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Probing Quantum Phase Transitions in the sdg-Interacting Boson Model Using von Neumann Entropy

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Abstract: In this work, the von Neumann entropy has been calculated and employed as a probe to analyse quantum phase transitions (QPTs) within the sdg-interacting boson model (sdg-IBM). The von Neumann entropy between the s-boson and dg-boson sectors is used as an indicator of QPTs and as a robust observable for the theoretical analysis of the $^{108-116}_{48}Cd$ isotopes. The von Neumann entropy correctly characterises the QPT in the $U_d(5)\otimes U_g(9) \leftrightarrow SO_{sdg}(15)$ transition region. The numerical results show that the $^{108}_{48}Cd$ and $^{116}_{48}Cd$ isotopes are located in the $U_d(5)\otimes U_g$ and $SO_{sdg}(15)$ limits, respectively.

Keywords: Quantum phase transitions, Von Neumann entropy, sdg- Interacting Boson Model

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I. INTRODUCTION

Recently, considerable effort has been devoted to understanding the relationship between entanglement in many-particle systems and QPTs. Quantum phase transitions arise from the constraints imposed by Heisenberg's uncertainty principle, in clear contrast to classical phase transitions, which are driven solely by thermal fluctuations and occur when a system crosses a critical point defined through macroscopic order parameters [1, 2]. QPTs are generated by changes in an external parameter or coupling constant. These transformations manifest as structural rearrangements in mesoscopic systems, including molecular clusters, semiconductors, atomic nuclei, and even quark matter. They occur at zero temperature and arise solely from quantum fluctuations induced by changes in control parameters. Such quantum phase transitions play a central role across diverse fields, from cosmology to quantum computation and studies of decoherence. Both classical and quantum systems exhibit critical points characterized by a diverging correlation length that governs their behavior near the transition. However, quantum systems exhibit additional correlations that have no classical counterparts [3−5]. This phenomenon, referred to as entanglement [6], serves as a fundamental resource that enables quantum computation and quantum communication [7]. Quantum entanglement is a physical phenomenon that occurs when a group of particles interacts such that the quantum state of each particle cannot be described independently of the states of the others, even when the particles are separated by large distances. The contribution of entanglement to phase transitions cannot be fully described within the framework of conventional statistical mechanics [8]. Quantum entanglement is one of the foundational concepts of quantum physics, first brought into focus through the seminal work of Einstein, Podolsky, and Rosen in 1935 [9]. Quantifying entanglement in multipartite systems remains a challenging problem, and in recent years considerable attention has been directed toward understanding various aspects of entanglement [10−13]. Although the characterization of entanglement in pure states is relatively straightforward, identifying and quantifying entanglement in mixed states remains a complex and largely unresolved mathematical challenge. In recent years, quantum information theory has provided new tools and perspectives for studying entangled nuclear systems, such as atomic nuclei, which inherently behave as multipartite systems [14]. This perspective opens promising possibilities for gaining deeper insight into phenomena governed by the strong interaction [15]. Nuclear many-body systems are self-bound assemblies composed of four types of fermions—protons and neutrons with spin up and spin down—and exhibit a wide range of complex correlation phenomena [16]. If no correlations are present among the nucleons, each particle acts independently, and the nucleus resides in an uncorrelated state. However, describing realistic nuclear systems remains a central challenge in nuclear many-body

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theory. Consequently, significant efforts are now focused on developing new quantum algorithms capable of addressing the complexities of multi-particle nuclear systems [17−22]. Quantum entanglement is central to virtually all applications of quantum mechanics, which naturally leads to the growing integration of quantum-information concepts and methods into nuclear physics. As a result, both quantum-information researchers and nuclear physicists have become increasingly interested in understanding the entanglement characteristics inherent in nuclear systems and processes [23, 24].

In nuclear physics, quantum phase transitions [25−30] represent sudden structural changes within a nuclear system, affecting both the ground state and the accompanying low-lying excitations. These transitions are most effectively explored through algebraic methods [31], including frameworks such as the *sd*-IBM [32−36] and its extension, the *sdg*-IBM [37−46].

In this work, we examine quantum entanglement within the broader context of nuclear structure and quantum information. A principal aim of this study is to characterize quantum phase transitions and nuclear structural evolution through the degree of entanglement. By analyzing how entanglement behaves near quantum critical points, we use it as a diagnostic tool to identify and describe QPTs. The entanglement characteristics of multipartite systems also enable the exploration of various dynamical symmetry limits, offering explicit expressions that can be readily compared with experimental observations.

To probe the phase transition, we compute exact entanglement entropy values across the transitional region of the sdg-IBM, employing the dual algebraic structure of a three-level pairing model formulated within the affine $SU(1,1)$ Lie algebra and using Schmidt decomposition techniques. Experimental indications of the $U_d(5)\otimes U_g(9)\leftrightarrow SO_{sdg}(15)$ transition in even–even nuclei are presented, and the corresponding analysis of the ${}^{108-116}_{48}Cd$ isotopes is carried out. The von Neumann entropy values for their low-lying states are also reported.

II. THEORETICAL FRAMEWORK

The *sdg*-IBM, which describes collective excitations in even–even nuclei through nucleon pairs with angular momenta $L=0$ (*s*-boson), $L=2$ (*d*-boson), and $L=4$ (*g*-boson), is described in detail in Refs. [37−48]. In our previous work [49], we proposed an algebraic method based on the dual algebraic structure of a three-level pairing model within the framework of the *sdg*-IBM for nuclei in the transitional region from the spherical to the γ-unstable quantum shape phase transition. To clarify how possible simplifications provide insight into exactly solvable models, the reader is referred to Refs. [47−53]. In the following sections, we briefly introduce the energy spectrum and von Neumann entropy.

