Astrophysical aspects of complexity factor in charged compact fluid configuration in $f(R, G)$ gravity

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Abstract: We aim to evaluate the complexity of self-gravitating matter distributions with anisotropic pressure and an electric field. The modified field equations associated with $f(R,G)$ gravity, including the effects of electric charge, and the expressions for the mass function are derived, where R is the Ricci scalar and G denotes the Gauss--Bonnet invariant. The concept of complexity, suggested by Herrera and collaborators in 2018, is further elaborated in the context of a charged cylindrically symmetric system by considering the framework of $f(R,G)$ theory and carrying out a comprehensive evaluation of the orthogonal splitting of the Riemann curvature tensor. Among the derived scalars, Y_{TF} is referred to as the complexity factor; it includes the impacts of specific intrinsic features of the fluid, such as variations in energy density, anisotropic pressure, and electric charge. Furthermore, the results obtained in the presence of additional curvature terms corresponding to $f(R,G)$ gravity are studied by establishing a relationship between the Tolman mass, the Weyl scalar, and the complexity factor. This analysis is relevant from an astrophysical perspective, since compact stellar interiors are expected to involve dense matter, pressure anisotropy, electromagnetic effects, and strong-field relativistic corrections. In this context, the complexity factor Y_{TF} provides a geometric diagnostic for measuring how density inhomogeneity, local anisotropy, charge distribution, and modified Gauss--Bonnet corrections influence the internal organization and stability of neutron-star-like compact configurations. The consideration of a polytropic equation of state further connects the proposed scenario with the effective modeling of dense compact-star matter.

Keywords: dense matter, polytropic equation of state, complexity factor, pressure anisotropy, charged compact structures

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I. INTRODUCTION

The structural composition of distinct celestial objects is highly relevant for assessing the dynamics of the cosmos. The diverse compositional features of a self-gravitating system may affect its structure, which is characterized by the system's complexity. Distinct physical attributes determine the complexity of dense matter, each reflecting a unique characteristic related to the stability and internal composition of the fluid. The fluid distribution from the center to the external boundary is crucial for analyzing complexity. A uniform density distribution implies the absence of tidal forces, fewer internal fluctuations, and, consequently, minimal complexity within the system. In contrast, a gradual outward decrease in density indicates deformations in the inner curvature, which increase complexity. Furthermore, additional degrees of freedom resulting from disruptions of internal matter symmetries consequently increase the complexity of the system.

Gravitational entropy, which measures a system's intrinsic variations, directly affects the complexity of matter. In addition, gravitational entropy plays a crucial role in determining the curvature and inhomogeneous nature

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of matter caused by gravity. The beginning of the universe is characterized by minimal degrees of freedom and lower curvature and is accurately described by considering the basic characteristics of matter. The minimal gravitational entropy arising from the homogeneous composition of matter corresponds to the early phase of the universe. As the universe evolved, gravity induced instabilities in its structural configuration, and the formation of irregular compact objects increased cosmic gravitational entropy. This describes the evolution of gravity from its most fundamental state to complex celestial formations. Therefore, it can be asserted that a larger value of gravitational entropy corresponds to an increase in the complexity of any system.

The concept of complexity has significant implications in many scientific domains. Recently, multiple attempts have been made to develop an appropriate criterion for evaluating the degree of complexity in various research disciplines [1−4]. Despite these efforts, there is still no consensus on a specific interpretation of complexity. Any system consists of organized and interrelated components. Complexity within such systems may arise from slight or severe disruptions among their components. Entropy and information were once considered important components in determining the intricacy of any system. However, the analysis of a perfect crystal and an ideal gas is particularly relevant for evaluating complexity within the context of mathematical physics. A perfect crystal has a symmetrical arrangement of atoms throughout its entire structure. In this case, limited information is required because only a small section explains the structure of the complete system. In an ideal gas, however, each particle is randomly dispersed, providing maximal information even within a small portion. Consequently, one can state that complexity is not manifested solely through the composition of a system, which emphasizes the need to include additional elements in describing the complexity of a system. Furthermore, the pressure associated with fluid distribution is a critical component in establishing the basic attributes and evolution of astronomical systems.

Within the domain of GR, Herrera [5] developed a new concept of complexity, according to which the complexity of self-gravitating cosmic structures can be described by using a specific combination of the physical features of the system. The trace-free component (Y_{TF}) was derived through the orthogonal decomposition of the Riemann curvature tensor together with the aforementioned parameters. This scalar variable, referred to as the CF, governs the degree of complexity of cosmological formations. The unique idea that the complexity factor vanishes in relation to static self-gravitating configurations was extensively studied by Herrera [6]. In this case, the complexity factor includes dispersive factors; therefore, the approach of fundamental expansion suggested by the associated constraint must be considered for a complete analysis of complexity. In recent years, research has motivated the evaluation of self-gravitating fluid distributions in the static context of complexity. A thorough consideration of the limitations related to this attempt, as discussed in [7−9], provides the main reason for challenging the understanding of complexity. Numerous studies on the intricacy of the composition of the universe are available in the literature [10−25]. Bayin [26] derived mathematical solutions for studying the impact of anisotropy on spinning and radiating spherical compact objects. Nasir *et al.* [27] determined the CF by considering anisotropic fluid dispersion as well as the presence of an electric field. They determined the structural scalars as a consequence of the orthogonal decomposition of the Riemann curvature tensor and identified one of them as a complexity factor.

Herrera *et al.* [28] investigated hyperbolically symmetric structural compositions and assessed the relevance of structure scalars obtained from the orthogonal decomposition of the Riemann tensor. They established the relations between the Tolman mass and the Misner-Sharp mass by considering the peculiar configuration and concluded that the Tolman mass represents the negative aspect of the system. Sharif and Butt [29] implemented the same technique as that considered in GR to compute the CF corresponding to the spherically symmetric composition. Herrera and Santos [30] analyzed the effect of the anisotropic nature of matter on the development of celestial objects. Herrera *et al.* [31] considered the orthogonal representation of the Riemann curvature tensor together with the expression for the curvature invariant to characterize the attributes of the spherically symmetric composition of celestial objects. Elizalde *et al*. [32] studied the dynamics of dark energy cosmology within the context of Gauss-Bonnet gravity framework in the presence or absence of distinct scalar fields and explained the CDM cosmological phase without the consideration of cosmological constant. Bhatti *et al.* [33] considered the concept of intricacy to evaluate non-static structures in the formalism of modified gravity, derived formulas related to the mass function in the presence of anisotropic configurations, and suggested that the presence of dark source terms increases the complexity of the system.

Albert Einstein proposed a novel theory of gravity, termed general relativity (GR), in 1915, establishing the relationship between matter and geometry in space-time through the well-known field equations. He determined that our universe is fundamentally static, implying that it neither expands nor contracts over time. Later, after studying galaxy clusters, Edwin Hubble detected red-shifted light emitted by them and concluded that galaxies are moving away from one another. As a result, he asserted that the cosmos was expanding at an accelerated rate. The Λ-CDM model, which was established in GR, is con-

sidered an exceptionally plausible formalism for explaining the development of compact objects at the galactic level. Subsequently, numerous conceptual and observational problems were identified within the framework of GR, which prompted the idea of its generalization. Furthermore, recently discovered evidence suggests the existence of an extremely large amount of dark matter (DM) and dark energy (DE), emphasizing the importance of a comprehensive understanding of the fundamental composition of our universe. The presence of DM on both galactic and cosmic scales, the emergence of singularities at the centers of black holes, and the ambiguity associated with the role of DE are some unresolved issues in GR. Despite the importance of GR in astrophysical studies, it is considered inadequate for explaining the accelerated expansion of the universe. Consequently, modifications of GR could provide an appropriate description of the presence of DM and DE, along with accelerated cosmic expansion.

Several modified theories of gravity have been proposed to explain the fundamental features of the expanding cosmos. The $f(R)$ theory of gravity is considered a highly generalized case of GR, in which the Ricci scalar R replaces the general action function. In contrast to GR, Nojiri and Odintsov [34] asserted the validity of $f(R)$ gravity within several frameworks that are consistent with certain solar-system findings for analyzing the intriguing characteristics of the cosmos. Many researchers have considered the framework of $f(R)$ theory [35] to analyze the relevance of numerous factors that lead to anomalies in self-gravitating formations and to obtain precise solutions for the appropriate equations of motion. Bamba *et al.* [36] studied the $\Lambda - CDM$ model of the cosmos within the framework of $f(R)$ theory, as well as the specific features of DE. The physical properties of relativistic compact objects have been widely studied using modified gravity theories. In particular, researchers have explored the formation and stability of electrically charged compact cosmic structures by applying alternative approaches within theories that include coupling between matter and geometry [37−48]. M Yousaf *et al*. [49] studied the structural characteristics of wormhole geometries in the framework of modified gravity after the consideration of Visser cut-and-paste approach to construct thin-shell wormholes. These theories often involve modifications to the gravitational action, such as adding a quadratic term in the Ricci scalar. Moreover, the influence of complexity on anisotropic charged fluid distributions has been analyzed in static, spherically symmetric spacetimes, revealing the importance of modified gravity in establishing the internal configuration of such systems. These studies have derived the hydrostatic equilibrium and field equations, resulting in the Misner-Sharp mass function and a related complexity factor [50]. In addition, a growing number of studies have examined how different models of modified gravity affect the formation, evolution, and structure of astrophysical systems [51−67]. Together, these studies enhance our understanding of how gravitational modifications influence various cosmic phenomena [68−82].

The Gauss-Bonnet theory of gravity has also been proposed as another alternative theory that includes a commonly defined function $f(G)$, in which G is expressed as the Gauss-Bonnet invariant with the expression $G = R^2 - 4R_{\zeta\eta}R^{\zeta\eta} + R_{\zeta\eta\mu\nu}R^{\zeta\eta\mu\nu}$. In the context of string theory [83], the Gauss-Bonnet invariant, which is a higher-dimensional topological constant, may be effective in describing several intriguing cosmic phenomena. Taking this into consideration, Nojiri *et al.* [84] reviewed various specific frameworks of $f(G)$ theory to illustrate the mysterious nature of DE and its prominent significance in cosmic expansion. Cognola *et al.* [85] established that, within the framework of certain models, the Gauss-Bonnet theory imposes constraints on solar-system tests. Bhatti *et al.* [86, 82] determined a feasible solution related to a compact object and evaluated its physical composition with the help of graphs in modified gravity. Bhatti *et al.* [87] studied the factors that are important for maintaining the durability of a spherically symmetric composition and calculated the collapse equation as a consequence of the disruption of the revised equations of motion.

Another important generalization of GR is the $f(R,G)$ theory, which effectively provides a description of accelerating cosmic expansion without considering dark energy. This gravitational framework can play a crucial role in determining solutions to issues associated with the preliminary era of cosmic inflation because both the Gauss-Bonnet invariant G and the Ricci scalar R specify two different phases of cosmic expansion. Furthermore, different transition phases in the life span of celestial objects are precisely described by assuming $f(R,G)$ theory. In high-density compact-star interiors, curvature corrections may act as effective geometric contributions that modify the balance among gravitational attraction, pressure gradients, anisotropic stresses, and electromagnetic repulsion. Therefore, the study of charged anisotropic compact configurations in $f(R,G)$ gravity can provide useful insight into how modified gravity effects may alter the structure and stability of dense astrophysical objects. This makes the present framework relevant to nuclear astrophysics, where compact stars are commonly used to probe the behavior of matter in extreme regimes.

The relationship between the Ricci scalar R and the Gauss-Bonnet invariant G corresponds to significant distinctive physical aspects that establish the difference between $f(R,G)$ theory and GR. This interaction contributes to the physical features of gravity through the incorporation of extra degrees of freedom together with curvature terms, resulting in a detailed description of cos-

mic development and various astrophysical events. However, if the basic formalism of $f(R,G)$ gravity does not satisfy the required stability criteria, these extra degrees of freedom can lead to ghost instabilities. Specifically, an inadequate functional dependence on R and G, or a mismatch between these curvature invariants, can produce kinetic terms with a negative sign, indicating the possibility of ghost models that correspond to the inconsistency of the considered theory of gravity. Consequently, it can be asserted that the study of ghost-free conditions is imperative for determining the mathematical compatibility of $f(R,G)$ theory of gravity. Distinct mathematical approaches have been proposed in the literature to avoid ghost instabilities in the formalism of $f(R,G)$ gravity. The key objective is to guarantee that the incorporation of extra degrees of freedom determined by higher-order curvature invariants corresponds to positive kinetic energy and does not cause Ostrogradsky instabilities. To develop a dynamically feasible $f(R,G)$ framework, stability constraints must be applied to the derived form of the action. A basic condition is that the kinetic matrix related to the additional scalar modes is strictly positive, ensuring that the propagating degrees of freedom possess positive energy. The consideration of specific constraints on functional derivatives, particularly $f_R > 0$ and $f_{RR} > 0$, ensures a positive functional gravity-based interaction and prevents the scalaron from becoming a ghost or tachyon. Similarly, the derivatives related to the Gauss-Bonnet invariant should satisfy appropriate positivity criteria to ensure the physical stability of the Gauss-Bonnet scalar mode.

