\eta},\eta) = \mathcal{Z}_{\rm CSC}(J,\overline{\eta}',\eta')$ and $\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta) = \mathcal{W}_{\rm CSC}(J,\overline{\eta}',\eta')$ (this functional is defined after Eq. (36)), where the primed sources are given in Eq. (A10). Using the second of these invariance relations, one can determine how the classical diquark fields $\Delta_{a_{cl}}(x)$ and $\Delta^*_{b_{cl}}(x)$ transform under $SU_c(3)$. Indeed,

$$\begin{aligned}\Delta'_{a_{cl}} &= \frac{\delta\mathcal{W}_{\rm CSC}(J,\overline{\eta}',\eta')}{\delta\overline{\eta}'_a(x)} = \int {\rm d}z \frac{\delta\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta)}{\delta\overline{\eta}_b(z)}\frac{\delta\overline{\eta}_b(z)}{\delta\overline{\eta}'_a(x)}\\ &= \Delta_{b_{cl}}U_{ba} = (U^*)^{-1}_{ab}\Delta_{b_{cl}},\\ \Delta'^*_{a_{cl}} &= \frac{\delta\mathcal{W}_{\rm CSC}(J,\overline{\eta}',\eta')}{\delta\eta'_a(x)} = \int {\rm d}z \frac{\delta\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta)}{\delta\eta_b(z)}\frac{\delta\eta_b(z)}{\delta\eta'_a(x)}\\ &= \Delta^*_{b_{cl}}U^*_{ba} = U^{\dagger}_{ab}\Delta^*_{b_{cl}},\end{aligned} \tag{A11}$$

where we have also used the relations (A10) and their consequences, $\frac{\delta\overline{\eta}_b(z)}{\delta\overline{\eta}'_a(x)} = U_{ba}\delta(z-x)$ and $\frac{\delta\eta_b(z)}{\delta\eta'_a(x)} = U^*_{ba}\delta(z-x)$.

Now, using the source transformations (A10) together with the transformations (A11) of the classical diquark fields under $SU_c(3)$, one can straightforwardly establish that the quantity $(\overline{\eta_a}\Delta_{a_{cl}} + \eta_b\Delta^*_{b_{cl}})$, which appears in the definition of the full effective action $\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{l_{cl}})$ of the NJL model (33)–(34) (see Sec. IV), is invariant under the simultaneous transformations (A10)–(A11). Finally, recalling that the functional $\mathcal{W}_{\rm CSC}(J,\overline{\eta},\eta)$ is also $SU_c(3)$ invariant, we conclude that the full effective action $\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{l_{cl}})$ of the NJL model (33) (introduced in Sec. IV) is invariant with respect to the color transformations (A11) of the classical scalar diquark fields, *i.e.*

$$\Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta'_{k_{cl}},\Delta'^*_{l_{cl}}) = \Gamma_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{l_{cl}}).$$

Consequently, the total TDP $\Omega_{\rm CSC}(\sigma_{cl},\pi_{a_{cl}},\Delta'_{k_{cl}},\Delta'^*_{l_{cl}})$ (39) of the model is also invariant under $SU_c(3)$. Therefore, it must depend only on the combination $\Delta \equiv \sqrt{\Delta_{a_{cl}}\Delta^*_{a_{cl}}}$, which is invariant under $SU_c(3)$, as can be readily verified using Eqs. (A11).

APPENDIX B: EFFECTIVE ACTION OF THE NJL MODEL (1) IN THE LEADING LARGE-N_c ORDER

Starting from the partition function (18), it is convenient to use the steepest-descent method (see, *e.g.*, Ref. [72]) to calculate the effective action (13) or (A1) of the simplest NJL model (1). First, suppose that the quantities $\sigma_0 \equiv \sigma_0(x;J)$ and $\vec{\pi}_0 \equiv \vec{\pi}_0(x;J)$ constitute a solution to the following system of four equations:

$$\begin{aligned}&\frac{\delta S_{\rm eff}(\sigma,\vec{\pi})}{\delta\sigma(x)} + J_0(x) = 0,\\ &\frac{\delta S_{\rm eff}(\sigma,\vec{\pi})}{\delta\pi_k(x)} + J_k(x) = 0, \quad k = 1,2,3,\end{aligned} \tag{B1}$$