A. The sdg-IBM based on the affine SU(1,1) algebra

To define the Hamiltonian within the framework of *sdg*-IBM using the affine $SU(1,1)$ Lie algebra, the algebraic structure of $SU(1,1)$ must first be introduced. The $SU(1,1)$ algebra is generated by S^ν, with $\nu=0$ and $\pm$, which satisfy the following commutation relations:

$$[S^0,S^\pm]=\pm S^\pm,\qquad [S^+,S^-]=-2S^0 \tag{1}$$

The quadratic Casimir operator of $SU(1,1)$ is given by

$$C_2=S_0(S_0-1)-S^+S^- \tag{2}$$

The basis states of an irreducible representation (irrep) of $SU(1,1)$, $|k\mu\rangle$, are determined by a single parameter, k, which can take any positive value, with $\mu=k,k+1,\ldots$. Therefore

$$C_2(SU(1,1))|k\mu\rangle=k(k-1)|k\mu\rangle,\qquad S^0|k\mu\rangle=\mu|k\mu\rangle \tag{3}$$

In the *sdg*-IBM, the generators of the *s*-, *d*-, and *g*-boson pairing algebras form the $SU_s(1,1)$, $SU_d(1,1)$, and $SU_g(1,1)$ algebras, respectively. They are defined as follows:

$$\begin{aligned}
&S^+(s)=\frac{1}{2}(s^\dagger.s^\dagger),\quad S^-(s)=\frac{1}{2}\tilde{s}.\tilde{s},\quad S^0(s)=\frac{1}{2}(n_s+\frac{1}{2})\\
&S^+(d)=\frac{1}{2}(d^\dagger.d^\dagger),\quad S^-(d)=\frac{1}{2}\tilde{d}.\tilde{d},\quad S^0(d)=\frac{1}{2}(n_d+\frac{5}{2})\\
&S^+(g)=\frac{1}{2}(g^\dagger.g^\dagger),\quad S^-(g)=\frac{1}{2}\tilde{g}.\tilde{g},\quad S^0(g)=\frac{1}{2}(n_g+\frac{9}{2})
\end{aligned} \tag{4}$$

In Eq(4), the $b^\dagger$ ($\tilde{b}$) and n_b with $b:=s,d,g$ are the *b*-boson creation operator (the *b*- boson annihilation operator) and the number of *b*- boson operator.

The vector space corresponding to the basis vectors of the *s*-, *d*-, and *g*- bosons is a 15-dimensional vector space. The generators of these bosons in this space form the $U_{sdg}(15)$ algebra. The subalgebras considered in this study are given as follows:

$$U_{sdg}(15)\supset U_d(5)\otimes U_g(9)\supset SO_{dg}(14)\supset SO_{dg}(5)\supset SO_{dg}(3)$$

and

$$U_{sdg}(15)\supset SO_{sdg}(15)\supset SO_{dg}(14)\supset SO_{dg}(5)\supset SO_{dg}(3)$$

The bases of the $U_d(5)\otimes U_g(9)\supset SO_{dg}(14)$ and

$SO_{sdg}(15) \supset SO_{dg}(14)$ subalgebras are simultaneously the bases of $SU_d(1,1) \otimes SU_g(1,1) \supset U(1)$ and $SU_{sdg}(1,1) \supset U(1)$, respectively. The Casimir operators of $SO_d(5) \otimes SO_g(9)$ and $SO_{sdg}(15)$ can be expressed in terms of the Casimir operators of $SU_d(1,1) \otimes SU_g(1,1)$, and $SU_{sdg}(1,1)$ as follows:

$$\begin{aligned} C_2(SU_{sdg}(1,1)) &= \frac{165}{4} + \frac{1}{4}C_2(SO_{sdg}(15)) \\ C_2(SU_g(1,1)) &= \frac{45}{16} + \frac{1}{4}C_2(SO_g(9)) \\ C_2(SU_d(1,1)) &= \frac{5}{16} + \frac{1}{4}C_2(SO_d(5)) \end{aligned} \tag{5}$$

Moreover, the infinite-dimensional $SU(1,1)$ algebra is generated by

$$\begin{aligned} S_n^{\pm} &= c_s^{2n+1}S^{\pm}(s) + c_d^{2n+1}S^{\pm}(d) + c_g^{2n+1}S^{\pm}(g), \\ S_n^{0} &= c_s^{2n}S^{0}(s) + c_d^{2n}S^{0}(d) + c_g^{2n}S^{0}(g) \end{aligned} \tag{6}$$

where c_s, c_d and c_g are the real parameters and n can be $0, \pm1, \pm2, \ldots$. These generators satisfy the commutation relations $[S_m^0, S_n^{\pm}] = \pm S_{m+n}^{\pm}$, $[S_m^+, S_n^-] = -2S_{m+n+1}^0$. By employing the generators of algebra $SU(1,1)$, the proposed Hamiltonian for the transitional region between $U_d(5) \otimes U_g(9) \leftrightarrow SO_{sdg}(15)$ (vibrational to γ-unstable transition) limits is

$$\begin{aligned} H = {} & S_0^+S_0^- + \alpha S_1^0 + \gamma C_2(SO_g(9)) + \delta C_2(SO_d(5)) \\ & + \kappa C_2(SO_d(3)) + \eta C_2(SO_g(3)) + \beta C_2(SO_{dg}(3)) \end{aligned} \tag{7}$$