The explicit relationship between the Gauss-Bonnet invariant and the Ricci scalar induces new physical implications that are not specified in GR or $f(R)$ theories of gravity. This linkage establishes a direct interaction between the dynamical characteristics of matter and the structural configuration of spacetime and further strengthens the significance of the dynamical evolution of matter in gravitational dynamics. The presence of the Gauss-Bonnet invariant in the action indicates an explicit effect on spatial curvature implied by basic physical parameters such as pressure, anisotropy, and energy density. This indicates an alternative framework of the hydrostatic equilibrium constraint corresponding to self-gravitating fluid compositions, revising the mass-radius relationships and complexity evaluation for extremely dense matter configurations. The consideration of particular dynamical or parametric constraints results in a simplified form of $f(R,G)$ gravity as the basic theory of gravity. The $f(R,G)$ theory of gravity can be simplified to $f(R)$ gravity by removing the dependence on the Gauss-Bonnet invariant G. Subsequently, the gravitational Lagrangian depends only on the Ricci scalar, and the resulting formalism of gravity preserves the conserved framework of the energy-momentum tensor, with additional curvature implications providing deviations from GR. Moreover, the combined consideration of the Ricci scalar and the Gauss-Bonnet invariant accurately characterizes the dynamical properties of mysterious dark components, such as dark matter and dark energy, without invoking exotic matter.

Odintsov *et al.* [88] considered the $f(R,G)$ formalism to establish distinct dark energy frameworks. The specific constraints related to weak fields in the background of $f(R,G)$ theory were evaluated by Laurentis and Revelles [89]. They also evaluated particular frameworks corresponding to $f(R)$ and $f(G)$ theories and suggested that a generic model similar to the Green function could be derived within the Newtonian limit of $f(R,G)$ gravity. Atazadeh and Darabi [90] determined certain energy constraints and the stability of cosmic findings within the framework of several models associated with the modified theory. Laurentis *et al.* [91] addressed the inflationary phase of the cosmos by considering the framework of $f(R,G)$ theory and evaluating the linkage between various astronomical findings within the context of two specific expressions for masses associated with the Ricci scalar and the Gauss-Bonnet invariant. Bhatti *et al.* [92] investigated the formation and emergence of gravastars through the formalism of $f(R,G)$ theory and described their distinct regions by considering a specific equation of state. Compact stars, especially neutron-star-like objects, provide important astrophysical laboratories for examining matter under extremely high-density conditions. In such systems, the equation of state (EOS), pressure anisotropy, electromagnetic contributions, and relativistic hydrostatic balance play crucial roles in determining the internal structure, mass distribution, and stability of the stellar configuration. Since the microscopic behavior of dense nuclear matter is still not uniquely fixed, effective relativistic models based on anisotropic fluids and polytropic EOS remain useful for exploring the macroscopic consequences of dense-matter physics in strong gravitational fields.

The Tolman-Oppenheimer-Volkoff (TOV) equation is a well-known equation that specifies the relationship between stellar formation and the metric in the case of conserved energy. The EOS is also significant for assessing the fundamental components of a compact object. It is essential to note that the anisotropic form of the EOS is significant for calculating solutions to the hydrostatic equilibrium equations. Furthermore, the polytropic EOS has considerable importance because it defines the relationship between pressure and density, and its solution leads to the derivation of the Lane Emden equation (LEEq) [93, 94]. Thirukkanesh and Ragel [95] devised a framework in which the Einstein field equations were amended to establish two separate models in an identical form after taking into account the polytropic EOS. They assessed that the effects on mass and energy density were

consistent with the observed findings. The primary goal of this manuscript and the challenges it addresses are given below:

- We assess the concept of complexity within the formalism of $f(R,G)$ gravity, taking into account the findings presented in [5]. Anisotropic matter distribution and non-uniform energy density are two particularly important factors in analyzing the complexity of the structural composition. The structural parameters are determined through the orthogonal splitting of the Riemann tensor. The scalar Y_{TF}, which incorporates the effects of non-uniform energy density, anisotropic matter distribution, and the dark source terms arising from $f(R,G)$ gravity on the total energy of the matter composition, is identified as the complexity factor.

- The consistency of the diminishing complexity condition is assessed by considering two specific frameworks proposed by Gokhroo and Mehra [96], thereby establishing the significance of the basic parameters in $f(R,G)$ gravity.

- We discuss the astrophysical relevance of the obtained complexity factor by relating it to dense compact-star matter, pressure anisotropy, electromagnetic effects, polytropic EOSs, and hydrostatic equilibrium. Thus, the diminishing-complexity constraint is interpreted as an effective restriction on the internal organization and stability of neutron-star-like compact configurations.

Our manuscript is arranged as described below: Particular terms and notations are specified in Sec II. In Sec. III, the baisc formation of $f(R,G)$ gravity, as well as certain incredibly important aspects having significant relevance in the study, are discussed. We determine the specific relations between gravitational mass, Tolman mass, and the conformal tensor in Sec. IV. Moreover, the importance of these relations in assessing the complexity is also discussed. In Sec. V, orthogonal splitting of the Riemann tensor yields multiple structural scalars out of which one is referred as the CF and is significant in assessing the system's complexity. The particular solutions associated with the alternative field equations for $f(R,G)$ theory in the context of zero CF restraint are derived in the context of $f(R,G)$ theory in Sec. VI. The astrophysical implications of our findings are expressed as Sec. VII. Finally, we provide a summary as closing remarks in Sec. VIII.

II. IMPORTANT TERMINOLOGIES AND NOTATIONS:

We begin by specifying the key terms and symbolic representations used to describe our results. We consider $u^{\zeta}=\left(\mathcal{A}^{-1},0,0,0\right)$; and $l^{\zeta}=\left(0,\mathcal{B}^{-1},0,0\right)$, as the four-velocity and the four-vector associated with the cylindrical fluid distribution, respectively. These quantities satisfy the constraints $l^{\zeta}l_{\zeta}=-1$, $u^{\zeta}u_{\zeta}=1$, and $l^{\zeta}u_{\zeta}=0$. The cylindrical matter distribution implies the vanishing of the magnetic part of the conformal tensor, while the corresponding electric part is written as $\mathbb{E}_{\zeta\eta}=C_{\zeta\xi\eta\alpha}u^{\xi}u^{\alpha}$, $\xi,\alpha=0,1,2,3$. It is also subject to the constraints $\mathbb{E}^{\zeta}_{\zeta}=0$, $\mathbb{E}_{\zeta\eta}=\mathbb{E}_{(\zeta\eta)}$, and $\mathbb{E}_{\zeta\eta}u^{\eta}=0$. In this manuscript, we consider structure scalars to evaluate the CF corresponding to the charged, highly dense matter distribution. We consider the following representation of tensors [31, 97]

$$Y_{\zeta\eta}=R_{\zeta\tau\eta\lambda}u^{\tau}u^{\lambda}, \tag{1}$$

$$Z_{\zeta\eta}=R_{\zeta\tau\eta\lambda}u^{\tau}u^{\lambda}=\frac{1}{2}\eta_{\zeta\tau\lambda\xi}R^{\tau\xi}_{\eta\mu}u^{\lambda}u^{\mu}, \tag{2}$$

$$X_{\zeta\eta}={}^{\star}R_{\zeta\tau\eta\lambda}u^{\tau}u^{\lambda}=\frac{1}{2}\eta^{\lambda\xi}_{\zeta\tau}R^{*}_{\lambda\xi\eta\mu}u^{\tau}u^{\mu}, \tag{3}$$

where $\star$ denotes the dual tensor and $\eta^{\lambda\xi}_{\zeta\tau}$ is the Levi-Civita symbol, which takes the values -1, 1, and 0, specifying the negative, positive, and zero fluctuations, along with $R^{*}_{\lambda\eta\zeta\delta}=\frac{1}{2}\eta_{\lambda\beta\zeta\delta}R^{\mu\beta}_{\mu\eta}$. These tensors define the orthogonal decomposition of the Riemann curvature tensor [98]. It is important to note that, in our case study, Greek indices run from 0 to 3.

III. REVISED EQUATIONS OF MOTION AND PRIMARY VARIABLES

A cylindrically symmetric matter configuration with anisotropic pressure and electric charge is considered to establish the mathematical framework of this study. The modified equations of motion associated with $f(R,G)$ gravity are solved, leading to a description of the cylindrically symmetric structure in terms of various kinematical parameters that are important for defining the structure scalars. The action associated with $f(R,G)$ gravity has the following form [99]

$$\mathbb{S}_{f(R,G)}=\frac{1}{2k}\int\sqrt{-g}f(R,G)\mathrm{d}^4x+\int\left(\mathcal{L}_m+\mathcal{L}_{\mathrm{EM}}\right)\sqrt{-g}\mathrm{d}^4x, \tag{4}$$

Here, $\mathcal{L}_m$ and $\mathcal{L}_{\mathrm{EM}}$ denote the matter Lagrangian and electromagnetic Lagrangian, respectively. Furthermore, g denotes the determinant of the metric tensor

$$G\equiv R_{\gamma\delta\alpha\beta}R^{\gamma\delta\alpha\beta}-4R_{\gamma\delta}R^{\gamma\delta}+R^2,$$

denotes the Gauss–Bonnet invariant. Varying the action with respect to $g_{\zeta\eta}$ yields the modified field equation as

$$\Sigma_{\zeta\eta}+\kappa\mathcal{T}^{(\mathrm{mat})}_{\zeta\eta}=R_{\zeta\eta}-\frac{1}{2}g_{\zeta\eta}R, \tag{5}$$

where the expression for $\Sigma_{\zeta\eta}$ is described as

$$\begin{aligned}\Sigma_{\zeta\eta}=&\nabla_\zeta\nabla_\eta f_R-g_{\zeta\eta}\Box f_R+2R\nabla_\zeta\nabla_\eta f_G\\&-2g_{\zeta\eta}\Box R f_G-4R^{\gamma}_{\zeta}\nabla_\gamma\nabla_\eta f_G-4R^{\gamma}_{\eta}\\&\times\nabla_\gamma\nabla_\zeta f_G+4R_{\zeta\eta}\Box f_G+4g_{\zeta\eta}R^{\mu\nu}\nabla_\mu\\&\times\nabla_\nu f_G+4R_{\zeta\mu\nu\eta}\nabla^\mu\nabla^\nu f_G-\frac{1}{2}g_{\zeta\eta}V\\&+(1-f_R)\left(R_{\zeta\eta}-\frac{1}{2}g_{\zeta\eta}R\right).\end{aligned} \tag{6}$$

Here, f_R and f_G represent the derivatives with respect to the Ricci scalar R and the Gauss-Bonnet invariant G, respectively. The energy-momentum tensor $\mathcal{T}^{\zeta(\mathrm{mat})}_{\eta}$ characterizes the anisotropic distribution of the fluid and is mathematically expressed as [100, 101]

$$\mathcal{T}^{\zeta(\mathrm{mat})}_{\eta}=\rho u^\zeta u_\eta-\mathcal{P}h^\zeta_\eta+\Pi^\zeta_\eta, \tag{7}$$

where

$$\Pi^\zeta_\eta=\Pi\left(j^\zeta j_\eta+\frac{1}{3}h^\zeta_\eta\right);\quad \mathcal{P}=\frac{\mathcal{P}_r+2\mathcal{P}_\perp}{3}, \tag{8}$$

and

$$\Pi=\mathcal{P}_r-\mathcal{P}_\perp;\quad h^\zeta_\eta=\delta^\zeta_\eta-u^\zeta u_\eta, \tag{9}$$

where j^ζ denotes the circular dimensions, Π denotes the anisotropic tensor, and h^ζ_η specifies the projection tensor. In the anisotropic-pressure scenario considered in this manuscript, $\mathcal{P}_\perp$ and $\mathcal{P}_r$ represent the tangential and radial pressure components. The consideration of cylindrically symmetric spacetime yields simplified field equations, making them useful for assessing estimation techniques, numerical methods, stability, and anisotropic matter distributions in the context of more restricted symmetries than spherical spacetime. It demonstrates the nonspherical relationships between geometry and matter and refers to the violation of the uniqueness theorem, specifying an expanded solution space in the formalism of GR. Furthermore, cylindrically symmetric spacetime plays a crucial role in GR and alternative gravity theories because of its capacity to describe precisely the fundamental characteristics of compact objects and to serve as a benchmark for gravitational waves and anisotropic matter distributions.

They provide the required analytical and mathematical formalism for establishing the relationship between gravity theories and cosmic phenomena. The nonlinear representation of modified gravity theories complicates the solution of field equations, whereas the consideration of cylindrically symmetric spacetime leads to a simplified form of these equations. In the formalism of alternative gravity theories, cylindrically symmetric spacetime offers a thorough explanation of topological effects, particularly conical singularities. In GR, the cosmic string represents a flat and conical spacetime, and the corresponding deficit angle specifies substantial physical attributes. In the formalism of modified gravity theories, this deficit angle may vary because of additional fields or extra curvature terms.

Cylindrical spacetime in the framework of revised gravity theories offers a comprehensive interpretation of global characteristics. The consideration of cylindrical spacetime in the formalism of alternative theories simplifies the analysis of physical characteristics related to light cones and confined timelike curves.

The analysis of gravitational waves within the formalism of modified gravity theories includes additional polarization phases, and the wave equations related to these phases and their relationships with matter can be studied efficiently using cylindrical wave solutions. The cylindrically static metric is used to study the interactions of dynamically included fields from various gravity theories. The evaluation of the cylindrical distribution of mass and energy over a significantly extended area is important in astrophysics and cosmology. The assumption of cylindrically symmetric spacetime, together with the formalism of alternative gravity theories, anticipates the emergence of cosmic strings in early cosmic evolution. A cosmic string specifies a thin, extremely dense thread of energy that stretches across the universe. Massive molecular clouds are usually compressed into string-like structures, and cylindrical spacetime provides a description of such developments in astrophysics. Galaxies with significant borders may have a cylindrical core instead of a spherical one, so the gravitational potential related to the center of these galaxies can be determined through cylindrically symmetric spacetime.