where $S_{\rm eff}(\sigma,\vec{\pi})$ is defined in Eq. (19). The set $(\sigma_0,\vec{\pi}_0)$ is also referred to as the saddle point of the functional $S_{\rm eff}(\sigma,\vec{\pi})$. Note that the quantities σ_0 and π_{0k} do not depend on N_c; that is, they are of order N_c^0 for large N_c. Then, expanding the functional $S_{\rm eff}(\sigma,\vec{\pi})$ in a Taylor series around its saddle point $(\sigma_0,\vec{\pi}_0)$

$$\begin{aligned}S_{\rm eff}(\sigma,\vec{\pi}) = S_{\rm eff}(\sigma_0,\vec{\pi}_0) &+ \int {\rm d}^4x \frac{\delta S_{\rm eff}}{\delta\sigma(x)}\Big|_{(\sigma_0,\vec{\pi}_0)}\tilde{\sigma}(x)\\ &+ \int {\rm d}^4x \frac{\delta S_{\rm eff}}{\delta\pi_k(x)}\Big|_{(\sigma_0,\vec{\pi}_0)}\tilde{\pi}_k(x)\\ &+ \Delta S_{\rm eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x)).\end{aligned} \tag{B2}$$

(The Taylor series expansion of the effective action in the simplest field-theory model involving a single scalar field is presented, for example, in Ref. [58], Eq. (11.58).) Here, $\tilde{\sigma}(x) = \sigma(x) - \sigma_0(x;J)$ and $\tilde{\pi}_k(x) = \pi_k(x) - \pi_{k_0}(x;J)$, and we denote by $\Delta S_{\rm eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x))$ the sum of all higher-order contributions in the quantities $\tilde{\sigma}(x)$ and $\tilde{\pi}_k(x)$. Using the saddle-point equations (B1), we obtain from Eq. (B2)

$$\begin{aligned}&S_{\rm eff}(\sigma,\vec{\pi}) + \int {\rm d}^4x\left[\sigma(x)J_0(x) + \vec{\pi}(x)\cdot\vec{J}(x)\right]\\ &= S_{\rm eff}(\sigma_0,\vec{\pi}_0) + \int {\rm d}^4x\left[\sigma_0(x)J_0(x) + \vec{\pi}_0(x)\cdot\vec{J}(x)\right]\\ &\quad + \Delta S_{\rm eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x)).\end{aligned} \tag{B3}$$

Substituting this expression into the partition function (18) and changing the path-integral variables as $\sigma \to \tilde{\sigma}(x)$ and $\pi_k \to \tilde{\pi}_k(x)$, we obtain

$$\begin{aligned}\exp(iN_c\mathcal{W}_{\mathcal{M}}(J)) = N' \exp\big(iN_c\big\{&S_{\rm eff}(\sigma_0,\vec{\pi}_0)\\ &+ \int {\rm d}^4x\left[\sigma_0(x)J_0(x) + \vec{\pi}_0(x)\cdot\vec{J}(x)\right]\big\}\big)\\ &\times \int \mathcal{D}\tilde{\sigma}\prod_{a=1}^{3}\mathcal{D}\tilde{\pi}_a \exp\left(iN_c\left\{\Delta S_{\rm eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x))\right\}\right).\end{aligned} \tag{B4}$$

We now perform another change of variables, this time in the path integral (B4): $\tilde{\sigma} \to \tilde{\sigma}/\sqrt{N_c}$ and $\tilde{\pi}_k \to \tilde{\pi}_k(x)/\sqrt{N_c}$. It then takes the form

$$\int \mathcal{D}\tilde{\sigma}\prod_{a=1}^{3}\mathcal{D}\tilde{\pi}_a \exp\left(\mathrm{i}N_c\left\{\Delta S_{\rm eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x))\right\}\right)$$
$$=\frac{1}{N_c^2}\int \mathcal{D}\tilde{\sigma}\prod_{a=1}^{3}\mathcal{D}\tilde{\pi}_a \exp\left(\mathrm{i}\left\{A(\tilde{\sigma}(x),\tilde{\pi}_k(x))+O\left(\frac{1}{\sqrt{N_c}}\right)\right\}\right), \tag{B5}$$