In Eq.(7) α, γ, δ, κ, η, and β are the real parameters. In this equation for $c_s = c_d = c_g$ describe the $SO(15)_{sdg}$ limit and also, for $c_s = 0$ and $c_d \neq c_g \neq 0$ describe the $U_d(5) \otimes U_g(9)$ limit. Also, the $c_s \neq c_d \neq c_g \neq 0$ situation corresponds to $U_d(5) \otimes U_g(9) \leftrightarrow SO_{sdg}(15)$ transitional region [47–49]. Here, α,γ, δ, κ, η, β are fit parameters and c_s and c_g are control parameters. For obtaining the eigenvalues of sdg-Hamiltonian, Eq.(7) the eigenstates are given as

$$|m; \nu_g, \nu_d, \nu_s, n_\Delta LM\rangle = \sum_{n_i \in z} a_{n_1 n_2 \ldots n_m} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m} S_{n_1}^+ S_{n_2}^+ \ldots S_{n_m}^+ |lw\rangle \tag{8}$$

The eigenstates of Eq. (7) can be obtained using the Fourier–Laurent expansion of the eigenstates and $SU(1,1)$ generators in terms of unknown c-number parameters x_i, with $i = 1, 2, \ldots, m$. Thus, the eigenstates can be expressed as follows:

$$|m; \nu_g, \nu_d, \nu_s, n_\Delta LM\rangle = AS_{x_1}^+ S_{x_2}^+ \ldots S_{x_m}^+ |lw\rangle \tag{9}$$

where A is the normalization factor and $S_{x_i}^+$ is

$$S_{x_i}^+ = \frac{c_s}{1 - c_s^2 x_i}S^+(s) + \frac{c_d}{1 - c_d^2 x_i}S^+(d) + \frac{c_g}{1 - c_g^2 x_i}S^+(g) \tag{10}$$

Let $|lw\rangle$ be the lowest weight state of $SU(1,1)$, which should satisfy $S^-(s)|lw\rangle = 0, S^-(d)|lw\rangle = 0, S^-(g)|lw\rangle = 0$. In the most general form, the $|lw\rangle$ is given by:

$$\begin{aligned} |lw\rangle = {} & |N, k_g = \frac{1}{2}(\nu_g + \frac{9}{2}), \mu_g = \frac{1}{2}(n_g + \frac{9}{2}), \\ & k_d = \frac{1}{2}(\nu_d + \frac{5}{2}), \mu_d = \frac{1}{2}(n_d + \frac{5}{2}), \\ & k_s = \frac{1}{2}(\nu_s + \frac{1}{2}), \mu_s = \frac{1}{2}(n_s + \frac{1}{2}), n_\Delta, L_d, L_g, L_{dg}, M_{dg}\rangle \end{aligned} \tag{11}$$

where $N = \nu_s + \nu_d + \nu_g$, $n_s = \nu_s$, $n_d = \nu_d$, $n_g = \nu_g$. Hence, we have

$$\begin{aligned} & S_n^0|lw\rangle = \Lambda_n^0|lw\rangle, \\ & \Lambda_n^0 = c_s^{2n}(\nu_s + \frac{1}{2})\frac{1}{2} + c_d^{2n}(\nu_d + \frac{5}{2})\frac{1}{2} + c_g^{2n}(\nu_g + \frac{9}{2})\frac{1}{2} \end{aligned} \tag{12}$$

The eigenvalues $E^{(m)}$ of the Hamiltonian (7) can be expressed as

$$\begin{aligned} E^{(m)} = {} & h^{(m)} + \alpha\Lambda_1^0 + \gamma(\nu_g(\nu_g + 7)) + \delta(\nu_d(\nu_d + 3)) \\ & + \kappa(L_d(L_d + 1)) + \eta(L_g(L_g + 1)) + \beta(L(L + 1)) \end{aligned} \tag{13}$$

Here, $h^{(m)} = \sum_{i=1}^{m} \frac{\alpha}{x_i}$, and the quantum number (m) is related to the total boson number N by $N = 2m + \nu_s + \nu_d + \nu_g$. To obtain numerical results for the energy spectra of the nuclei considered, a set of nonlinear Bethe Ansatz equations (BAEs) with m unknowns for m-pair excitations must be solved. The constants in Eq.(13) (γ,δ, κ, η, and β) can also be obtained using the least-squares method (LSM). The set of BAEs using the transformed variables $C_s = \frac{c_s}{c_d} \leq 1, C_g = \frac{c_g}{c_d} = 1$, $g = 1, y_i = c_d^2 x_i$ is given as follows:

$$\frac{\alpha}{y_i} = \frac{C_s^2(\nu_s + \frac{1}{2})}{1 - C_s^2 y_i} + \frac{(\nu_d + \frac{5}{2})}{1 - y_i} + \frac{(\nu_g + \frac{9}{2})}{1 - y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j} \tag{14}$$

Here, C_s varies between 0 and 1. To calculate the roots of the BAEs for specified values of ν_s, ν_d, and ν_g, we solve Eq. (14) using fixed values of C_s and α. We then repeat this procedure for different values of C_s and α to obtain energy spectra that minimize the deviation from the experimental values [54], $\sigma = (\frac{1}{N_{tot}} \sum_i^{N_{tot}} |E_{exp}(i) - E_{cal}(i)|^2)^{\frac{1}{2}}$, where N_{tot} is the number of energy levels.