The emergence of an intense magnetic field can disrupt spherical symmetry and develop a matter configuration with cylindrical symmetry. Furthermore, the cylindrical metric thoroughly illustrates the dynamics of the gravitational field and the structural configuration of spacetime surrounded by cosmic strings. The cylindrical formalism assumes a conical spacetime, resulting in a particular gravitational lensing configuration instead of deformed images. Our cosmos is formed as a cosmic web, with galaxy clusters, filaments, and voids. The filaments are enormous, uninterrupted clusters of dark matter and galaxies, and on broader scales these filaments are specified as infinite cylinders. The assumption of cylindrical metrics provides a description of the stability of

such filaments at observed scales. Subsequently, it can be concluded that the cylindrically symmetric framework is an exceptionally effective approach for determining fundamental physical characteristics from cosmic strings and huge cosmic filaments, particularly in the context of analyzing unconventional impacts anticipated by alternative theories of gravity. In our case study, the inner composition is evaluated by considering the following cylindrically symmetric metric [102]

$$\mathrm{d}s^2 = \mathcal{A}^2(r)\mathrm{d}t^2 - \mathcal{B}^2(r)\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\alpha^2\mathrm{d}z^2. \tag{10}$$

The coordinate ranges $0 \le r$, $0 \le \theta \le 2\pi$, $-\infty < z < +\infty$, $-\infty \le t \le +\infty$ are used to describe a static cylindrical matter distribution. The modified field equations for $f(R,G)$ gravity are

$$G_\eta^\zeta = 8\pi\mathcal{T}_\eta^{\zeta(\mathrm{eff})} = 8\pi\left(\mathcal{T}_\eta^{\zeta(\mathrm{mat})} + \mathcal{T}_\eta^{\zeta(D)} + \mathbb{E}_\eta^\zeta\right), \tag{11}$$

The value of $T_\eta^{\zeta(D)}$ is determined by considering Eq. (5),

$$\begin{aligned}\mathcal{T}_\eta^{\zeta(D)} =& \nabla^\zeta\nabla_\eta f_R - g_\eta^\zeta\Box f_R + 2R\nabla^\zeta\nabla_\eta f_G - 2g_\eta^\zeta \\ &\times\Box R f_G - 4R_\gamma^\zeta\nabla^\gamma\nabla_\eta f_G - 4R_\eta^\gamma\nabla_\gamma\nabla^\zeta f_G \\ &+ 4R_\eta^\zeta\Box f_G + 4g_\eta^\zeta R^{\mu\nu}\nabla_\mu\nabla_\nu f_G + (1-f_R) \\ &\times\left(R_\eta^\zeta - \frac{1}{2}g_\eta^\zeta R\right) + 4R_{\mu\nu\eta}^\zeta\nabla^\mu\nabla^\nu f_G - \frac{1}{2}g_\eta^\zeta V.\end{aligned} \tag{12}$$

This is the energy–momentum tensor associated with self-gravitating matter, which is governed by the internal configuration of the celestial object within the framework of $f(R,G)$ gravity.

The electromagnetic energy–momentum tensor is given by the following expression:

$$\mathbb{E}_{\zeta\eta} = \frac{1}{4\pi}\left\{-F_\zeta^\delta F_{\eta\delta} + \frac{1}{4}F^{\delta\tau}F_{\delta\tau}g_{\zeta\eta}\right\}, \tag{13}$$

where $F_{\zeta\eta} = \psi_{\eta,\zeta} - \psi_{\zeta,\eta}$ and ψ_ζ denote the electromagnetic field tensor and the four-potential, respectively. The relevant electromagnetic field equations are

$$F_{;\eta}^{\zeta\eta} = \zeta_0 J^\zeta, \quad F_{[\zeta\eta;\delta]} = 0, \tag{14}$$

where J^ζ and ζ_0 are specified as the four-current and magnetic permeability, respectively. It is assumed that stationary charges correspond to the absence of a magnetic field. Thus,

$$\psi_\zeta = \psi(r)\delta_\zeta^0, \tag{15}$$

where ψ is a scalar potential that specifies the radial component of the electric field and depends only on the radial coordinate r. The electric charge contained within a cylindrically symmetric charged matter distribution can be determined as

$$q(r) = 4\pi\int_0^r \lambda r^2\sqrt{\mathcal{B}}\mathrm{d}r. \tag{16}$$

Here, λ denotes the electric charge density, and $\mathcal{B}(r)$ is a metric function associated with the radial component of the spatial configuration. Within the framework of $f(R,G)$ gravity, the modified equations of motion are

$$8\pi\left[\rho + \psi_0^{0(D)} - \frac{3q^2}{8\pi r^4}\right] = \frac{1}{r^2\mathcal{B}^2} - \frac{2\mathcal{B}'}{r\mathcal{B}^3}, \tag{17}$$

$$8\pi\left[-\mathcal{P}_r + \psi_1^{1(D)} - \frac{3q^2}{8\pi r^4}\right] = \frac{2\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2} + \frac{1}{r^2\mathcal{B}^2}, \tag{18}$$

$$\begin{aligned}8\pi\left[-\mathcal{P}_\perp + \psi_2^{2(D)} - \frac{q^2}{8\pi r^4}\right] =& \frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^2} - \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3} \\ &- \frac{\mathcal{B}'}{r\mathcal{B}^3} + \frac{\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2}.\end{aligned} \tag{19}$$

where $\psi_0^{0(D)}$, $\psi_1^{1(D)}$, and $\psi_2^{2(D)}$ are the dark source terms associated with $f(R,G)$ gravity, and their expressions are given in the Appendix. The hydrostatic equilibrium equation is derived from the conservation of the energy-momentum tensor

$$\begin{aligned}&\frac{\rho\mathcal{A}'}{\mathcal{A}\mathcal{B}^2} + \frac{\mathcal{A}\mathcal{A}'}{\mathcal{B}^2}\psi^{00(D)} + \frac{\mathcal{A}'\mathcal{P}_r}{\mathcal{A}\mathcal{B}^2} + \frac{\mathcal{P}_r'}{\mathcal{B}^2} + \frac{4\mathcal{B}'}{\mathcal{B}^3}\mathcal{P}_r \\ &+ \left(1 + \frac{2}{r} + \frac{\mathcal{A}'}{\mathcal{A}}\right)\psi^{11(D)} + \frac{3}{8\pi}\left(\frac{q^2}{r^4\mathcal{B}^2}\right)' \\ &+ \frac{2\mathcal{B}'}{\mathcal{B}}\psi^{11(D)} + \frac{3q^2\mathcal{B}'}{4\pi r^4\mathcal{B}^3} - \left(\alpha^2+1\right)\frac{\mathcal{P}_\perp}{r\mathcal{B}^2} \\ &- \frac{2r}{\mathcal{B}^2}\psi^{22(D)} - \frac{q^2}{4\pi\mathcal{B}^2r^5} + \frac{2\mathcal{P}_r}{r\mathcal{B}^2} + \frac{3q^2}{4\pi r^5\mathcal{B}^2} = 0.\end{aligned} \tag{20}$$

The above equation characterizes the anisotropic development of a fluid and is known as the TOV equation, which can aid in analyzing the fundamental physical structure of a cylindrically symmetric system. Moreover, the TOV equation is commonly used to determine the substantial characteristics of astronomical objects. The use of Eq. (18) results

$$\frac{\mathcal{A}'}{\mathcal{A}} = \left(\frac{4r^2}{r^2 - mr + 4q^2}\right)\left(-4\pi\mathcal{P}_r + 4\pi r\psi_1^{1(D)} - \frac{3q^2}{2r^3}\right) - \frac{1}{r}. \tag{21}$$

After substituting $\frac{\mathcal{A}'}{\mathcal{A}}$, together with the specific relations for the radial and tangential pressure components derived from the field equations, the hydrostatic equilibrium equation takes the following form:

$$\left(\frac{\mathcal{P}_r}{\mathcal{B}^2}\right)'+\frac{\mathcal{A}\mathcal{A}'}{\mathcal{B}^2}\psi^{00(D)}+\left(1+\frac{2}{r}\right)\psi^{11(D)}+\frac{3}{8\pi}\left(\frac{q^2}{r^4\mathcal{B}^2}\right)'+\frac{2\mathcal{B}'}{\mathcal{B}}\psi^{11(D)}+\frac{3q^2\mathcal{B}'}{4\pi r^4\mathcal{B}^3}-\frac{\mathcal{P}_\perp}{r\mathcal{B}^2}-\frac{2r}{\mathcal{B}^2}\times\psi^{22(D)}-\frac{q^2}{4\pi\mathcal{B}^2r^5}+\frac{2\mathcal{P}_r}{r\mathcal{B}^2}+\frac{3q^2}{4\pi r^5\mathcal{B}^2}-\frac{\alpha\mathcal{P}_\perp}{r\mathcal{B}^2}+\left[\frac{\rho}{\mathcal{B}^2}+\frac{\mathcal{P}_r}{\mathcal{B}^2}+\psi^{11(D)}\right]\left[\left(\frac{4r^2}{r^2-mr+4q^2}\right)\times\left(-4\pi r\mathcal{P}_r-4\pi r\psi^{11(D)}-\frac{3q^2}{2r^3}+\frac{1}{4r}-\frac{2m}{r^2}\right)\right]=0. \tag{22}$$

IV. RELATIONSHIP BETWEEN MASS FUNCTION AND THE CURVATURE TENSOR

The Riemann tensor $R^{\alpha}_{\beta\zeta\eta}$ is generally expressed in terms of the Weyl tensor $C^{\alpha}_{\beta\zeta\eta}$ and the Ricci tensor R^{η}_{ζ}

$$R^{\alpha}_{\beta\zeta\eta}=C^{\alpha}_{\beta\zeta\eta}+\frac{1}{2}R^{\alpha}_{\zeta}g_{\beta\eta}-\frac{1}{2}R_{\beta\zeta}\delta^{\alpha}_{\eta}+\frac{1}{2}R_{\beta\eta}\delta^{\alpha}_{\zeta}-\frac{1}{2}R^{\alpha}_{\eta}g_{\beta\zeta}-\frac{1}{6}R\left(\delta^{\alpha}_{\zeta}g_{\beta\eta}-g_{\beta\zeta}\delta^{\alpha}_{\eta}\right). \tag{23}$$

The conformal tensor provides information about the gravitational effects of a body moving along geodesics. These effects can be inferred by examining variations in the separation between adjacent geodesics. The conformal tensor is regarded as an associated curvature quantity related to the propagation of gravitational waves in matter-free regions. Another important feature is its trace-free nature: it comprises 256 components, of which 10 can be described independently in 4D. Furthermore, the inclusion of a four-velocity vector can decompose these ten components into two rank-two tensors.

A. Tolman mass

The mass contained in a cylindrical configuration is derived by considering C-energy [103] and the Tolman formalism [104]. These mathematical frameworks yield identical results at the boundary and have several implications for the internal energy in the presence of an inhomogeneous fluid structure. C-energy specifies the energy per unit length contained within the cylindrical configuration, whereas Tolman mass measures the gravitational mass. Tolman mass is defined as a volume quantity integrated within the three-dimensional volume, whereas C-energy is regarded as a linear density quantity along the symmetry axis. In addition, Tolman mass corresponds to the initial intensity of test particles, while C-energy is related to the cylindrical singularity and predicts the linear mass density required for the deficit angle. Thus, Tolman mass can be regarded as a constrained quantity that describes the total mass confined within the geometry, whereas C-energy is a per-unit-length quantity that represents the linear mass density of the gravitational distribution. The consideration of Tolman mass for a cylindrical matter distribution is appropriate for finite configurations in the context of mass per unit length rather than absolute mass. The combined consideration of Tolman mass and C-energy plays a crucial role in analyzing the underlying mass-energy distribution of finite cylindrical configurations. The expression related to the C-energy is written as

$$m(r)=\tilde{E}=E\hat{\ell}=\frac{-r}{2\mathcal{B}^2}\left(1-\frac{\mathcal{B}^2}{4}\right)+\frac{q^2}{2r}. \tag{24}$$

Combining the derivation of the preceding equation with Eq. (17) yields

$$m=\frac{r}{8}+\frac{1}{\mathcal{B}}+4\pi\int_0^r r^2\left(\rho+\psi_0^{0(D)}\right)\mathrm{d}r+\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r. \tag{25}$$

Eqs. (17)–(19), together with the C-energy result

$$m=\frac{r}{8}+\frac{9q^2}{4r}-\frac{r}{2\mathcal{B}^2}-2\pi r^3\left[\rho-\mathcal{P}_r-2\mathcal{P}_\perp+\psi_0^{0(D)}+\psi_1^{1(D)}+\psi_2^{2(D)}\right]+r^3\mathbb{E}. \tag{26}$$

Consequently, the conformal scalar $\mathbb{E}$, fluid components, and correction terms associated with $f(R,G)$ theory are important for calculating the mass function. This indicates that the nonlinear and irregular aspects of density have significant implications for the analysis of stellar configurations. Substituting the electric component of the conformal tensor into Eq. (26) yields the following representation of the mass function:

$$m=\frac{r}{8}+\frac{9q^2}{4r}-\frac{r}{2\mathcal{B}^2}-2\pi r^3\left[\rho-\mathcal{P}_r-2\mathcal{P}_\perp+\psi_0^{0(D)}+\psi_1^{1(D)}+\psi_2^{2(D)}\right]-\frac{r^3}{2\mathcal{A}\mathcal{B}^2}\left[\mathcal{A}''-\frac{\mathcal{A}'\mathcal{B}'}{\mathcal{B}}+\frac{\mathcal{A}\mathcal{B}'}{r\mathcal{B}}-\frac{\mathcal{A}'}{r}+\frac{\mathcal{A}}{r^2}\right]. \tag{27}$$