where $A(\tilde{\sigma}(x),\tilde{\pi}_k(x))$ denotes the part of the functional $\Delta S_{eff}(\tilde{\sigma}(x),\tilde{\pi}_k(x))$ that is quadratic in $\tilde{\sigma}(x)$ and $\tilde{\pi}_k(x)$, *i.e.*

$$A(\tilde{\sigma}(x),\tilde{\pi}_k(x))=\int \mathrm{d}^4x\mathrm{d}^4y\left\{\frac{1}{2}\frac{\delta^2 S_{\rm eff}}{\delta\sigma(x)\delta\sigma(y)}\Big|_{(\sigma_0,\vec{\pi}_0)}\tilde{\sigma}(x)\tilde{\sigma}(y)\right.$$
$$+\sum_{k=1}^{3}\frac{\delta^2 S_{\rm eff}}{\delta\sigma(x)\delta\pi_k(y)}\Big|_{(\sigma_0,\vec{\pi}_0)}\tilde{\sigma}(x)\tilde{\pi}_k(y)$$
$$\left.+\frac{1}{2}\sum_{k,l=1}^{3}\frac{\delta^2 S_{\rm eff}}{\delta\pi_k(x)\delta\pi_l(y)}\Big|_{(\sigma_0,\vec{\pi}_0)}\tilde{\pi}_l(x)\tilde{\pi}_k(y)\right\}. \tag{B6}$$

Using, at $N_c\to\infty$, the general relation $\exp(A+O(1/\sqrt{N_c}))=\exp(A)\{1+O(1/\sqrt{N_c})\}$, one can take the logarithm of both sides of Eq. (B4) and obtain (up to irrelevant constants) the functional $W(J)$ at leading order in the large-N_c expansion:

$$\mathcal{W}_{\mathcal{M}}(J)=S_{\rm eff}(\sigma_0,\vec{\pi}_0)+\int \mathrm{d}^4x\left[\sigma_0(x)J_0(x)\right.$$
$$\left.+\vec{\pi}_0(x)\cdot\vec{J}(x)\right]+O(1/N_c). \tag{B7}$$

Recall that the saddle point $(\sigma_0,\vec{\pi}_0)$ depends on the external sources $J_0(x)$ and $J_k(x)$. We now determine the effective action $\Gamma(\sigma_{cl},\vec{\pi}_{cl})$ in (A1) (or (13)) starting from Eq. (B7). To this end, we first find the classical fields $\sigma_{cl}(x)\equiv\delta\mathcal{W}_{\mathcal{M}}(J)/\delta J_0(x)$ and $\pi_{k_{cl}}(x)\equiv\delta\mathcal{W}_{\mathcal{M}}(J)/\delta J_k(x)$. From Eq. (B7), it follows that

$$\sigma_{cl}(x)=\sigma_0(x;J)+\int \mathrm{d}^4y\left(\frac{\delta S_{\rm eff}}{\delta\sigma_0(y;J)}+J_0(y)\right)\frac{\delta\sigma_0(y;J)}{\delta J_0(x)}$$
$$+\int \mathrm{d}^4y\left(\frac{\delta S_{\rm eff}}{\delta\pi_{k_0}(y;J)}+J_k(y)\right)\frac{\delta\pi_{k_0}(y;J)}{\delta J_0(x)}+O(1/N_c). \tag{B8}$$

However, the expressions in parentheses in Eq. (B8) vanish by the saddle-point equations (B1). Thus, in the large-N_c limit, $\sigma_{cl}(x)=\sigma_0(x;J)$. Similarly, one can show that, within the same approximation, $\vec{\pi}_{cl}(x)=\vec{\pi}_0(x;J)$. Applying this result to Eq. (B7) and to the definition of the effective action $\Gamma(\sigma_{cl},\vec{\pi}_{cl})$ in Eq. (13), we find that

$$\Gamma(\sigma_{cl},\vec{\pi}_{cl})=S_{\rm eff}(\sigma_{cl},\vec{\pi}_{cl})+O(1/N_c). \tag{B9}$$

A comparison of our result (B9) with the expression obtained for the effective action (potential) in the mean-field approximation (see, for example, Refs. [66−71]) shows that the two are identical. Therefore, in what follows, we use the notation $\Gamma_{mf}(\sigma_{cl},\vec{\pi}_{cl})$ for the effective action calculated both in the mean-field approximation and at leading order in the $1/N_c$ approximation (see Eq. (20)).