B. Effective Order Parameter of Von Neumann entropy in sdg-IBM

We focus on the behavior of the von Neumann entropy in the transitional region and outline the procedure for calculating this quantity in the sdg-IBM. As an appropriate measure of entanglement, the von Neumann entropy provides a straightforward means of quantifying the entanglement between two subsystems. It can be expressed in terms of the singular values obtained from the Schmidt decomposition of the system's quantum state. In general, any bipartite pure state can be written as [55]:

$$|\psi\rangle_{AB} = \sum_{i=1}^{m} a_i |\upsilon_i\rangle_A \otimes |\nu_i\rangle_B \tag{15}$$

where a_i is the Schmidt coefficient, and $|\upsilon_i\rangle_A$ and $|\nu_i\rangle_B$ are orthonormal states in subsystems A and B, respectively. Thus, the von Neumann entropy can be expressed using the Schmidt decomposition as [56–58]

$$S_A = S_B = -\sum_{i=1}^{m} a_i^2 log(a_i^2) \tag{16}$$

To construct the Von Neumann entropy within the *sdg*-IBM framework, we treat the *s*, *d* and *g*, sectors as distinct quantum subsystems with corresponding Hilbert spaces H_s, H_d and H_g. The full Hilbert space of the system is then given by the tensor product $H = H_s \otimes H_d \otimes H_g$, where each space represents the single-particle states of the *s*-, *d*-, and *g*-bosons. Our goal is to evaluate the Von Neumann entropy associated with these bosonic degrees of freedom. To begin, we define the operators $S^+_{x_i}$ as follows:

$$S^+_{x_i} = \alpha_i S^+_s + \beta_i S^+_d + \gamma_i S^+_g \tag{17}$$

where $\alpha_i = \frac{c_s}{1-c_s^2 x_i}$, $\beta_i = \frac{c_d}{1-c_d^2 x_i}$, and $\gamma_i = \frac{c_g}{1-c_g^2 x_i}$. We use the following relation [59]:

$$(S^+)^m |k,k\rangle = \sqrt{\frac{m!\Gamma(m+2k)}{\Gamma(2k)}} |k,k+m\rangle \tag{18}$$

The Schmidt decomposition in Eq. (15) can be written as:

$$\begin{aligned}
|\psi\rangle^B_{\nu,L,M_L} &= \Theta \prod_{i=1}^{m} (\alpha_i S^+_s + \beta_i S^+_d + \gamma_i S^+_g) |k_s,k_s,L,M_L\rangle |k_d,k_d,L,M_L\rangle |k_g,k_g,L,M_L\rangle \\
&= \Theta \sum_{l_1,l_2,l_3=0}^{m} \delta_{m,l_1+l_2+l_3} A_{l_1,l_2,l_3} \sqrt{\frac{l_1!\Gamma(l_1+2k_s)}{\Gamma(2k_s)} \times \frac{l_2!\Gamma(l_2+2k_d)}{\Gamma(2k_d)} \times \frac{l_3!\Gamma(l_3+2k_g)}{\Gamma(2k_g)}} \\
&\quad |k_s,k_s+l_1,L,M_L\rangle |k_d,k_d+l_2,L,M_L\rangle |k_g,k_g+l_3,L,M_L\rangle \\
&= \sum_{l_1=0}^{m} \sum_{l_2,l_3=0}^{m} \delta_{m-l_1,l_2+l_3} b_{l_1,l_2,l_3,k_s,k_d,k_g} |k_s,k_s+l_1,L,M_L\rangle |k_d,k_d+l_2,L,M_L\rangle |k_g,k_g+l_3,L,M_L\rangle
\end{aligned} \tag{19}$$

where the normalization factor (Θ), A_{l_1,l_2,l_3}, and $b_{l_1,l_2,l_3,k_s,k_d,k_g}$ are defined as follows:

$$\Theta = \frac{1}{\sqrt{\sum_{l_1,l_2,l_3=0}^{m} \delta_{m,l_1+l_2+l_3} A^2_{l_1,l_2,l_3} \left(\frac{l_1!\Gamma(l_1+2k_s)}{\Gamma(2k_s)}\right) \times \left(\frac{l_2!\Gamma(l_1+2k_d)}{\Gamma(2k_d)}\right) \times \left(\frac{l_3!\Gamma(l_1+2k_g)}{\Gamma(2k_g)}\right)}} \tag{20}$$

$$A_{l_1,l_2,l_3} = \frac{1}{l_1!l_2!l_3!} \sum_{\pi \in S_m} \alpha_{\pi(1)} \alpha_{\pi(2)} \ldots \alpha_{\pi(l_1)} \beta_{\pi(l_1+1)} \ldots \beta_{\pi(l_1+l_2)} \gamma_{\pi(l_1+l_2+1)} \ldots \gamma_{\pi(m)} \tag{21}$$

$$b_{l_1,l_2,l_3,k_s,k_d,k_g} = \Theta A_{l_1,l_2,l_3} \sqrt{\frac{l_1!\Gamma(l_1+2k_s)}{\Gamma(2k_s)} \times \frac{l_2!\Gamma(l_2+2k_d)}{\Gamma(2k_d)} \times \frac{l_3!\Gamma(l_3+2k_g)}{\Gamma(2k_g)}} \tag{22}$$

In Eq.(19), the states $|k_s,k_s+l_1,L,M_L\rangle$, $|k_d,k_d+l_2,L,M_L\rangle$, and $|k_g,k_g+l_3,L,M_L\rangle$ correspond to the basis states of the *s*-,

d-, and g-boson subspaces, respectively. In general, the specific choice of basis for each Hilbert space does not affect the entanglement calculation, since any basis can ultimately be transformed into the Schmidt basis. Conveniently, for the bosonic system considered here, the chosen basis already coincides with the Schmidt basis. Consequently, the von Neumann entropy in the sdg-IBM can be written as:

$$|\psi\rangle^{B}_{\nu,L,M_L} = \sum_{l_1=0}^{m} |k_s, k_s + l_1, L, M_L\rangle \Big(\sum_{l_2,l_3=0}^{m} \delta_{m-l_1,l_2+l_3} b_{l_1,l_2,l_3,k_s,k_d,k_g} |k_d, k_d + l_2, L, M_L\rangle |k_g, k_g + l_3, L, M_L\rangle \Big) \tag{23}$$