By comparing Eq. (25) with Eq. (26), the value of $\mathbb{E}$ becomes

$$\mathbb{E}=-\frac{4\pi}{3r^3}\int_0^r\frac{r^3}{3}\left(\rho+\psi_0^{0(D)}\right)'\mathrm{d}r+2\pi\left[-\mathcal{P}_r-2\mathcal{P}_\perp+\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right]-\frac{9q^2}{4r^4}$$

$$+\frac{1}{r^2\mathcal{B}}\left(\frac{1}{r}+\frac{1}{2\mathcal{B}}\right)+\frac{1}{r^3}\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r +\frac{10\pi}{3}\left(\rho+\psi_0^{0(D)}\right). \tag{28}$$

Eq. (28) establishes the relationship between the conformal tensor and key structural characteristics, such as inhomogeneous energy density and isotropic pressure. Furthermore, substituting Eq. (28) into Eq. (26) yields

$$m=\frac{r}{8}+\frac{4\pi r^3}{3}\left(\rho+\psi_0^{0(D)}\right)-\frac{4\pi}{3}\int_0^r\frac{r^3}{3}\left(\rho+\psi_0^{0(D)}\right)'\mathrm{d}r +\frac{1}{\mathcal{B}}+\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r. \tag{29}$$

It demonstrates a specific relationship between the homogeneous energy density and the mass function. In the context of the Weyl tensor and $f(R,G)$ gravity, the two factors, anisotropic pressure and energy-density inhomogeneity, highlight the importance of the fluid distribution. Nevertheless, in the context of homogeneous mass dispersion, the essential variation arises from the inhomogeneity in the mass function associated with a regular disruption of the mass density, as described in Eq. (29).

Tolman proposed an innovative formalism for evaluating the generation of energy associated with the surface of matter. This energy was assumed to be the accumulated energy of the configuration and is referred to as the "active gravitational mass" [105]. Furthermore, it provides a pertinent modification of the mass formalism for examining the anisotropic configuration of the fluid [104]

$$m_T=4\pi\int_0^{r_\Sigma}\mathcal{A}\mathcal{B}r^2\left(\psi_0^{0(D)}-\psi_1^{1(D)}-2\psi_2^{2(D)}\right)\mathrm{d}r. \tag{30}$$

Tolman's concept focuses on computing the total mass–energy distribution associated with the system's composition. We now investigate the mass distribution of a cylindrical configuration with radius r in the context of $f(R,G)$ theory

$$m_T=4\pi\int_0^{\tilde{r}}\mathcal{A}\mathcal{B}\tilde{r}^2\left(\psi_0^{0(D)}-\psi_1^{1(D)}-2\psi_2^{2(D)}\right)\mathrm{d}r. \tag{31}$$

The surface energy was used to evaluate the apparent expression for the inertial mass, denoted by m_T, as discussed in [30, 31]. Substituting Eqs. (17)–(19), together with the expression for $\mathcal{A}'$ in Eq. (31), yields

$$m_T=\frac{1}{2}\int\left[\frac{-r\mathcal{A}\mathcal{B}'}{\mathcal{B}^2}-\frac{r^2\mathcal{A}''}{\mathcal{B}}+\frac{q^2\mathcal{A}\mathcal{B}}{r^2}-\left(1-\frac{r\mathcal{B}'}{\mathcal{B}}\right)\right.$$
$$\times\left(\frac{12r^3\mathcal{A}}{r^2\mathcal{B}-mr\mathcal{B}+4q^2\mathcal{B}}\left(-4\pi\mathcal{P}_r+4\pi r\psi_1^{1(D)}\right.\right.$$
$$\left.\left.-\frac{3}{2}\frac{q^2}{r^3}\right)-\frac{3\mathcal{A}}{\mathcal{B}}\right)-8\pi r^2\mathcal{A}\mathcal{B}\psi_2^{2(D)}\Bigg]\mathrm{d}r. \tag{32}$$

The above equation emphasizes the significance of the gravitational source m_T, which is often defined as the inertial mass. Additionally, the expression for the Tolman mass is given as

$$m_T=M\left(\frac{r}{r_\Sigma}\right)^3+\frac{3r^3}{4}\int\left(\frac{\mathcal{A}'\mathcal{B}'}{r\mathcal{B}^2}-\frac{2\mathcal{A}'}{r^2\mathcal{B}}-\frac{\mathcal{A}''}{r\mathcal{B}}\right)\mathrm{d}r +r^3\int\frac{1}{r^2}\left(\frac{q^2\mathcal{A}\mathcal{B}}{r^2}\right)'\mathrm{d}r+r^3\int\frac{1}{r^3}\eta_2^2\mathrm{d}r +9r^3\int\frac{\mathcal{A}}{r^2-mr+4q^2}\left(4\pi r\psi_1^{1(D)}-\frac{3q^2}{2r^3}\right)\mathrm{d}r -\frac{27r^3}{16}\int\frac{1-\mathcal{A}}{r^3\mathcal{B}}\mathrm{d}r+\frac{81r^3}{8}\int\frac{q^2}{r^5\mathcal{B}}\mathrm{d}r+\frac{9r^3}{4}\int\frac{1}{r\mathcal{B}} \times\left(\mathbb{E}-\frac{1}{r^2\mathcal{B}^2}\right)\mathrm{d}r-9\pi r^3\int\frac{1}{r\mathcal{B}}\left(\rho-\mathcal{P}_r-2\mathcal{P}_\perp +\psi_0^{0(D)}+\psi_1^{1(D)}+\psi_2^{2(D)}\right)\mathrm{d}r-3r^3\int\frac{q^2\mathcal{A}\mathcal{B}}{r^6}\mathrm{d}r +3r^3\int\frac{1}{r^4}\eta_2^{2(D)}\mathrm{d}r. \tag{33}$$

Considering Eq. (28) in this equation determines

$$m_T=M\left(\frac{r}{r_\Sigma}\right)^3+\frac{3r^3}{4}\int\left(\frac{\mathcal{A}'\mathcal{B}'}{r\mathcal{B}^2}-\frac{2\mathcal{A}'}{r^2\mathcal{B}}-\frac{\mathcal{A}''}{r\mathcal{B}}\right)\mathrm{d}r +r^3\int\frac{1}{r^2}\left(\frac{q^2\mathcal{A}\mathcal{B}}{r^2}\right)'\mathrm{d}r+r^3\int\frac{1}{r^3}\eta_2^2\mathrm{d}r +9r^3\int\frac{\mathcal{A}}{r^2-mr+4q^2}\left(4\pi r\psi_1^{1(D)}-\frac{3q^2}{2r^3}\right)\mathrm{d}r -\frac{27r^3}{16}\int\frac{1-\mathcal{A}}{r^3\mathcal{B}}\mathrm{d}r+\frac{81r^3}{8}\int\frac{q^2}{r^5\mathcal{B}}\mathrm{d}r -\frac{9r^3}{4}\int\frac{1}{r^3\mathcal{B}^3}\mathrm{d}r-9\pi r^3\int\frac{1}{r\mathcal{B}}\left(\rho-\mathcal{P}_r-2\mathcal{P}_\perp +\psi_0^{0(D)}+\psi_1^{1(D)}+\psi_2^{2(D)}\right)\mathrm{d}r-3r^3\int\frac{q^2\mathcal{A}\mathcal{B}}{r^6}\mathrm{d}r +3r^3\int\frac{1}{r^4}\eta_2^{2(D)}\mathrm{d}r-\frac{9r^3}{4}\int\left[\frac{1}{r^4\mathcal{B}^2}+\frac{1}{2r^3\mathcal{B}^3}\right. +\frac{1}{r^4\mathcal{B}}\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r+\frac{10\pi}{3r\mathcal{B}}\left(\rho+\psi_0^{0(D)}\right) -\frac{4\pi}{3r^4\mathcal{B}}\int\frac{r^3}{3}\left(\rho+\psi_0^{0(D)}\right)\mathrm{d}r-\frac{9q^2}{4r^5\mathcal{B}} \left.+\frac{2\pi}{r\mathcal{B}}\left(-\mathcal{P}_r-2\mathcal{P}_\perp+\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right)\right]. \tag{34}$$

In $f(R,G)$ theory, this expression is significant be-

cause it identifies the effects of correction terms, inhomogeneous energy density, and anisotropic pressure on the Tolman mass.

V. ORTHOGONAL DECOMPOSITION OF RIEMANN TENSOR AND STRUCTURE SCALARS

We now consider several key variables that determine the properties of anisotropic fluids. The orthogonal splitting of the Riemann tensor yields expressions for these parameters. Herrera [5] comprehensively described the fundamental characteristics of the matter composition in terms of structure scalars. These scalars are the trace and trace-free components of three distinct tensors that are compatible with the evaluation of the model. Considering the modified equations of motion together with Eq. (23) implies

$$R^{\zeta\alpha}_{\eta\beta} = C^{\zeta\alpha}_{\eta\beta} + 28\pi\mathcal{T}^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]} + 8\pi\mathcal{T}^{(eff)}\left(\frac{1}{3}\delta^{\zeta}_{[\eta}\delta^{\alpha}_{\beta]} - \delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}\right). \quad (35)$$

Considering Eq. (12) completes the corresponding interpretation of the orthogonal splitting of the Riemann tensor

$$R^{\zeta\alpha}_{\eta\beta} = R^{\zeta\alpha}_{(I)\eta\beta} + R^{\zeta\alpha}_{(II)\eta\beta} + R^{\zeta\alpha}_{(III)\eta\beta} + R^{\zeta\alpha}_{(IV)\eta\beta} + R^{\zeta\alpha}_{(V)\eta\beta}, \quad (36)$$

where

$$R^{\zeta\alpha}_{(I)\eta\beta} = 16\pi\rho u^{[\zeta}u_{[\eta}\delta^{\alpha]}_{\beta]} + \frac{q^2}{2r^4}\left[u^{\zeta}u_{[\eta}\delta^{\alpha}_{\beta]}\right] - 16Ph^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]} + 8\pi(\rho - 3p)\left(\frac{1}{3}\delta^{\zeta}_{[\eta}\delta^{\alpha}_{\beta]}\right), \quad (37)$$

$$R^{\zeta\alpha}_{(II)\eta\beta} = 16\Pi^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}, \quad (38)$$

$$R^{\zeta\alpha}_{(III)\eta\beta} = 4u^{[\zeta}u_{[\eta}\mathbb{E}^{\alpha]}_{\beta]} - \eta^{\zeta\alpha}_{\beta}\eta_{\lambda\eta\vartheta}\mathbb{E}^{\vartheta\lambda} + \frac{q^2}{2r^4}\left[u^{\alpha}u_{[\eta}\delta^{\zeta}_{\beta]}\right] + \frac{1}{2}\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}f(R,G), \quad (39)$$

$$R^{\zeta\alpha}_{(IV)\eta\beta} = \frac{1}{2}\left[\left(\Box f_R + 2\Box R f_G + \frac{1}{2}f_G G\right)\right]\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]} + \frac{5}{3}\frac{q^2}{r^4}\left[h^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}\right] + \left(\nabla^{\zeta}\nabla_{\eta}f_R + 2R\nabla^{\zeta}\nabla_{\eta}f_G - 4R^{\varsigma}_{\eta}\nabla_{\varsigma}\nabla^{\zeta}f_G + 4R^{\zeta}_{\eta}\Box f_G + 4R^{\zeta}_{\mu\nu\eta}\nabla^{\mu}\nabla^{\nu}f_G + R^{\zeta}_{\eta} - R^{\zeta}_{\eta}\times f_R\right)\delta^{\alpha}_{\beta} + \left(-\nabla^{\zeta}\nabla_{\beta}f_R - 2R\nabla^{\zeta}\nabla_{\beta}f_G + 4R^{\varsigma}_{\beta}\nabla_{\varsigma}\nabla^{\zeta}f_G - 4R^{\zeta}_{\beta}\Box f_G - 4R^{\zeta}_{\mu\nu\beta}\nabla^{\mu}\nabla^{\nu}f_G - R^{\zeta}_{\beta} + R^{\zeta}_{\beta}f_R\right)\delta^{\alpha}_{\eta}, \quad (40)$$

$$R^{\zeta\alpha}_{(V)\eta\beta} = 4\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}R^{\mu\nu}\nabla_{\mu}\nabla_{\nu}f_R + \frac{q^2}{2r^4}\left[\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}\right] + \frac{1}{2}\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}f_R R + \frac{1}{2}\delta^{[\zeta}_{[\eta}\delta^{\alpha]}_{\beta]}R + \left(-\nabla^{\alpha}\nabla_{\eta}f_R - 2R\nabla^{\alpha}\nabla_{\eta}f_G + 4R^{\alpha}_{\varsigma}\nabla^{\varsigma}\nabla_{\eta}f_G + 4R^{\varsigma}_{\eta}\nabla_{\varsigma}\times\nabla^{\alpha}f_G - 4R^{\alpha}_{\eta}\Box f_G - 4R^{\alpha}_{\mu\nu\eta}\nabla^{\mu}\nabla^{\nu}f_G + R^{\alpha}_{\eta} + R^{\alpha}_{\eta}f_R\right)\delta^{\zeta}_{\beta} + \left(\nabla^{\alpha}\nabla_{\beta}f_R + 2R\nabla^{\alpha}\nabla_{\beta}f_G - 4R^{\alpha}_{\varsigma}\nabla^{\varsigma}\nabla_{\beta}f_G - 4R^{\varsigma}_{\beta}\nabla_{\varsigma}\nabla^{\alpha}f_G + 4\times R^{\alpha}_{\beta}\Box f_G + 4R^{\alpha}_{\mu\nu\beta}\nabla^{\mu}\nabla^{\nu}f_G + R^{\alpha}_{\beta}f_R\right)\delta^{\zeta}_{\eta}. \quad (41)$$