APPENDIX C: DUAL INVARIANCE OF PATH INTEGRATION MEASURE

We show that the path-integral measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ in the partition function (4) is invariant under the Pauli–Gursey (or duality) transformation (23)–(24). Recall that $\psi\equiv\psi(x)$ is a flavor doublet; that is, $\psi=\begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}$. Moreover, since $\psi_{u,d}$ are four-component spinors, $\psi(x)$ is an eight-component spinor: $\psi^T(x)=(\psi_{u1},...,\psi_{u4},\psi_{d1},...,\psi_{d4})$. For simplicity, we ignore the fact that $\psi(x)$ is also a color triplet. Thus, the path integration measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ can be written explicitly in the following form

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi\equiv\prod_{\alpha=1}^{8}\mathcal{D}\bar{\psi}_\alpha\prod_{\beta=1}^{8}\mathcal{D}\psi_\beta$$
$$=\prod_{\alpha=1}^{4}\mathcal{D}\bar{\psi}_{u\alpha}\prod_{\beta=1}^{4}\mathcal{D}\psi_{u\beta}\prod_{\alpha=1}^{4}\mathcal{D}\bar{\psi}_{d\alpha}\prod_{\beta=1}^{4}\mathcal{D}\psi_{d\beta}. \tag{C1}$$

First, we note that for vanishing chemical potentials, the Lagrangian $L_{\rm NJL}$ (1) is invariant under the transformations $\psi(x)\to\psi'(x)=U(\vec{\theta})\psi(x)$, where the 2×2 unitary matrix $U(\vec{\theta})$ in two-flavor space is given by

$$U(\vec{\theta})\equiv U(\theta_1,\theta_2,\theta_3)=\exp(\mathrm{i}\vec{\tau}\cdot\vec{\theta})=\cos|\theta|+\frac{\mathrm{i}\vec{\tau}\cdot\vec{\theta}}{|\theta|}\sin|\theta|$$
$$=\begin{pmatrix}\cos|\theta|+\dfrac{\mathrm{i}\theta_3}{|\theta|}\sin|\theta|, & \dfrac{\mathrm{i}\theta_1+\theta_2}{|\theta|}\sin|\theta|\\ \dfrac{\mathrm{i}\theta_1-\theta_2}{|\theta|}\sin|\theta|\,, & \cos|\theta|-\dfrac{\mathrm{i}\theta_3}{|\theta|}\sin|\theta|\end{pmatrix}, \tag{C2}$$

and $\vec{\tau}\cdot\vec{\theta}=\tau_1\theta_1+\tau_2\theta_2+\tau_3\theta_3$, $|\theta|^2=\theta_1^2+\theta_2^2+\theta_3^2$. Altogether, the matrices $U(\vec{\theta})$ form the $SU_V(2)$ group.

It is also well known that $L_{\rm NJL}$ is invariant under axial transformations $\psi(x)\to\psi'(x)=U_A(\vec{\omega})\psi(x)$, where the 2×2 unitary matrix $U_A(\vec{\omega})$ in the two-flavor space has the form

$$U_A(\vec{\omega})=\exp(\mathrm{i}\gamma^5\vec{\tau}\cdot\vec{\omega})=\cos|\omega|+\frac{\mathrm{i}\gamma^5\vec{\tau}\cdot\vec{\omega}}{|\omega|}\sin|\omega|. \tag{C3}$$

These transformations constitute the axial $SU_A(2)$ group. We have presented expressions for the matrices $U(\vec{\theta})$ and