In this calculation, the entanglement between the s boson and the dg bosons is considered. Therefore, the general form of the wave function is:

$$|\psi\rangle_m = \sum_{l_1=0}^{m} \sqrt{p_{l_1}} |l_1\rangle_s |\phi_{l_1}(d,g)\rangle \tag{24}$$

where $|l_1\rangle_s$ are states constructed from s-bosons, and $|\phi_{l_1}(d,g)\rangle$ are the corresponding states in the dg-sector. The total angular momentum of the s-bosons is zero ($L=0$). Therefore, any angular momentum in the full state originates solely from the d- and g-bosons. This is consistent with the structure of the lowest-weight state (Eq. (11)), in which the angular momentum content is carried entirely by the dg-sector. The entanglement discussed in the von Neumann entropy analysis is between subspaces rather than individual particles. The s-boson sector forms collective scalar pairs, since they have $L=0$, whereas the dg-boson sector forms collective excitations with angular momentum. Thus, the entanglement between the s-boson sector and the dg-boson sector is indeed between collective modes, such as pairs, condensates, or coherent superpositions, rather than between single s-bosons and individual d or g bosons. $\phi_{l_1}(d,g)$ and p_{l_1} are written as follows:

$$|\phi_{l_1}(d,g)\rangle = \frac{\left(\sum_{l_2,l_3=0}^{m} \delta_{m-l_1,l_2+l_3} b_{l_1,l_2,l_3,k_s,k_d,k_g} |k_d, k_d + l_2, L, M_L\rangle |k_g, k_g + l_3, L, M_L\rangle\right)}{\sqrt{p_{l_1}}} \tag{25}$$

$$p_{l_1} = \sum_{l_2,l_3=0}^{m} \delta_{m-l_1,l_2+l_3} b^2_{l_1,l_2,l_3,k_s,k_d,k_g} \tag{26}$$

Finally, the von Neumann entropy (S) is given by

$$S := S_s = S_{dg} = -\sum_{l_1=0}^{m} p_{l_1} ln p_{l_1} \tag{27}$$

III. RESULT

In many transitional nuclei, such as Cd isotopes near the closed shell $Z=50$, as well as in strongly deformed nuclei, additional experimentally observed low-lying 0^+, 2^+, and 4^+ states cannot be explained within the $(sd)^N$ boson space. Other experimental evidence, such as the $k^\pi = 0^+_3, 3^+_1, 2^+_2$, and 4^+ bands, can be interpreted as bands built on hexadecupole vibrations and hexadecupole transition densities associated with the excitation of some 4^+ levels in Cd isotopes. Moreover, the sd-IBM is designed for low-lying states, usually with energies below 2.5 MeV. By introducing the g-boson, the investigation of higher-lying states becomes possible. For example, higher states such as 0^+_4, 2^+_4, 4^+_3 (g-boson state), 4^+_4,.... are investigated, and good results are obtained.

The fit parameters for the energy spectra of selected Cadmium isotopes ($^{108-116}_{48}Cd$), along with their standard deviations (σ), are listed in Table 1. The small values of σ indicate good agreement between the experimental spectra and those predicted by the sdg-IBM. The experimental and theoretical energies of the investigated levels of the $^{108-116}_{48}Cd$ isotopes, together with the quantum labels for each level, are presented in Tables 2a to 2e. The corresponding experimental and theoretical energy spectra are also shown in Figure 1. At first glance, the close similarity between the experimental and theoretical spectra indicates the accuracy and efficiency of the sdg-IBM. More quantitatively, the standard deviation values confirm this conclusion. Using energy ratios such as ($R_{4/2} = E(4^+_1)/E(2^+_1)$) and ($R_{0/2} = E(0^+_1)/E(2^+_1)$), one can make predictions about the shape-phase transition from spherical to γ-unstable nuclei. For example, $2.2 \leqslant R_{4/2} \leqslant 2.5$ characterises the spectrum of transitional nuclei in the spherical to γ-unstable region. These energy ratios have been obtained within the framework of the sdg-IBM for $^{108-116}_{48}Cd$ and compared with experimental data. The results are shown in Figure 2. The calculated values of $2.29 \leqslant R^{cal}_{4/2} \leqslant 2.65$ for this chain confirm the transition region from spherical to γ-unstable.

Table 1. Parameters of Hamiltonian used in the calculation of the Cd Isotopes. All parameters are given in keV.

nucleus	N	C	α	β	γ	δ	κ	η	σ
^{108}Cd	6	0.26	17.5812	57.4572	18.6345	−45.7015	−15.5771	0.4504	137.8519
^{110}Cd	7	0.37	14.5522	26.2164	7.6178	−40.4789	−15.3354	2.2183	182.2283
^{112}Cd	8	0.44	16.6395	37.6761	16.8718	−22.4835	−11.3060	0.5086	174.8445
^{114}Cd	9	0.54	7.0611	45.0905	7.3047	−34.5723	−12.2082	1.9206	168.66
^{116}Cd	8	0.64	21.5989	29.7029	18.1880	−6.7820	−8.8425	1.7389	176.0712

Table 2a. The calculated entanglement entropy for ^{108}Cd isotope with N = 6.