Further,

$$\epsilon_{\zeta\eta\beta} = v^{\mu}\eta_{\mu\zeta\eta\beta}, \quad \epsilon_{\zeta\eta\beta}v^{\beta} = 0, \quad (42)$$

and

$$\epsilon^{\alpha\beta\xi}\epsilon_{\xi\zeta\eta} = \delta^{\beta}_{\zeta}h^{\alpha}_{\eta} - \delta^{\alpha}_{\zeta}h^{\beta}_{\eta} + u_{\zeta}\left(u^{\alpha}\delta^{\beta}_{\eta} - u^{\beta}\delta^{\alpha}_{\eta}\right). \quad (43)$$

In the context of a cylindrical matter distribution, the magnetic component of the conformal tensor has a considerable effect on the orthogonal decomposition of the curvature tensor. Three distinct tensors are obtained by considering the above-mentioned equations in terms of the related quantities $X_{\zeta\eta}$, $Y_{\zeta\eta}$, and $Z_{\zeta\eta}$, which are expressed as

$$X_{\zeta\eta} = 4\pi\Pi_{\zeta\eta} - \mathbb{E}_{\zeta\eta} - \frac{q^2}{2r^4}\left(\delta_{\zeta}\delta_{\eta} + \frac{1}{3}h_{\zeta\eta}\right) + \frac{8\pi}{3}\times\rho h_{\zeta\eta} + \mathcal{S}^{(A)}_{\zeta\eta}, \quad (44)$$

$$Y_{\zeta\eta} = 4\pi\Pi_{\zeta\eta} + \mathbb{E}_{\zeta\eta} + \frac{q^2}{r^4}h_{\zeta\eta} - \frac{q^2}{2r^4}\left(\delta_{\zeta\eta} + \frac{1}{3}h_{\zeta\eta}\right) + \frac{4\pi}{3}(\rho + 3P)h_{\zeta\eta} + \mathcal{S}^{(B)}_{\zeta\eta}, \quad (45)$$

$$Z_{\zeta\eta} = \mathcal{S}^{(C)}_{\zeta\eta}. \quad (46)$$

The expressions for $\mathcal{S}^{(A)}_{\zeta\eta}$, $\mathcal{S}^{(B)}_{\zeta\eta}$, and $\mathcal{S}^{(C)}_{\zeta\eta}$ are provided in the Appendix. The tensors defined by these equations are characterized as structural scalars, and a comprehensive interpretation of these scalars is presented in [31]. In addition, the trace and trace-free parts of these structural scalars can be exceptionally valuable for identifying the various physical characteristics of the cylindrically symmetric configuration.

$$X_T = 8\pi\rho + \frac{3q^2}{2r^4} + \mathbb{F}. \tag{47}$$

The expression for $\mathbb{F}$ is given in the Appendix, and the trace-free part can be written as

$$X_{TF} = 4\pi\Pi - \mathbb{E} - \frac{q^2}{2r^4}. \tag{48}$$

Substituting $\mathbb{E}$ from Eq. (28) into this equation yields

$$\begin{aligned} X_{TF} = {} & 4\pi\Pi + \frac{4\pi}{9r^3}\int_0^r r^3\left(\rho+\psi_0^{0(D)}\right)'\mathrm{d}r \\ & - 2\pi\left[-\mathcal{P}_r - 2\mathcal{P}_\perp + \psi_0^{0(D)} + \psi_1^{1(D)}\right. \\ & \left. + \psi_2^{2(D)}\right] - \frac{q^2}{2r^4} - \frac{1}{r^3\mathcal{B}} - \frac{1}{2r^2\mathcal{B}^2} \\ & + \frac{1}{r^3}\int\frac{q}{r}\left(q' - \frac{2q}{r}\right)\mathrm{d}r - \frac{9q^2}{4r^4}, \end{aligned} \tag{49}$$

and

$$Y_T = 4\pi(\rho + p_r + 2p_\perp) + \frac{3q^2}{r^4} + \mathbb{H}. \tag{50}$$

Here, the value of $\mathbb{H}$ is provided in the Appendix

$$Y_{TF} = 4\pi\Pi + \mathbb{E} - \frac{q^2}{2r^4} + \mathbb{L}_{\zeta\eta}. \tag{51}$$

Here, $\mathbb{L}_{\zeta\eta} = \dfrac{\mathcal{S}_{\zeta\eta}}{\delta_\zeta\delta_\eta + \dfrac{1}{3}h_{\zeta\eta}}$. Consideration of Eq. (28) yields

$$\begin{aligned} Y_{TF} = {} & 4\pi\Pi - \frac{q^2}{2r^4} + 2\pi\left[-\mathcal{P}_r - 2\mathcal{P}_\perp + \psi_1^{1(D)}\right. \\ & \left. + \psi_2^{2(D)}\right] - \frac{4\pi}{9r^3}\int_0^r r^3\left(\rho+\psi_0^{0(D)}\right)'\mathrm{d}r \\ & + \frac{10\pi}{3}\left(\rho+\psi_0^{0(D)}\right) + \mathbb{L}_{\zeta\eta} - \frac{9q^2}{4r^4} + \frac{1}{r^3\mathcal{B}} \\ & + \frac{1}{2r^2\mathcal{B}^2}. \end{aligned} \tag{52}$$

The trace-free components given in Eqs. (48) and (51) are taken to represent the anisotropic pressure as

$$X_{TF} + Y_{TF} = 8\pi\Pi - \frac{q^2}{r^4} + \mathbb{L}_{\zeta\eta}. \tag{53}$$

Structural scalars associated with a cylindrical matter distribution are determined through the orthogonal decomposition of the Riemann tensor. These scalars are important for assessing the complexity of a system, as they characterize its energy density and anisotropic pressure.

- Y_T plays an important role in estimating the total energy density, which determines the stability, mass distribution, and gravitational interactions of the matter composition.

- Y_{TF} effectively characterizes the nonuniform energy density and pressure anisotropy.

- X_T plays a decisive role in evaluating the mass–energy composition and evolution of compact objects. In particular, it describes variations in the fluid volume over time.

- The scalar X_{TF} characterizes the irregular energy dispersion within the fluid configuration.

The matter distribution and its internal inhomogeneities are highly relevant to the analysis of the complexity of any structure. This complexity effectively evaluates the dynamics of self-gravitating matter by analyzing the extent to which it deviates from its homogeneous and isotropic state. In classical gravitational frameworks, Y_{TF} is obtained through the orthogonal splitting of the Riemann curvature tensor and is referred to as the complexity factor, as suggested by Herrera [5]. A matter distribution without any complexity is considered the most fundamental configuration, and equality between the tangential and radial pressure components plays an important role in defining the zero complexity of the system. In this context, constant pressure corresponds to the absence of pressure anisotropy. Pressure anisotropy, together with nonhomogeneous energy density, is regarded as a key factor in evaluating the complexity of the matter distribution, as it generates stresses in different directions and influences the stability of the matter configuration. The structural scalar Y_{TF} represents the implications of anisotropic and nonhomogeneous matter distributions for the Tolman mass, which is regarded as the basic parameter for establishing the intensity of the gravitational field associated with the matter. It is important to mention that the complexity factor should be a scalar to avoid coordinate restrictions and is expected to vanish for the simplest systems, including homogeneous matter distributions. The presence of Y_{TF} as an integrand in the formalism for the Tolman mass describes deviations from uniformity and measures the local contribution to the gravitational mass induced by the internal complexity of the system. The formalism for Y_{TF} incorporates the term $4\pi(P_\perp - P_r)$, which represents the anisotropic pressure distribution, whereas the derivative representation of the energy density corresponds to its nonhomogeneous distribution. For a simple matter configuration, $Y_{TF} = 0$ effectively corresponds to zero complexity by ensuring equality between the pressure components and a uniform distribution of energy density. Moreover, Y_{TF} has a direct relationship with

the Tolman mass, which is assumed to be the most important quantity for describing the characteristics of the gravitational field. Consequently, we deduce that dense matter exhibits less stable and more complex behavior due to the nonuniform fluid distribution and greater anisotropies implied by the increasing value of the complexity factor.

The combined consideration of X_{TF} and Y_{TF} is highly relevant to determining the fundamental characteristics of mass-energy distributions. Y_{TF} is referred to as the complexity factor because of its specific relationship with the Tolman mass, whereas X_{TF} provides a detailed description of stresses within the matter and of the electric part of the Weyl scalar $\mathbb{E}$ in the mathematical representation of X_{TF}, describing the tidal forces associated with the fundamental mass distribution. The combination of these structure scalars provides a detailed description of mass-energy interactions, as they correspond to two different representations of the underlying matter distribution. It can be concluded that X_{TF} is not merely an integral factor but also effectively describes the mass-energy distribution. The vanishing of X_{TF} corresponds to a specific constraint on the matter distribution, as it specifies equality between pressure anisotropy and the electric component of the Weyl tensor, which has significant implications for mass-energy dispersion, the gravitational field, and the stability of the system. In the basic relationship between X_{TF} and Y_{TF}, $X_{TF}=0$ minimizes the anisotropic effects on Y_{TF}, corresponding to complete dependence on the density gradient. Subsequently, the nonhomogeneous distribution of energy density explicitly defines the complexity of matter and corresponds to the simplest distribution of matter. The insertion of Eq. (52) into Eq. (34) illustrates the relationship between the gravitational mass and Y_{TF},

$$\begin{aligned} m_T =& M\left(\frac{r}{r_\Sigma}\right)^2+\frac{3r^3}{4}\int\left(\frac{\mathcal{A}'\mathcal{B}'}{r\mathcal{B}^2}-\frac{2\mathcal{A}'}{\mathcal{B}r^2}-\frac{\mathcal{A}''}{\mathcal{B}r}\right)\mathrm{d}r\\ &+r^3\int\frac{1}{r^3}\eta_2^{2(D)}\mathrm{d}r+r^3\int\frac{1}{r^2}\left(\frac{q^2\mathcal{A}\mathcal{B}}{r^2}\right)'+9r^3\\ &\times\int\frac{\mathcal{A}}{r^2-mr+4q^2}\left(4\pi r\psi_1^{1(D)}-\frac{3q^2}{2r^3}\right)\mathrm{d}r\\ &-\frac{27r^3}{16}\int\left(\frac{1-\mathcal{A}}{r^3\mathcal{B}}\right)\mathrm{d}r+\frac{81r^3}{8}\int\frac{q^2}{r^5\mathcal{B}}\mathrm{d}r-\frac{9r^3}{2}\\ &\times\int\frac{r^3}{\mathcal{B}^3}\mathrm{d}r-2r^3\int\frac{1}{r\mathcal{B}}\left(\rho+Y_{TF}+\frac{q^2}{2r^4}\right.\\ &\left.-\mathbb{L}_{\zeta\eta}+\psi_0^{0(D)}+\psi_1^1+\psi_2^{2(D)}\right)\mathrm{d}r+3r^3\\ &\times\int\frac{1}{r^4}\eta_2^{2(D)}\mathrm{d}r-3r^3\int\frac{q^2\mathcal{A}\mathcal{B}}{r^6}\mathrm{d}r. \end{aligned} \tag{54}$$

Eq. (54) implies that Y_{TF} describes how structural complexities, energy-density inhomogeneity, and pressure anisotropy affect the Tolman mass within the framework of $f(R,G)$ gravity. Moreover, the expression for Y_{TF} indicates that the interpretation of the Tolman mass in this framework differs from that for a perfect fluid with homogeneous energy density. The Tolman mass can be expressed as

$$\begin{aligned} m_T =& 4\pi\int_0^r\mathcal{A}\mathcal{B}r^2\left(Y_T-\mathbb{H}-\frac{11q^2}{4\pi r^4}\right)\mathrm{d}r\\ &+4\pi\int_0^r\mathcal{A}\mathcal{B}r^2\left(\psi_0^{0(D)}-\psi_1^{1(D)}-2\psi_2^{2(D)}\right)\mathrm{d}r. \end{aligned} \tag{55}$$

Equation (55) shows that the fundamental factors are linked to the dark source terms associated with $f(R,G)$ gravity. It is important to note that the analysis of mass density is significantly influenced by Y_T. Herrera *et al.* [106, 107] and Bhatti *et al.* [108] investigated the influence of Y_T on the Raychaudhuri equation, which is widely used to describe the evolution of scalars. Therefore, Eq. (55) confirms that the Raychaudhuri equation can be described by considering the expression for m_T, specifically in the context of $f(R,G)$ gravity.