$U_A(\vec{\omega})$ in the two-dimensional flavor space $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$. In reality, they act in the space of eight-component Grassmann spinors, $\psi^T(x) = (\psi_{u1}, ..., \psi_{u4}, \psi_{d1}, ..., \psi_{d4})$, where they are represented by 8×8 unitary matrices with unit determinant. Suppose that $\psi(x) \to \psi'(x) = U(\vec{\theta})\psi(x)$, where $U(\vec{\theta}) \in SU_V(2)$. Then we have

$$\begin{aligned} \mathcal{D}\overline{\psi}'\mathcal{D}\psi' &\equiv \prod_{\alpha=1}^{8}\mathcal{D}\overline{\psi}'_\alpha \prod_{\beta=1}^{8}\mathcal{D}\psi'_\beta \\ &= \prod_{\alpha=1}^{8}\mathcal{D}\overline{\psi}_\alpha (\det U^\dagger(\vec{\theta}))^{-1} \prod_{\beta=1}^{8}\mathcal{D}\psi_\beta (\det U(\vec{\theta}))^{-1} \\ &= \mathcal{D}\overline{\psi}\mathcal{D}\psi, \end{aligned} \tag{C4}$$

i.e., the path-integral measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ (see, *e.g.*, Eq. (17)) is invariant under transformations of the $SU_V(2)$ group. Here, we have used the above-mentioned identity $\det U(\vec{\theta})U^\dagger(\vec{\theta}) = 1$. (If we take into account that $\psi(x)$ is also a color triplet, then, using the same method, one can easily prove that the path-integral measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ is invariant under the transformations $\psi(x) \to \psi'(x) = U\psi(x)$, where $U \in SU_c(3)$.)

Similarly, this measure is also invariant under an arbitrary transformation $U_A(\vec{\omega}) \in SU_A(2)$, where $U_A(\vec{\omega})$ is given in Eq. (C3). Indeed, suppose that $\psi'(x) = U_A(\vec{\omega})\psi(x)$. Then, using the identity $\gamma^0 \exp(\mathrm{i}\gamma^5\vec{\omega}\cdot\vec{\tau}) = \exp(-\mathrm{i}\gamma^5\vec{\omega}\cdot\vec{\tau})\gamma^0$, we have $\overline{\psi}'(x) = \overline{\psi}(x)U_A(\vec{\omega})$ and hence the path-integral measure transforms as follows:

$$\begin{aligned} \mathcal{D}\overline{\psi}'\mathcal{D}\psi' &= \prod_{\alpha=1}^{8}\mathcal{D}\overline{\psi}_\alpha \prod_{\beta=1}^{8}\mathcal{D}\psi_\beta (\det U_A(\vec{\omega}))^{-2} \\ &= \prod_{\alpha=1}^{8}\mathcal{D}\overline{\psi}_\alpha \prod_{\beta=1}^{8}\mathcal{D}\psi_\beta \, \mathrm{e}^{-\mathrm{i}N_c \mathrm{tr}\{\tau^a\}\int \mathrm{d}^4x \omega_a(x)\frac{\mathcal{A}}{2}} \\ &= \mathcal{D}\overline{\psi}\mathcal{D}\psi, \end{aligned} \tag{C5}$$

where $\mathcal{A} = -2\mathrm{tr}[\gamma_5]\delta(x-x)$ is the same expression as that for the chiral anomaly of $U_A(1)$, and it is finite when regularization is used. We have also used the additional factor $\mathrm{tr}(\tau^a) = 0$. Hence, the path-integral measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ is also invariant under transformations from the $SU_A(2)$ group.

We can therefore conclude that the integration measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ is invariant with respect to arbitrary transformations $\psi(x) \to \psi'(x) = U(\vec{\theta})U_A(\vec{\omega})\psi(x)$. For $\vec{\theta} = -\vec{\omega} = (\pi/4, 0, 0)$, it follows that the path-integral measure is invariant with respect to the discrete Pauli–Gursey transformation (24), namely $\psi(x) \to \psi'(x) = \psi_R(x) + \mathrm{i}\tau_1\psi_L(x)$. Moreover, the integration measure over the auxiliary meson fields $\mathcal{D}\sigma\prod_{i=1}^{3}\mathcal{D}\pi_i$ in the generating functional (10) is also invariant under the PG transformation (32) of these fields.

• Based on this result, we are ready to consider how the dual symmetry (31)−(32) of the auxiliary Lagrangian $\widetilde{L}_{\rm NJL}$ (8) is reflected in the properties of the effective action (A1) of the NJL model (1) and its thermodynamic potential.