J^π	m	v_s	v_d	v_g	$E_{exp}(keV)$	$E_{sdg-cal}(keV)$	x_i	S
0_1^+	3	0	0	0	0	0	[13.9359 14.6447 9.3352]	0.1660
2_1^+	2	1	1	0	632.988	634.8	[12.7330 9.7674]	0.2657
0_2^+	3	0	0	0	1720.646	1662.1	[0.0774 0.3157 0.0132]	0.0136
2_2^+	2	0	2	0	1601.833	1446.1	[14.1687 0.0109]	0.0136
4_1^+	2	1	0	1	1508.466	1597.9	[12.7330 0.0668]	0.0386
2_3^+	2	0	1	1	2162.738	2025.8	[5.4561 0.0620]	0.0149
0_3^+	2	0	1	1	1913.420	1681.1	[5.4561 0.0620]	0.0149
3_1^+	2	0	1	1	2145.850	2085.1	[14.1066 0.0119]	0.0149
4_2^+	2	0	2	0	2239.35	2032.4	[14.1687 0.0109]	0.0136
6_1^+	2	0	1	1	2541.367	2397.7	[14.1066 0.2618]	0.0149
0_4^+	3	0	0	0	2374.57	2401.7	[13.9359 0.0150 0.0150]	0.0741
2_4^+	2	0	0	2	2365.781	2474.0	[0.0613 0.0119]	0.0148
4_3^+	2	0	1	1	2645.62	2544.8	[14.1066 0.0119]	0.0149
4_4^+	2	0	0	2	2738.72	2999.2	[14.1066 0.0119]	0.0148

Table 2b. The calculated entanglement entropy for ^{110}Cd isotope with N = 7.

J^π	m	v_s	v_d	v_g	$E_{exp}(keV)$	$E_{sdg-cal}(keV)$	x_i	S
0_1^+	3	1	0	0	0	0	[6.2444 4.7779 6.9930]	0.3164
2_1^+	3	0	1	0	657.76	650.3	[6.9515 7.0088 4.9141]	0.3074
0_2^+	3	1	0	0	1473.12	1634.5	[0.0650 6.4148 0.0099]	0.1258
2_2^+	2	1	2	0	1475.79	1336.1	[6.4388 0.0089]	0.1179
4_1^+	3	0	0	1	1542.45	1554.217	[0.0584 0.0566 0.0549]	0.2620
2_3^+	2	1	1	1	1783.48	1678.0	[6.4388 0.0089]	0.1174
0_3^+	2	1	1	1	1731.33	1800.6	[0.0516 0.0089]	0.1174
3_1^+	2	1	1	1	2162.816	2115.2	[0.0516 0.0089]	0.1174
4_2^+	2	1	2	0	1809.48	1488.4	[6.4388 0.0089]	0.1179
6_1^+	2	1	1	1	2479.95	2621.8	[6.4388 0.0089]	0.1174
0_4^+	2	1	0	0	2078.80	1690.2	[6.4388 0.0089]	0.1174
2_4^+	2	1	0	2	2355.79	2250.7	[0.0516 0.0089]	0.1174
4_3^+	3	1	1	1	2220.06	2167.0	[0.0650 0.0625 0.0112]	0.1174
4_4^+	2	1	0	2	2250.55	2368.9	[6.4388 0.0089]	0.1174

Table 2c. The calculated entanglement entropy for ^{112}Cd isotope with N = 8.

J^{π}	m	v_s	v_d	v_g	$E_{exp}(keV)$	$E_{sdg-cal}(keV)$	x_i	S
0_1^+	4	0	0	0	0	0	[4.9093 3.9732 3.5239 5.1639]	0.4400
2_1^+	3	1	1	0	617.57	620.4	[4.5482 4.9543 3.3259]	0.5057
0_2^+	4	0	0	0	1224.06	1203.8	[4.9093 0.0968 0.0239 0.0890]	0.3511
2_2^+	3	0	2	0	1312.32	1406.6	[4.9599 5.1298 0.0117]	0.1855
4_1^+	3	1	0	1	1415.38	1521.6	[0.0882 0.0844 0.0811]	0.4970
2_3^+	3	0	1	1	1468.73	1545.2	[0.0809 0.0779 0.0190]	0.3440
0_3^+	3	0	1	1	1433.16	1319.1	[0.0809 0.0779 0.0190]	0.3440
3_1^+	3	0	1	1	2064.53	1771.2	[0.0809 0.0779 0.0190]	0.3440
4_2^+	3	0	2	0	1870.68	1775.8	[4.9599 5.1298 0.0117]	0.1855
6_1^+	3	0	1	1	2167	2087.6	[0.0809 0.0779 0.2786]	0.3440
0_4^+	4	0	0	0	1870.94	1636.1	[4.9093 0.0968 0.0239 0.0237]	0.3029
2_4^+	3	0	0	2	2121.6	2195.7	[0.0809 4.1693 0.0117]	0.1817
4_3^+	3	0	1	1	2081	1862.3	[0.0809 5.1298 0.0190]	0.3440
4_4^+	3	0	0	2	2454	2730.3	[0.0809 4.1693 0.0117]	0.1817

Table 2d. The calculated entanglement entropy for ^{114}Cd isotope with N = 9.

J^{π}	m	v_s	v_d	v_g	$E_{exp}(keV)$	$E_{sdg-cal}(keV)$	x_i	S
0_1^+	4	1	0	0	0	0	[3.0134 2.4415 2.2456 3.4119]	0.6745
2_1^+	4	0	1	0	558.465	627.5	[3.2909 2.7756 2.4438 3.4097]	0.6403
0_2^+	4	1	0	0	1134.532	1316.5	[0.0499 0.0461 0.0071 1.7127]	0.5635
2_2^+	3	1	2	0	1209.708	1124.4	[3.0908 3.1293 0.0057]	0.5114
4_1^+	4	0	0	1	1283.739	1512.2	[0.0457 0.0428 0.0442 0.3661]	1.0507
2_3^+	3	1	1	1	1364.344	1425.5	[3.0908 3.3138 0.0057]	0.8131
0_3^+	3	1	1	1	1305.609	1155.0	[3.0908 3.3138 0.0057]	0.8131
3_1^+	3	1	1	1	1864.262	1696.1	[3.0908 3.3138 0.0057]	0.8131
4_2^+	3	1	2	0	1732.246	1584.8	[3.0908 3.1293 0.0057]	0.5114
6_1^+	3	1	1	1	1990.3	1835.4	[3.0908 2.5922 0.2226]	0.8131
0_4^+	4	1	0	0	1859.698	1503.9	[0.0499 0.0461 0.0071 0.0383]	0.8763
2_4^+	3	1	0	2	1784	1683.4	[3.0908 3.3138 0.0057]	0.7247
4_3^+	3	1	1	1	1932.077	2056.8	[3.0908 3.3138 0.0057]	0.8131
4_4^+	3	1	0	0	2152.266	2341.6	[3.0908 3.3138 0.0057]	0.7247