VI. MATTER CONFIGURATION WITH VANISHING COMPLEXITY FACTOR

In this section, we consider certain mathematical models to analyze the vanishing complexity constraint. Given the extensive study of complexity in the literature, we relate a structure scalar, obtained through the orthogonal splitting of the Riemann curvature tensor, to the complexity of the matter distribution. In this manuscript, Y_{TF} is considered to be the CF. This structure scalar accounts for energy-density inhomogeneity, pressure anisotropy, and dark source terms associated with $f(R,G)$ gravity. Since the alternative field equations contain five unknown variables, two additional constraints are required to solve the system, and the vanishing complexity condition may be regarded as one such constraint that closes the system

$$\begin{aligned} \Pi =& -\frac{1}{2}\left[-\mathcal{P}_r-2\mathcal{P}_\perp+\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right]\\ &+\frac{1}{9r^3}\int_0^r r^3\left(\rho+\psi_0^{0(D)}\right)'\mathrm{d}r+\frac{11q^2}{16\pi r^4}-\frac{1}{4\pi r^3}\\ &\times\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r-\frac{1}{4\pi r^3\mathcal{B}}-\frac{1}{8\pi r^2\mathcal{B}^2}-\frac{5}{6}\\ &\times\left(\rho+\psi_0^{0(D)}\right)-\frac{\mathbb{L}_{\zeta\eta}}{4\pi}. \end{aligned} \tag{56}$$

It is important to note that vanishing complexity also characterizes homogeneous or inhomogeneous energy density, together with the isotropic and anisotropic properties of the fluid, as specified in Eq. (56). Moreover, this equation may be regarded as the non-local equation of

state corresponding to $f(R,G)$ gravity. This system is analyzed in detail within the framework of the following models.

A. The Gokhroo-Mehra models

A varying inhomogeneous density distribution proposed within the Gokhroo-Mehra framework [96] helps determine the "assembled" and "distributed" nature of matter. It implies that an increase in the radial extent of matter corresponds to a decrease in density, and vice versa. The higher density at the center indicates greater disturbances and, consequently, increased fluid complexity. Additionally, the Gokhroo-Mehra model allows for an anisotropic pressure distribution by decomposing pressure into tangential and radial components. This leads to dynamical and structural irregularities, which produce a complex fluid composition. The consideration of anisotropic pressure together with a significant density distribution makes the Gokhroo-Mehra model particularly suitable for the analysis of complexity. In this case, the energy density is calculated as

$$\rho=\left(1-\frac{\mathcal{K}r^2}{r_\Sigma^2}\right)\rho_0, \tag{57}$$

where $\mathcal{K}$ is a constant taking values in the range $(0,1)$. For an uncharged matter distribution, the value of $\mathcal{K}\in(0,1)$ specifies the linear decrease in energy density, positive energy dispersion, and consistency with the energy conditions, particularly the dominant energy condition. In contrast, $\mathcal{K}>1$ represents negative density at the boundary, which is fundamentally indefinite. However, the emergence of an electric field determines variations in the behavior of the energy density, thereby affecting the gravitational effects of the matter distribution. The positive electromagnetic contribution indicates the presence of a significant electric field, ensuring a positive energy density despite the lower matter distribution at the boundary. Moreover, the development of energy density caused by the presence of electric charge confirms the viability of the dominant energy condition. In the case of a charged fluid, $\mathcal{K}>1$ is conceivable, although the upper limit is not specified. The repulsive electromagnetic energy must be sufficiently strong to cancel the negative matter energy density corresponding to $\mathcal{K}>1$. For $1<\mathcal{K}\leq 1.5$, a substantial electric field is required at the boundary to sustain the positive energy density. In this context, the negative matter density should be interpreted as tangible binding energy exceeding the rest mass held together by strong electromagnetic stresses, and this range often correlates with stable fluid distributions. By contrast, $\mathcal{K}\geq 1.5-2$ refers to exceptionally increased negative matter density. A substantial electric field must be present, resulting in a high charge-to-mass ratio, which is considered untenable for highly dense matter compositions. Furthermore, it should be noted that $\mathcal{K}\gg 2$ is physically implausible, as the electric field cannot overcome such a large negative matter density without violating physical laws. After inserting this result in Eq. (25), we obtain

$$m=\frac{4\pi r^3}{3}\left(1-\frac{3\mathcal{K}r^2}{5r_\Sigma^2}\right)\rho_0+4\pi\int_0^r r^2\psi_0^{0(D)}\mathrm{d}r +\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r+\frac{r}{8}+\frac{1}{\mathcal{B}}. \tag{58}$$

The analysis of Eq. (58), together with Eq. (24), yields

$$\mathcal{B}^{-1}=\sqrt{\frac{q^2}{r^2}-\frac{2}{r\mathcal{B}}-\frac{2}{r}\int\frac{q}{r}\left(q'-\frac{2q}{r}\right)\mathrm{d}r} \times\sqrt{-\frac{8\pi r^2}{3}\left(1-\frac{3\mathcal{K}r^2}{5r_\Sigma^2}\right)\rho_0-\frac{8\pi}{r}\int r^2\psi_0^{0(D)}\mathrm{d}r}. \tag{59}$$

Assuming the 2nd and 3rd field equations yields

$$8\pi\left\{\mathcal{P}_r-\mathcal{P}_\perp+\psi_2^{2(D)}-\psi_1^{1(D)}+\frac{q^2}{4\pi r^4}\right\}=\frac{1}{\mathcal{B}^2} \times\left[\frac{\mathcal{A}''}{\mathcal{A}}-\frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}}-\frac{\mathcal{B}'}{r\mathcal{B}}-\frac{\mathcal{A}'}{r\mathcal{A}}-\frac{1}{r^2}\right]. \tag{60}$$

The following newly recommended factors will play an important role in assessing subsequent findings

$$\mathcal{A}^2=\mathrm{e}^{\int\left(2z-\frac{2}{r}\right)\mathrm{d}r},\quad \mathcal{B}^{-2}=y(r). \tag{61}$$

Eq. (60) takes the following form in terms of the variables considered above:

$$y'+y\left[\frac{2z'}{z}+2z+\frac{4}{r^2z}-\frac{6}{r}\right]=\frac{16\pi}{z}\left[\Pi+\frac{q^2}{4\pi r^4} -\psi_1^{1(D)}+\psi_2^{2(D)}\right]. \tag{62}$$

The aforementioned equation yields the same result as that obtained from the Riccati equation and is derived by considering the function $\mathcal{B}$ associated with the metric constrained by $y(r)$, as expressed in Eq. (59), together with the value of Π given in Eq. (56). Integrating this equation gives a metric expressed as a combination of z and Π within the framework of $f(R,G)$ theory [109]

$$\mathrm{d}s^2=\mathrm{e}^{2\int\left(z-\frac{1}{r}\right)\mathrm{d}r}\mathrm{d}t^2-r^2\mathrm{d}\theta^2-r^2\alpha^2\mathrm{d}z^2 -\frac{\frac{z^2}{r^6}\mathrm{e}^{2\int\left(z+\frac{2}{r^2z}\right)\mathrm{d}r}}{4\pi\int\frac{z}{r^6}\left[\Pi+\frac{q^2}{4\pi r^4}+\psi_2^{2(D)}-\psi_1^{1(D)}\right]} \times\frac{1}{\mathrm{e}^{2\int\left(z+\frac{2}{r^2z}\right)\mathrm{d}r}\mathrm{d}r+\mathcal{G}}, \tag{63}$$

Here, $\mathcal{G}$ is written as a constant of integration. Furthermore, the relevant parameters take the following form after the inclusion of the newly specified variables

$$\frac{m'}{r^2}=4\pi\left[\rho+\frac{1}{32\pi r^2}-\frac{\mathcal{B}'}{4\pi r^2\mathcal{B}^2}+\psi_0^{0(D)}+\frac{qq'}{4\pi r^3}-\frac{q}{2\pi r^3}\right], \tag{64}$$

$$4\pi\mathcal{P}_r=4\pi\psi_1^{1(D)}-\left[\frac{z\left(r^2-4mr\right)+\left(4m-\frac{r}{2}\right)}{r^3}\right]-\frac{3q^2}{8\pi r^4}, \tag{65}$$

$$8\pi\mathcal{P}_\perp=8\pi\psi_2^{2(D)}-\frac{q^2}{r^4}+\left(\frac{z}{r}-\frac{1}{4r}\right)m'+\left(\frac{2m}{r}-\frac{1}{4}\right)z'+\left(\frac{2m}{r}-\frac{1}{4}\right)z^2+\left(\frac{1}{4r}-\frac{3m}{r^2}\right)z-\frac{r^2}{4}+\frac{2m}{r^3}+\frac{m}{4r^2}. \tag{66}$$

Specific approaches for interpreting the gravitational structure are consistent with positive energy density under the constraint $\rho>P_r,P_\perp$. These approaches can effectively illustrate the complex dynamics of cylindrical matter distributions. Furthermore, Di Prisco [110] confirmed these findings within the framework of GR.

B. Polytropic equation of state

The polytropic EOS is widely used as an effective description of compact-star matter when the exact microscopic nuclear EOS is not uniquely specified. In the present model, it provides a relation between pressure and density and enables us to examine how the complexity factor constrains the internal behavior of dense matter under the combined influence of anisotropy, electric charge, and $f(R,G)$ curvature corrections. This treatment strengthens the astrophysical interpretation of the model by linking the geometric complexity of the configuration to dense stellar interiors.

The EOS plays a crucial role in evaluating the matter composition of compact objects. The hydrostatic equilibrium equation can be readily derived by considering the anisotropic form of the EOS. The polytropic EOS specifies the relationship between matter density and pressure. The consideration of the mass function, hydrostatic equilibrium, and the polytropic EOS yields the Lane-Emden equation in dimensionless form. The polytropic index governs the internal behavior of pressure in response to density fluctuations, taking into account the composition and stability of the fluid. Larger values of the polytropic index, corresponding to more compact mass distributions, imply a more complex, highly condensed, and destabilized matter configuration. Different values of the polytropic index may be considered to describe distinct forms of complexity. The polytropic EOS is significant for assessing the complexity of matter by describing the fundamental configuration of the system. It indicates that stability and the interior mass distribution are directly related to the EOS parameters. Moreover, it serves as a benchmark for comparing structures of increasing complexity.

In this manuscript, we consider a polytropic EOS in combination with the preceding CF constraint. We now consider two different polytropic contexts [111−113]. We commence from the following

$$\mathcal{P}_r=K\rho^\chi=K\rho^{1+\frac{1}{n}},\quad \chi=1+\frac{1}{n}. \tag{67}$$

The emergence of dark source terms associated with alternative theories of gravity in the TOV equation implies a discrepancy between the measured macroscopic EOS and the microphysical EOS. In $f(R)$ gravity, curvature terms correspond to increased stiffness, allowing larger values of the polytropic index while maintaining the stability of the matter composition. In $f(R,G)$ gravity, the relationship between the Ricci scalar and the Gauss-Bonnet invariant yields a modified framework of hydrostatic equilibrium. This framework consequently reduces the effects of the gravitational force, allowing a specific variation in pressure that could support a higher mass and thus suggests larger values of n. The framework of $f(R,G)$ gravity implies that negative pressure can serve as a significant substitute for stiff-EOS pressure. Therefore, the consideration of alternative theories of gravity results in larger values of the polytropic index while retaining compatibility, because the related dark source terms correspond to increased stiffness and reduced gravitational force. Since the solution of a non-dimensional system of equations can be derived easily, we consider certain variables to determine the non-dimensional representation of the mass function and the TOV equation

$$\omega=\frac{\mathcal{P}_{rc}}{\rho_c},\quad r=\frac{\xi}{C},\quad C^2=\frac{4\pi\rho_c}{\omega(n+1)}, \tag{68}$$

$$\vartheta(\xi)=\frac{C^3m(r)}{4\pi\rho_c},\quad \chi^n=\frac{\rho}{\rho_c}. \tag{69}$$

Here, the subscript c denotes evaluation at the center. We impose $\psi(\xi_\Sigma)=0$ at the boundary $r=r_\Sigma(\xi=\xi_\Sigma)$. In terms of dimensionless variables, the TOV equation becomes

$$\rho_c\chi^n\frac{\mathrm{d}\chi}{\mathrm{d}\xi}=\frac{\xi\omega}{\mathcal{P}_{rc}}\left[2\psi^{22(D)}-\left\{\chi^n\rho_c(1+\omega\chi)+\mathcal{B}^2\times\psi^{11(D)}\right\}\left\{\frac{16\pi}{\xi-\vartheta\xi\omega(n+1)+\frac{16\pi q^2\mathcal{P}_{rc}}{\omega^2(n+1)}}\times\left(\frac{\mathcal{P}_{rc}}{\omega^2(n+1)\xi}\left(\frac{\xi-32\pi\vartheta(n+1)\omega}{4\xi}\right)-\frac{4\pi\xi\omega^2(n+1)}{\mathcal{P}_{rc}}\left(\psi^{11(D)}-\omega\rho\chi\right)-\frac{3q^2\rho_c^2}{2\xi^3\omega^2(n+1)^2}\right)\right\}\right]+\frac{2\mathcal{B}'}{\mathcal{B}}\mathcal{K}\rho_c\chi^{n+1}-\frac{\chi\chi'\psi^{00(D)}}{\rho_c}-\frac{\mathcal{B}^2\omega}{\mathcal{P}_{rc}}\left(1+\frac{2\mathcal{P}_{rc}}{\xi\omega^2(n+1)}\right)\times\psi^{11(D)}-\frac{3\mathcal{B}^2}{8\pi\rho_c}\left(\frac{q^2\mathcal{P}_{rc}^2}{\omega^4(n+1)^2\xi^4\mathcal{B}^2}\right)'-2\omega\frac{\mathcal{B}\mathcal{B}'\psi^{11(D)}}{\mathcal{P}_{rc}}-\frac{3q^2\mathcal{B}'\mathcal{P}_{rc}}{4\pi\omega^3(n+1)^2\xi^4\mathcal{B}}-\frac{2\Pi}{\xi\omega}\frac{1}{n+1}-\frac{q^2\rho_c^2}{2\pi\xi^5\omega^3(n+1)^3}. \tag{70}$$