The dual symmetry (31)−(32) of the auxiliary Lagrangian $\widetilde{L}_{\rm NJL}$ (8) implies that both the total effective action $\Gamma(\sigma_{cl}, \pi_{a_{cl}})$ (13) and the total thermodynamic potential $\Omega(\sigma_{cl}, \pi_{a_{cl}})$ (15) of the model are symmetric with respect to the transformation

$$\begin{aligned} &\sigma_{cl}(x) \to -\pi_{1_{cl}}(x),\ \ \pi_{1_{cl}}(x) \to \sigma_{cl}(x), \\ &\pi_{2_{cl}}(x) \to \pi_{3_{cl}}(x),\ \ \pi_{3_{cl}}(x) \to -\pi_{2_{cl}}(x). \end{aligned} \tag{C6}$$

In terms of the variables Σ and Π (see the text preceding Eq. (16)), it has the form $\Sigma \leftrightarrow \Pi$ (with the simultaneous interchange $\mu_I \leftrightarrow \mu_{I5}$).

We first consider the transformation of the functional $\mathcal{W}_{\mathcal{M}}(J_a)$ (which appears in the expression for the generating functional (10)) under the dual transformations (31)−(32). As a first step, we interchange the chemical potentials, $\mu_I \leftrightarrow \mu_{I5}$, in the chemical-potential term $\mathcal{M}$ defined by Eq. (2). We denote the resulting functional by $\mathcal{W}_{\widetilde{\mathcal{M}}}(J_a)$, where $\widetilde{\mathcal{M}} = \frac{\mu_B}{3} + \frac{\mu_{I5}}{2}\tau_3 + \frac{\mu_I}{2}\gamma^5\tau_3 + \mu_5\gamma^5$. Next, in the functional integral $\mathcal{W}_{\widetilde{\mathcal{M}}}(J_a)$ (which is simply expression (10) with $\mathcal{M} \to \widetilde{\mathcal{M}}$), we transform the integration variables, both the spinor and auxiliary scalar fields, according to the dual transformations (31)−(32). Since the functional integral measure does not change under this procedure, the original functional $\mathcal{W}_{\widetilde{\mathcal{M}}}(J_a)$ transforms according to the following rule:

$$\mathcal{W}_{\mathcal{M}}(J_a) = \mathcal{W}_{\widetilde{\mathcal{M}}}(J'_a), \tag{C7}$$

where $J'_0 = J_1$, $J'_1 = -J_0$, $J'_2 = -J_3$, $J'_3 = J_2$. Based on the left- and right-hand sides of Eq. (C7), one can use the definition in Eq. (13) to introduce two formally different effective actions

$$\begin{aligned} &\Gamma_{\mathcal{M}}(\sigma_{cl}, \pi_{a_{cl}}) \\ &= \mathcal{W}_{\mathcal{M}}(J_a) - \int \mathrm{d}^4x \left\{ \sigma_{cl}(x)J_0(x) + \sum_{a=1}^{3}\pi_{a_{cl}}(x)J_a(x) \right\}, \end{aligned} \tag{C8}$$

$$\begin{aligned} &\Gamma_{\widetilde{\mathcal{M}}}(\sigma'_{cl}, \pi'_{a_{cl}}) \\ &= \mathcal{W}_{\widetilde{\mathcal{M}}}(J'_a) - \int \mathrm{d}^4x \left\{ \sigma'_{cl}(x)J'_0(x) + \sum_{a=1}^{3}\pi'_{a_{cl}}(x)J'_a(x) \right\}, \end{aligned} \tag{C9}$$

where, by definition (13), we have

$$\sigma_{cl}(x)=\frac{\delta\mathcal{W}_{\mathcal{M}}(J_a)}{\delta J_0(x)},\ \pi_{b_{cl}}(x)=\frac{\delta\mathcal{W}_{\mathcal{M}}(J_a)}{\delta J_b(x)},$$
$$\sigma'_{cl}(x)=\frac{\delta\mathcal{W}_{\widetilde{\mathcal{M}}}(J'_a)}{\delta J'_0(x)},\ \pi'_{b_{cl}}(x)=\frac{\delta\mathcal{W}_{\widetilde{\mathcal{M}}}(J'_a)}{\delta J'_b(x)}. \tag{C10}$$