Next, we compare these results with the von Neumann entropy. As explained in Sec. 2.2, the signature of the phase transition in even-even nuclei can be identified using the von Neumann entropy. Here, the von Neumann entropy is used to investigate the transition from spherical ($U_d(5)\otimes U_g(9)$) to γ-unstable ($SO(15)$) limits in even-even nuclei. We have calculated the von Neumann entropy within the sdg-IBM framework using the dual algebraic structure based on the affine $SU(1,1)$ Lie algebra and the Schmidt decomposition. The spherical limit occurs for $C_s=\frac{c_s}{c_d}=0$, $c_d=1$, $c_g=1$, and the γ-unstable limit occurs for $C_s=\frac{c_s}{c_d}=1$, $c_s=c_d=c_g=1$. Theoretically, the von Neumann entropy as a function of the control parameter C_s has been calculated for each state. It shows how the von Neumann entropy of the s- and dg-boson sectors changes from $U_d(5)\otimes U_g(9)$ to $SO(15)$.

To analyse QPT, the von Neumann entropy for the $^{108-116}_{48}Cd$ isotopes has been calculated. Using this observable, we examine whether these nuclei exhibit transitional behavior indicative of shapes intermediate between the spherical and γ-unstable limits. Studies have shown that these isotopes have $U_d(5)\otimes U_g(9)\leftrightarrow SO(15)$ transitional characteristics. The optimal sets of transitional Hamiltonian parameters have been obtained using the LSM. The

Table 2e. The calculated entanglement entropy for ^{116}Cd isotope with N = 8.

J^{π}	m	ν_s	ν_d	ν_g	$E_{exp}(keV)$	$E_{sdg-cal}(keV)$	x_i	S
0_1^+	4	0	0	0	0	0	[4.5008 3.6730 3.2723 4.7025]	1.0741
2_1^+	3	1	1	0	513.49	530.3	[4.1812 4.5397 3.0942]	1.3064
0_2^+	4	0	0	0	1282.56	1297.3	[4.5008 0.1176 0.0333 0.1077]	1.0742
2_2^+	3	0	2	0	1212.99	1406.5	[4.5447 4.6946 0.0169]	1.1315
4_1^+	3	1	0	1	1219.448	1447.7	[0.1075 0.1027 0.0985]	1.2629
2_3^+	3	0	1	1	1642.499	1585.0	[0.0989 0.0951 0.0267]	1.1121
0_3^+	3	0	1	1	1380.310	1406.8	[0.0989 0.0951 0.0267]	1.1121
3_1^+	3	0	1	1	1915.816	1763.2	[0.0989 0.0951 0.0267]	1.1121
4_2^+	3	0	2	0	1131.5	1698.6	[4.5447 4.6946 0.0169]	1.0397
6_1^+	3	0	1	1	2026.66	1917.0	[0.0989 0.0951 0.2981]	1.1121
0_4^+	4	0	0	0	1928.44	1573.6	[4.5008 0.1176 0.0333 0.0330]	1.0956
2_4^+	3	0	0	2	1951.33	2062.7	[0.0989 3.8445 0.0169]	1.0465
4_3^+	3	0	1	1	2041.91	1778.3	[0.0989 4.6946 0.0267]	1.1121
4_4^+	3	0	0	2	2376.45	2502.9	[0.0989 3.8445 0.0169]	1.0465

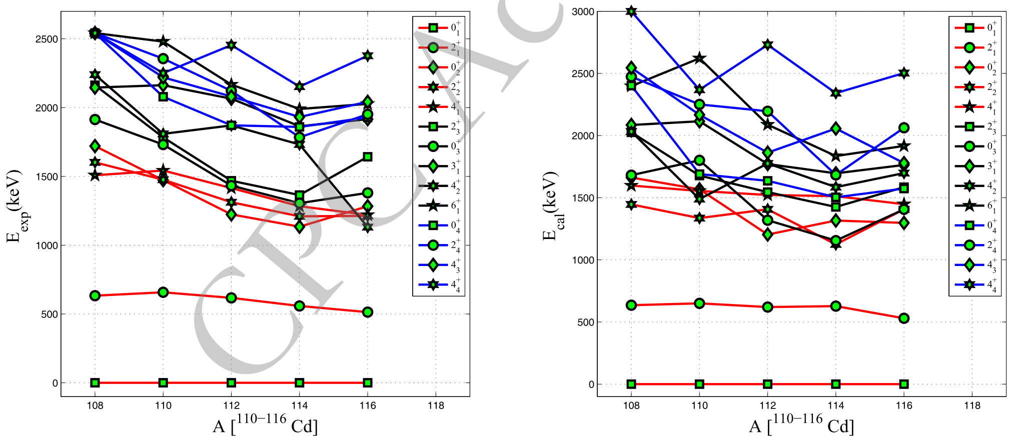

Fig. 1. (color online) Comparison of experimental and calculated energies for the Cd isotope.