In terms of dimensionless variables, Eq. (25) can be written as

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}=4\pi\xi^2\left(\chi^n+\frac{\psi_0^{0(D)}}{\rho_c}\right)+\frac{1}{8\omega(n+1)}-\frac{\mathcal{B}'}{\omega\mathcal{B}^2}\times\frac{1}{n+1}-\frac{qq'\rho_c}{\xi\omega^2(n+1)^2}-\frac{2q^2\rho_c}{\omega^2(n+1)^2}. \tag{71}$$

The inclusion of Π, ϑ, and χ as undetermined functions in both aforementioned equations indicates that additional conditions are required to determine the model's specific solutions. Therefore, we consider the diminishing CF condition together with the non-dimensional variables

$$\frac{\xi}{n\rho_c}\frac{\mathrm{d}\Pi}{\mathrm{d}\xi}+\frac{3\Pi}{n\rho_c}=\frac{11\xi}{n\rho_c}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{q^2}{r^4}\right)-\frac{\xi}{4\pi n\rho_c}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbb{L}_{\zeta\eta}-\frac{\xi}{4\pi n\rho_c}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{r^3\mathcal{B}}\right)-\frac{\xi}{8\pi n\rho_c}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{r^2\mathcal{B}^2}\right)-\frac{q}{4\pi r^3}\times\frac{q'}{n\rho_c}+\frac{41q^2}{16\pi r^4n\rho_c}-\frac{13\xi}{18n\rho_c}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\rho+\psi_0^{0(D)}\right)-\frac{\xi}{2n\rho_c}\times\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right)-\frac{\mathbb{L}_{\zeta\eta}}{4\pi n\rho_c}-\frac{3}{4\pi r^3n\rho_c\mathcal{B}}-\frac{5}{2n\rho_c}\left(\rho+\psi_0^{0(D)}\right)-\frac{3}{8\pi r^2n\rho_c\mathcal{B}^2}-\frac{3}{2n\rho_c}\left(\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right). \tag{72}$$

Consequently, we must address three differential equations that incorporate Π, ϑ, and χ as unknown functions. Under appropriate basic constraints, analytical solutions for these equations can be derived; alternatively, numerical integration can be performed by specifying values for n and ω. Furthermore, these equations can explain significant phenomena in our cosmos within the context of vanishing complexity constraints. For specific parametric values, these results may describe the density, mass, and pressure associated with compact structures. Additionally, the second polytropic EOS can also be considered

$$\mathcal{P}_r=K\rho_b^{\chi}=K\rho_b^{1+\frac{1}{n}}. \tag{73}$$

Here, ρ_b denotes the baryonic mass density. Considering this equation together with Eqs. (70) and (72) leads to

$$\rho_{bc}\chi_b^n\frac{\mathrm{d}\chi_b}{\mathrm{d}\xi}=\frac{\xi\omega}{\mathcal{P}_{rc}}\left[2\psi^{22(D)}-\left\{\chi_b^n\rho_{bc}(1+\omega\chi_b)+\mathcal{B}^2\psi^{11(D)}\right\}\left\{\frac{16\pi}{\xi-\vartheta\xi\omega(n+1)+\frac{16\pi q^2\mathcal{P}_{rc}}{\omega^2(n+1)}}\times\left(\frac{\mathcal{P}_{rc}}{\omega^2(n+1)\xi}\left(\frac{\xi-32\pi\vartheta(n+1)\omega}{4\xi}\right)-\frac{4\pi\xi\omega^2(n+1)}{\mathcal{P}_{rc}}\times\left(\psi^{11(D)}-\omega\rho_b\chi_b\right)-\frac{3q^2\rho_{bc}^2}{2\xi^3\omega^2(n+1)^2}\right)\right\}\right]+\frac{2\mathcal{B}'}{\mathcal{B}}\mathcal{K}\rho_{bc}\times\chi_b^{n+1}-\frac{\chi_b\chi_b'\psi^{00(D)}}{\rho_{bc}}-\frac{\mathcal{B}^2\omega}{\mathcal{P}_{rc}}\left(1+\frac{2\mathcal{P}_{rc}}{\xi\omega^2(n+1)}\right)\psi^{11(D)}-\frac{3\mathcal{B}^2}{8\pi\rho_{bc}}\left(\frac{q^2\mathcal{P}_{rc}^2}{\omega^4(n+1)^2\xi^4\mathcal{B}^2}\right)'-\frac{2\omega\mathcal{B}\mathcal{B}'\psi^{11(D)}}{\mathcal{P}_{rc}}-\frac{3q^2}{4\pi}\times\frac{\mathcal{B}'\mathcal{P}_{rc}}{\omega^3(n+1)^2\xi^4\mathcal{B}}-\frac{2\Pi}{\xi\omega(n+1)}-\frac{q^2\rho_{bc}^2}{2\pi\xi^5\omega^3(n+1)^3}, \tag{74}$$

and

$$\frac{\xi}{n\rho_{bc}}\frac{\mathrm{d}\Pi}{\mathrm{d}\xi}+\frac{3\Pi}{n\rho_{bc}}=\frac{11\xi}{n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{q^2}{r^4}\right)-\frac{\xi}{4\pi n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbb{L}_{\zeta\eta}-\frac{\xi}{4\pi n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{r^3\mathcal{B}}\right)-\frac{\xi}{8\pi n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{r^2\mathcal{B}^2}\right)-\frac{q}{4\pi}\times\frac{q'}{r^3n\rho_{bc}}+\frac{41q^2}{16\pi r^4n\rho_{bc}}-\frac{13\xi}{18n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\rho_{bc}+\psi_0^{0(D)}\right)-\frac{\xi}{2n\rho_{bc}}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right)-\frac{\mathbb{L}_{\zeta\eta}}{4\pi n\rho_{bc}}-\frac{3}{4\pi r^3n\rho_{bc}\mathcal{B}}-\frac{5}{2n\rho_{bc}}\left(\rho+\psi_0^{0(D)}\right)-\frac{3}{8\pi r^2n\rho_{bc}\mathcal{B}^2}-\frac{3}{2n\rho_{bc}}\left(\psi_1^{1(D)}+\psi_2^{2(D)}+\psi_3^{3(D)}\right), \tag{75}$$

where $\psi_b^n=\dfrac{\rho_b}{\rho_{bc}}$. Consequently, the emergence of cosmic

structures can be assessed by using the solutions of Eqs. (71), (74), and (75) for a specified EOS, together with the diminishing-complexity constraint. Equation (67) describes distinct eras of the expanding cosmos for different values of the polytropic constant K through the relation between radial pressure and energy density. Specifically, it identifies the matter-, radiation-, and stiff-fluid-dominated epochs corresponding to $K=0$, $\frac{1}{3}$, and 1, respectively. Values of $K\in(\frac{-1}{3},-1)$ characterize the quintessence era, and $K<-1$, is contemplated for the description of phantom phase. Furthermore, the different values of the polytropic index might be assumed for the interpretation of compact structures, i.e., the values having the range from $n=0.5$ to $n=1$ are appropriate for the description of neutron stars.

VII. ASTROPHYSICAL IMPLICATIONS OF OUR WORK:

The evaluation of complexity in gravitational frameworks is highly relevant to the comprehensive description of the internal configuration and assessment of various compact objects. In this manuscript, the orthogonal splitting of the Riemann curvature tensor is employed to analyze the implications of complexity in the framework of $f(R,G)$ gravity. The astrophysical significance of our results is addressed as follows:

● **Analysis of high-density matter distributions:** The decomposition of the Riemann curvature tensor into its fundamental components plays an important role in assessing deviations from uniform distributions in self-gravitating matter configurations. Our results demonstrate the importance of anisotropic pressure and inhomogeneous energy density in determining the complexity of astrophysical systems. The consideration of anisotropic pressure is highly relevant for characterizing the fundamental features of extremely dense gravitational fields associated with compact objects, including black holes and neutron stars.

● **Revised EOS constraints:** The inclusion of the complexity factor Y_{TF} in the evaluation of complexity provides an opportunity to develop a more specific EOS framework for assessing the dynamical characteristics of compact structures. Our proposed methodology imposes specific mathematical constraints on the EOS framework, as astrophysical observations suggest diverse anisotropies in the configurations of compact structures. In this manuscript, different values of the polytropic index can be used to describe various physical characteristics of compact structures; i.e., values between $n=0.5$ and $n=1$ correspond to the structural analysis of neutron stars. Moreover, the considered mathematical framework can contribute to determining the mass-radius relationship and the stability of extremely dense matter configurations.

● **Implications of modified gravity theory on compact structures:** The presence of dark source terms associated with $f(R,G)$ gravity in the mathematical framework significantly modifies the description of mass–pressure relationships in anisotropic matter distributions. The derived relation between the Tolman mass and the conformal tensor demonstrates the importance of correction terms in evaluating the structural configuration of celestial structures.

● **Association of the obtained results with cosmic findings:** The evaluation of complexity associated with the internal configuration of compact structures plays a crucial role in astronomy and cosmology. Gravitational-wave observations of binary neutron star interactions (e.g., GW170817) can be useful for studying the internal composition of neutron stars, while our findings illustrate the significance of the complexity factor in describing tidal distortion and the characteristics of gravitational waves. In addition, gravitational-wave findings combined with electric field analysis may substantiate our theoretical assumptions in the next few years.

● **Mechanism of gravitational collapse and development of celestial structures:** Complexity analysis is important for assessing the emergence and stability of compact objects, which ultimately determine their final fate. Our findings indicate that the development of complexity in matter configurations causes physical instabilities that could lead to gravitational collapse. This insight plays a crucial role in describing the evolution of a neutron star into a black hole and the emergence of different compact objects within different gravitational frameworks.

The practical implications include astronomical modelling, the propagation of gravitational waves, and interpretations of the emergence of celestial formations. Considering these results in empirical contexts can provide a comprehensive description of compact configurations in GR and alternative gravity theories.

VIII. SUMMARY

The analysis of self-gravitating fluid configurations has significantly influenced the study of astronomical objects and motivated researchers to evaluate fundamental features, such as luminosity, the mass–radius relationship, and system stability. These studies have revealed the complex dynamical behavior of such systems. Our manuscript focuses mainly on characterizing the CF cor-

responding to a cylindrically static configuration with anisotropic fluid within the context of $f(R,G)$ theory. This modified theory of gravity has attracted considerable interest from the scientific community because of its effective description of the accelerating evolution of the universe. We first reviewed the basic formulation of the modified theory and included several additional parameters. The modified field equations and conservation law were then determined in the presence of an electric field and an anisotropic fluid configuration. The techniques recommended by C-Energy and Tolman were considered for deriving expressions for m and m_T; these masses were then used to establish the specific relation between important variables and the conformal tensor. Irregular energy density, anisotropic pressure, and dark source terms associated with $f(R,G)$ gravity are correlated with the effects of C-Energy, the conformal tensor, and gravitational mass on the cylindrically symmetric matter configuration.

We evaluated the complexity of the system by considering a scalar function together with the relevance of comoving congruence. The orthogonal splitting of the Riemann curvature tensor yielded the derivation of structural scalars, and their significance in the evolution and stability of the homogeneous fluid configuration was reviewed. Moreover, a diminishing complexity constraint was considered to illustrate the linkage between energy density and pressure. Non-homogeneous energy density and anisotropic fluid pressure are important in estimating the complexity of the model. In fundamental gravitational systems, isotropic pressure and energy density are uniformly distributed. Consequently, these frameworks indicate the absence of complexity. We considered the orthogonal splitting of the Riemann curvature tensor to derive the dynamical features of the system recommended by Herrera *et al.* [31]. The decomposition of the electric part of the curvature tensor results in the dynamical parameter (Y_{TF}), which is recognized as CF in GR.

In our case study, five unknowns are introduced into the modified equations of motion, one of which was developed after considering the diminishing complexity constraint. The complexity of the model is reduced within the framework of the Gokhroo and Mehra models, as well as the polytropic EOS. We began by applying Gokhroo and Mehra's approach and then studied the polytropic EOS to establish a system of equations from which the TOV equation, the decreasing-complexity constraint, and the expression for mass, including the non-dimensional variables, were determined. Under specific restrictions, these equations describe the cylindrically symmetric configuration more effectively.

We summarize our manuscript with the following remarks:

- Anisotropic matter composition, together with inhomogeneous energy density, plays a crucial role in analyzing the complexity of any system. The orthogonal decomposition of the Riemann tensor yields the specific structural variables proposed by Herrera [31]. We evaluated the characteristics of four structure scalars.

- The component Y_{TF} incorporates the effects of spatial anisotropy, inhomogeneous energy density, and correction terms associated with $f(R,G)$ gravity on the overall energy distribution of the fluid. After the inclusion of these correction terms, the expression for Y_{TF} is assumed to indicate an increase in the complexity of the system.

- We assessed the vanishing complexity constraint associated with the specific models. The formalism proposed by the "Gokhroo and Mehra model" describes the primary variables in $f(R,G)$ gravity. In addition, the dimensionless forms of the TOV equation, vanishing CF, and mass are determined after considering the polytropic EOS. It is worth noting that the CF is eliminated under the assumptions of isotropic pressure and uniform density in GR [5]. In contrast to these constraints, our research is compatible with the vanishing of complexity, which illustrates the significance of the correction terms introduced by $f(R,G)$ gravity. We briefly addressed the electric and magnetic components of the Weyl tensor. In addition, we evaluated two distinct implications for stellar compositions after incorporating alternative corrections and the diminishing complexity factor.