However, from the preceding relations and by taking into account (C7), one can, for example, obtain

$$\sigma'_{cl}(x)=\frac{\delta\mathcal{W}_{\widetilde{\mathcal{M}}}(J'_a)}{\delta J'_0(x)}=\frac{\delta\mathcal{W}_{\mathcal{M}}(J_a)}{\delta J'_0(x)}$$
$$=\int \mathrm{d}^4z\frac{\delta\mathcal{W}_{\mathcal{M}}(J_a)}{\delta J_b(z)}\frac{\delta J_b(z)}{\delta J'_0(x)}=\pi_{1_{cl}}(x), \tag{C11}$$

where one must also use the relation $\dfrac{\delta J_b(z)}{\delta J'_0(x)}=\delta_{b1}\delta(z-x)$ (the connection between the primed sources J'_b and the unprimed sources J_a is given after Eq. (C7)). Similarly, one finds that

$$\pi'_{1_{cl}}(x)=-\sigma_{cl}(x),\ \pi'_{2_{cl}}(x)=-\pi_{3_{cl}}(x),\ \pi'_{3_{cl}}(x)=\pi_{2_{cl}}(x), \tag{C12}$$

and, using these relations, one can readily see that

$$\sigma_{cl}(x)J_0(x)+\sum_{a=1}^{3}\pi_{a_{cl}}(x)J_a(x)=\sigma'_{cl}(x)J'_0(x)+\sum_{a=1}^{3}\pi'_{a_{cl}}(x)J'_a(x). \tag{C13}$$

Consequently, Eq. (C7) implies that the effective actions (C8) and (C9) are identical, $\Gamma_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})=\Gamma_{\widetilde{\mathcal{M}}}(\sigma'_{cl},\pi'_{a_{cl}})$. The same property also holds for their TDPs

$$\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})=\Omega_{\widetilde{\mathcal{M}}}(\sigma'_{cl},\pi'_{a_{cl}}). \tag{C14}$$

As shown in Appendix A, owing to the $U_{I_3}(1)$ and $U_{AI_3}(1)$ invariance of the NJL model (1), the TDP $\Omega_{\mathcal{M}}(\sigma_{cl},\pi_{a_{cl}})$ depends on the invariants $\Sigma=\sqrt{\sigma_{cl}^2+\pi_{3_{cl}}^2}$ and $\Pi=\sqrt{\pi_{1_{cl}}^2+\pi_{2_{cl}}^2}$, respectively. However, the second TDP $\Omega_{\widetilde{\mathcal{M}}}(\sigma'_{cl},\pi'_{a_{cl}})$ depends effectively on $\Sigma'=\sqrt{\sigma_{cl}'^2+\pi_{3_{cl}}'^2}$ and $\Pi'=\sqrt{\pi_{1_{cl}}'^2+\pi_{2_{cl}}'^2}$. Using relations (C12) and (C13), we find that $\Sigma'=\Pi$, whereas $\Pi'=\Sigma$. Therefore, Eq. (C14) shows that the full TDP of the NJL model (1) satisfies the dual symmetry relation

$$\Omega_{\mathcal{M}}(\Sigma,\Pi)=\Omega_{\widetilde{\mathcal{M}}}(\Pi,\Sigma). \tag{C15}$$

That is, it is identical to that in the mean-field approximation (22).

Concerning the NJL model (33) with a color superconducting channel of quark interaction, we first note that its auxiliary Lagrangian $\widetilde{L}_{\mathrm{CSC}}$ (34) is invariant under the dual transformations $\mathcal{D}$ of both the spinor fields (31) and the auxiliary meson fields (32). However, as shown in Section IV, the dual transformation $\mathcal{D}$ leaves the diquark scalar fields $\Delta_a(x)$ and $\Delta_b^*(x)$ unchanged (see Eqs. (42) and (43)). Hence, in this case, using the same methods as those used to analyze the dual properties of the simplest NJL model (1) (see above in the present Appendix), it is straightforward to show that the effective action $\Gamma_{\mathrm{CSC}}(\sigma_{cl},\pi_{a_{cl}},\Delta_{k_{cl}},\Delta^*_{k'_{cl}})$ of this NJL model is invariant under transformations (C11) and (C12), supplemented by the condition that the classical diquark fields $\Delta_{k_{cl}}$ and $\Delta^*_{k'_{cl}}$ remain unchanged. In addition, the values of the chemical potentials μ_I and μ_{I5} must be interchanged.