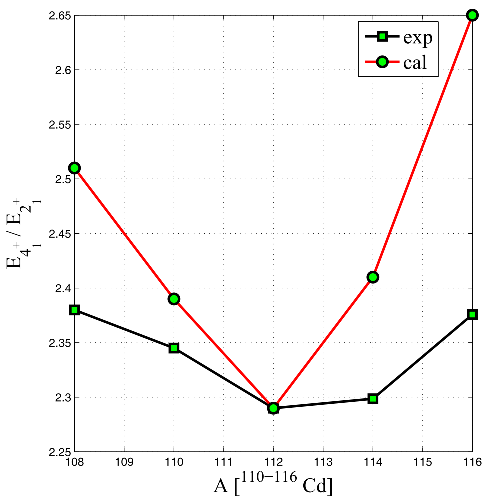

Fig. 2. (color online) Comparison of theoretical and experimental values of $R_{4/2} = \frac{E(4_1^+)}{E(2_1^+)}$ and $R_{0/2} = \frac{E(0_2^+)}{E(2_1^+)}$ for Cd isotopes.

experimental data for the $^{108-116}_{48}Cd$ nuclei are taken from Refs. [54]. The computed von Neumann entropies for the low-lying states of the $^{108-116}_{48}Cd$ isotopes, along with the predicted values for both low- and high-spin states, are shown in Fig. 3. The behavior of the control parameter C_s indicates structural evolution and shape-phase transitions across the Cd isotopic chain. According to Fig. 3 and Tables 2.a-2.e, the von Neumann entropy of the s- and dg-boson sectors is zero for vibrational nuclei and increases with increasing isotope mass. As shown in Tables 2.a-2.e, the control parameter C_s increases from 0.26 in $^{108}_{48}Cd$ to 0.64 in $^{116}_{48}Cd$, indicating a transition from spherical to γ-unstable structure.

IV. CONCLUSIONS

In this study, we examined the von Neumann entropy as an observable for investigating QPTs. Our analysis fo-

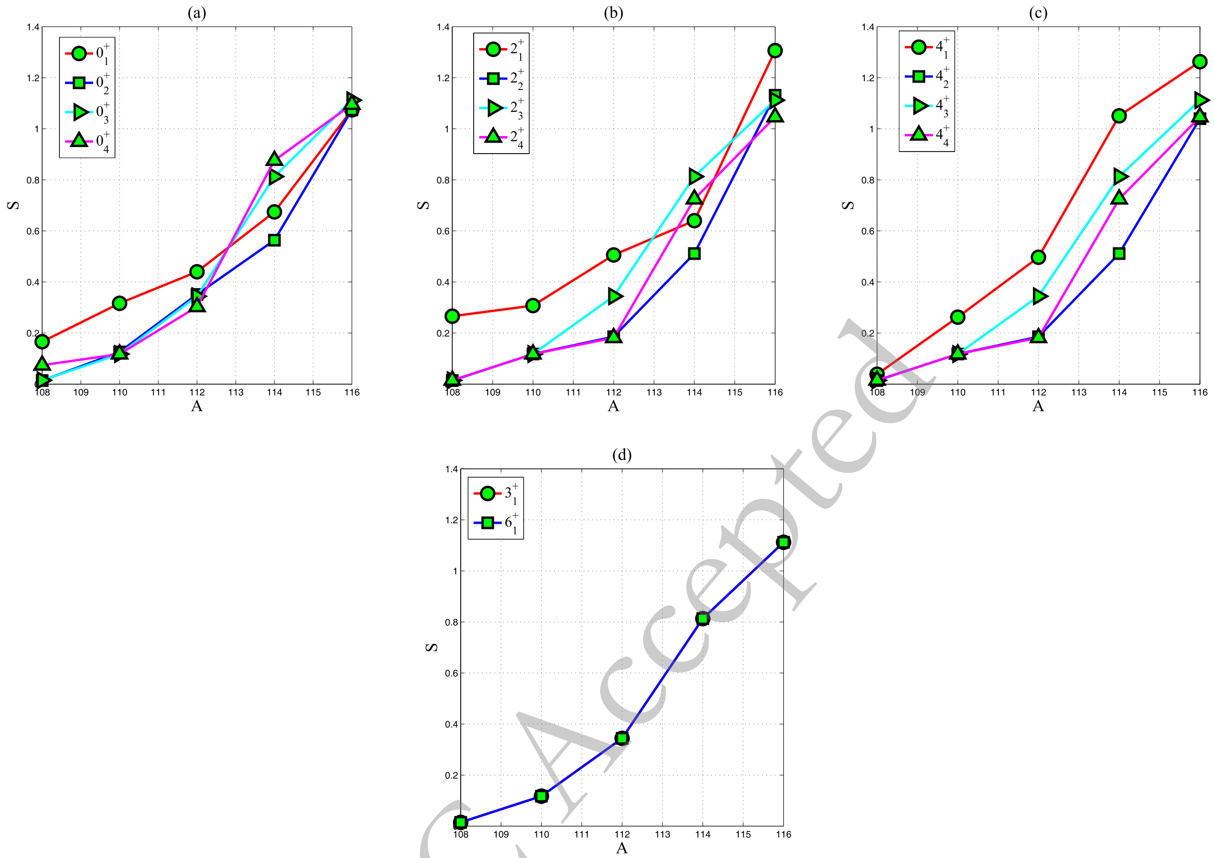

Fig. 3. (color online) Calculated entropies for all low and high states and for selected states of Cd isotopes.

cused on the sdg-IBM, a three-level pairing scheme that effectively represents a qubit-like interacting structure. We introduced a procedure for evaluating the von Neumann entropy in interacting nuclear systems and demonstrated that this quantity serves as an effective order parameter for identifying shape-phase transitions in nuclei. The von Neumann entropies were computed and analyzed for the low-lying states of the $^{108-116}_{48}Cd$ isotopes. The results show that, in the $U_d(5)\otimes U_g(9)$ limit, the s- and dg-bosons are not entangled, whereas the $SO(15)$ limit corresponds to maximal entanglement. These findings confirm that the von Neumann entropy is a reliable and meaningful signature for characterizing shape-phase transitions in nuclei.

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