APPENDIX D: REGULARIZATION AND DUAL INVARIANCE

Regularization is unavoidable in effective models such as the NJL model, and the regularization parameter cannot be removed. Therefore, the regularization scheme is part of the model definition. Let us show that the regularization procedure does not break the invariance of the measure in the path integral and, consequently, preserves the duality property of the effective model.

In principle, one can choose a regularization that breaks the invariance of the Lagrangian, thereby breaking the duality property of the model. However, this only indicates that such a regularization procedure is, at minimum, not suitable for studying duality properties. Let us consider, for example, the three-dimensional momentum cutoff regularization scheme.

It is instructive to work in momentum space

$$\psi(x)=\int\frac{\mathrm{d}p^0}{2\pi}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}p\cdot x}\,\tilde{\psi}(p^0,\mathbf{p}).$$

The measure in the path integral with three-momentum cutoff regularization can be written as

$$\mathcal{D}\bar{\psi}\,\mathcal{D}\psi=\prod_{|\mathbf{p}|<\Lambda}\prod_{p^0}\prod_{\alpha}d\bar{\tilde{\psi}}_\alpha(p^0,\mathbf{p})\,d\tilde{\psi}_\alpha(p^0,\mathbf{p}).$$

Let us consider the dual transformation (23). In momentum space, it takes the following form:

$$PG:\ \ \tilde{\psi}_R(p^0,\mathbf{p})\to\tilde{\psi}_R(p^0,\mathbf{p}),\ \ \tilde{\psi}_L(p^0,\mathbf{p})\to\mathrm{i}\tau_1\tilde{\psi}_L(p^0,\mathbf{p}),$$
$$\tilde{\psi}_R(p^0,\mathbf{p})\equiv\Pi_+\tilde{\psi}(p^0,\mathbf{p}),\ \ \tilde{\psi}_L(p^0,\mathbf{p})\equiv\Pi_-\tilde{\psi}(p^0,\mathbf{p}).$$

Although 3-momentum cutoff regularization breaks Lorentz invariance, it does not alter the decomposition of fermions into left-handed and right-handed components.

For generality, we now consider the more general transformation $\psi(x) \to \psi'(x) = U(\vec{\theta})U_A(\vec{\omega})\psi(x)$ discussed in Appendix C, evaluated at $\vec{\theta} = -\vec{\omega} = (\omega, 0, 0)$ (the dual transformation is obtained at $\omega = -\pi/4$). For small values of the parameter ω, it takes the following form

$$\begin{aligned} &\psi_R(x) \to \psi'_R(x) = \psi_R(x), \\ &\psi_L(x) \to \psi'_L(x) = \psi_L(x) - 2\mathrm{i}\tau_1\omega\psi_L(x). \end{aligned} \tag{D1}$$

In general, the dual transformation in momentum space is given by

$$\tilde{\psi}'_R(p) = \tilde{\psi}_R(p), \quad \tilde{\psi}'_L(p) = \tilde{\psi}_L(p) - 2\mathrm{i}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\,\tilde{\omega}(p-q)\tau_1\tilde{\psi}_L(q),$$

where $\tilde{\omega}(p) = \int \mathrm{d}^4 x\, \mathrm{e}^{-\mathrm{i}px}\omega$. However, if ω is constant, as in our case, then $\tilde{\omega}(p-q) = \omega(2\pi)^4\delta^{(4)}(p-q)$.

Thus, one obtains the transformation

$$\tilde{\psi}'_L(p) = \tilde{\psi}_L(p) - 2\mathrm{i}\omega\tau_1\tilde{\psi}_L(p).$$

One can see that the transformation does not mix the various momentum modes of the fermion field ψ. If we cut off the modes at a scale Λ, the measure remains invariant under this transformation, and the regularization procedure of the effective model does not stand in the way.

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