The findings of this study may be substantially relevant to the analysis of the fundamental properties and evolution of cosmic voids [114, 115]. Cosmic voids are regions of very low density within the large-scale distribution of matter. In our cosmos, they are regarded as fundamental components of cosmic structure. These outcomes may help refine interpretations of important celestial events, such as supernova explosions. Building on this work, distinct and significant models should be developed to enable a comprehensive evaluation of the association between the $f(R,G)$ framework and cosmic expansion. From an astrophysical perspective, the present results show that the complexity factor Y_{TF} may serve as an effective diagnostic for the internal organization of dense compact stellar matter. This scalar incorporates the effects of density inhomogeneity, local pressure anisotropy, electric charge, and modified Gauss-Bonnet corrections, all of which can influence the equilibrium and stability of neutron-star-like configurations. The incorporation of polytropic equations of state further supports the relevance of the present framework for modeling compact stellar interiors, where the exact microscopic equation of state remains uncertain. Thus, the proposed

formalism provides a useful geometric route for analyzing dense astrophysical systems within the scope of nuclear astrophysics.

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APPENDIX

The terms used in Eqs. (17)–(19) have the following mathematical forms

$$\begin{aligned}\psi_0^{0(D)} =& \frac{32\mathcal{A}'\mathcal{B}'}{r\mathcal{A}\mathcal{B}^5}f'_G + \frac{8\mathcal{A}'^2}{r\mathcal{A}^2\mathcal{B}^4}f'_G + \frac{4\mathcal{A}'}{r^2\mathcal{A}\mathcal{B}^4}f'_G + \frac{1}{\mathcal{B}^2}\times f''_R - \frac{\mathcal{B}'}{\mathcal{B}^3}f'_R + \frac{2}{r\mathcal{B}^2}f'_R + \frac{4\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^5}f''_G + \frac{16\mathcal{B}'}{r\mathcal{B}^5}\\ &\times f''_G - \frac{16\mathcal{A}'}{r\mathcal{A}\mathcal{B}^4}f''_G + \frac{2}{r\mathcal{B}^2}f'_R - \frac{4}{r^2\mathcal{B}^4}f''_G + \frac{12}{\mathcal{A}}\times\frac{\mathcal{B}'\mathcal{A}''}{\mathcal{B}^5}f'_G - \frac{16\mathcal{B}'^2}{r\mathcal{B}^6}f'_G + \frac{12\mathcal{B}'}{r^2\mathcal{B}^5}f'_G - \frac{16\mathcal{A}''}{r\mathcal{A}\mathcal{B}^4}\\ &\times f'_G + \frac{16\mathcal{B}'}{r^2\mathcal{B}^5}f'_G - \frac{16\mathcal{X}'}{r\mathcal{A}\mathcal{B}^4}f'_G - \frac{16}{r^3\mathcal{B}^4}f'_G + \frac{4}{\mathcal{B}^5}\times\mathcal{A}'\mathcal{B}'f''_G - \frac{4\mathcal{A}'}{r^2\mathcal{A}^2\mathcal{B}^3}f'_G - \frac{4\mathcal{A}''}{\mathcal{A}}\frac{\mathcal{B}'}{\mathcal{B}^5}f'_G + \frac{2\mathcal{A}''}{\mathcal{A}\mathcal{B}^2}\\ &- \frac{2\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3} + \frac{4\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2} - \frac{2\mathcal{B}'}{r\mathcal{B}^3} + \frac{1}{r^2\mathcal{B}^2} - \frac{\mathcal{A}''}{\mathcal{A}}\frac{f_R}{\mathcal{B}^2} + \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3}f_R - \frac{2\mathcal{A}'}{r\mathcal{A}}\frac{1}{\mathcal{B}^2}f_R - \frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^2}f_R + \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3}\\ &\times f_R + \frac{2\mathcal{B}'}{r\mathcal{B}^3}f_R - \frac{2}{r\mathcal{A}}\frac{\mathcal{A}'}{\mathcal{B}^2}f_R - \frac{2}{r^2\mathcal{B}^2}f_R - \frac{1}{2}f_G\times G + \frac{1}{2}f + \frac{\mathcal{A}''}{\mathcal{A}}\frac{1}{\mathcal{B}^2}f_R - \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3}f_R - \frac{2\mathcal{B}'}{r\mathcal{B}^3}f_R + \frac{2\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2}f_R + \frac{1}{r^2\mathcal{B}^2}f_R.\end{aligned}$$

$$\begin{aligned}\psi_1^{1(D)} =& \frac{4\mathcal{A}''}{\mathcal{A}\mathcal{B}^4}f''_G - \frac{8\mathcal{B}'}{r\mathcal{B}^5}f''_G - \frac{2}{r^2\mathcal{B}^4}f''_G - \frac{16\mathcal{B}'^2}{r\mathcal{B}^6}f'_G + \frac{16\mathcal{A}'\mathcal{B}'}{r\mathcal{A}\mathcal{B}^5}f'_G - \frac{2\mathcal{B}''}{r^2\mathcal{B}^5}f'_G + \frac{\mathcal{A}'}{\mathcal{A}}\frac{1}{\mathcal{B}^2}f'_R + \frac{2}{r\mathcal{B}^2}f'_R\\ &- \frac{8\mathcal{A}'\mathcal{A}''}{\mathcal{A}^2\mathcal{B}^4}f'_G + \frac{4\mathcal{A}'^2\mathcal{B}'}{\mathcal{A}^2\mathcal{B}^5}f'_G - \frac{16\mathcal{A}'^2}{r\mathcal{A}^2\mathcal{B}^4}f'_G + \frac{32\mathcal{B}'}{r^2\mathcal{B}^5}\times f'_G - \frac{8}{r}\frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^4}f'_G - \frac{16}{r\mathcal{A}}\frac{\mathcal{A}'}{\mathcal{B}^4}f'_G - \frac{16}{r^3\mathcal{B}^4}f'_G - \frac{4}{\mathcal{B}^2}\\ &\times\frac{\mathcal{A}'\mathcal{B}'}{\mathcal{B}^3}f''_G + \frac{16\mathcal{B}'^2}{r\mathcal{B}^6}f'_G - \frac{4}{\mathcal{A}}\frac{\mathcal{A}''}{\mathcal{B}^4}f''_G - \frac{8\mathcal{A}''}{r\mathcal{A}\mathcal{B}^4}f'_G + \frac{4\mathcal{A}'^2\mathcal{B}'}{\mathcal{A}\mathcal{B}^5}f'_G + \frac{4\mathcal{A}'\mathcal{B}'}{\mathcal{B}^5}f''_G + \frac{8\mathcal{A}'\mathcal{B}'}{r\mathcal{B}^5}f'_G - \frac{4\mathcal{A}'}{r^2\mathcal{A}}\\ &\times\frac{f'_G}{\mathcal{B}^4} + \frac{4\mathcal{B}'}{r^2\mathcal{B}^5}f'_G - \frac{4\mathcal{A}'}{r^2\mathcal{A}^2\mathcal{B}^3}f'_G + \frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^2} - \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{B}^3} - \frac{2\mathcal{B}'}{r\mathcal{B}^3} + \frac{2\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2} + \frac{1}{r^2\mathcal{B}^2} - \frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^2}f_R + \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3}f_R\\ &+ \frac{2\mathcal{B}'}{r\mathcal{B}^3}f_R - \frac{2\mathcal{A}'}{r\mathcal{A}}\frac{f_R}{\mathcal{B}^2} - \frac{1}{r^2\mathcal{B}^2}f_R - \frac{1}{2}f_G G + \frac{1}{2}f + \frac{\mathcal{A}''}{\mathcal{A}\mathcal{B}^2}f_R - \frac{\mathcal{A}'}{\mathcal{A}}\frac{\mathcal{B}'}{\mathcal{B}^3}f_R - \frac{2\mathcal{B}'}{r\mathcal{B}^3}f_R + \frac{2\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2}f_R.\end{aligned}$$

$$\begin{aligned}\psi_2^{2(D)} =& \frac{1}{r\mathcal{B}^2}f'_R + \frac{4\mathcal{A}''}{r\mathcal{A}\mathcal{B}^4}f'_G + \frac{4\mathcal{A}'\mathcal{B}'}{r\mathcal{A}\mathcal{B}^5}f'_G - \frac{8\mathcal{B}'}{r^2\mathcal{B}^5}f'_G + \frac{2\mathcal{A}'}{r^2\mathcal{A}\mathcal{B}^4}f'_G - \frac{8}{r^3\mathcal{B}^4}f'_G + \frac{\mathcal{A}'}{\mathcal{A}\mathcal{B}^2}f'_R + \frac{1}{\mathcal{B}^2}f''_R - \frac{\mathcal{B}'}{\mathcal{B}^3}f'_R - \frac{2\mathcal{A}'\mathcal{A}''}{\mathcal{A}^2\mathcal{B}^4}f'_G\\ &+ \frac{2\mathcal{A}'^2\mathcal{B}'}{\mathcal{A}^2\mathcal{B}^5}f'_G - \frac{6\mathcal{A}'^2}{r\mathcal{A}^2}\frac{f'_G}{\mathcal{B}^4} + \frac{2\mathcal{A}''}{\mathcal{A}\mathcal{B}^4}f''_G + \frac{2}{\mathcal{B}^2}\frac{2\mathcal{A}'\mathcal{B}'}{\mathcal{B}^3}f''_G + \frac{10\mathcal{B}'}{r\mathcal{B}^5}f''_G + \frac{2\mathcal{B}'\mathcal{A}''}{\mathcal{A}\mathcal{B}^5}\times f'_G - \frac{2\mathcal{A}'\mathcal{B}'^2}{\mathcal{B}^6}f'_G - \frac{4}{r}\frac{\mathcal{B}'^2}{\mathcal{B}^6}f'_G\\ &- \frac{2\mathcal{A}'}{r^2\mathcal{A}^2\mathcal{B}^3}f'_G + \frac{12\mathcal{B}'}{r^2\mathcal{B}^5}f'_G - \frac{4\mathcal{A}'}{r\mathcal{A}}\frac{1}{\mathcal{B}^4}f''_G - \frac{4}{r^2}\frac{f'_G}{\mathcal{B}^4} - \frac{4\mathcal{A}''}{r\mathcal{B}^4}f''_G - \frac{4\mathcal{B}'}{r^2\mathcal{B}^5}f''_G - \frac{1}{2}f_G G + \frac{1}{2}f + \frac{2}{\mathcal{A}}\frac{\mathcal{A}'\mathcal{B}'}{\mathcal{B}^3}f_R + \frac{B'}{r\mathcal{B}^3}f_R\\ &- \frac{\mathcal{A}'}{r}\frac{f_R}{\mathcal{A}\mathcal{B}^2} - \frac{3\mathcal{B}'}{r\mathcal{B}^3} + \frac{3\mathcal{A}'}{r\mathcal{A}\mathcal{B}^2} + \frac{2}{r^2\mathcal{B}^2} - \frac{1}{2}f_G G + \frac{1}{2}f + \frac{\mathcal{A}''}{\mathcal{A}}\frac{1}{\mathcal{B}^2}f_R - \frac{\mathcal{A}'\mathcal{B}'}{\mathcal{A}\mathcal{B}^3}f_R - \frac{2}{r}\frac{\mathcal{B}'}{\mathcal{B}^3}f_R + \frac{2}{r}\frac{\mathcal{A}'f_R}{\mathcal{A}\mathcal{B}^2} + \frac{1}{r^2\mathcal{B}^2}f_R.\end{aligned}$$

The mathematical expressions for the terms appearing in Eqs. (44)–(50) take the following form

$$\begin{aligned}\mathcal{S}^{(A)}_{\zeta\eta} =& \left[\frac{1}{4}R\nabla^\nu\nabla_\varrho f_G - \frac{1}{2}R^\lambda_\varrho\nabla_\lambda\nabla^\nu f_G + \frac{1}{2}R^\nu_\varrho\Box f_G + \frac{1}{2}R^\nu_{\alpha\epsilon\varrho}\nabla^\alpha\nabla^\epsilon f_G + \frac{1}{8}R^\nu_\varrho - \frac{1}{8}f_R R^\nu_\varrho - \frac{1}{2}R^\nu_\lambda\nabla^\lambda\right.\\ &\left.\times\nabla_\varrho f_G + \frac{1}{8}\nabla^\nu\nabla_\varrho f_R\right]\epsilon^{\varrho\beta}_\zeta\epsilon_{\nu\beta\eta} + \left[\frac{-1}{8}\nabla^\nu\nabla_\beta f_R - \frac{1}{4}R\nabla^\nu\nabla_\beta f_G - \frac{1}{2}R^\lambda_\beta\nabla_\lambda\nabla^\nu f_G - \frac{1}{2}R^\nu_\beta\Box f_G\right.\\ &\left.- \frac{1}{2}R^\nu_{\alpha\epsilon\beta}\nabla^\alpha\nabla^\epsilon f_G - \frac{1}{8}R^\nu_\beta + \frac{1}{8}R^\nu_\beta f_R - \frac{1}{2}R^\nu_\lambda\nabla_\beta\times\nabla^\lambda f_G\right]\epsilon^{\varrho\beta}_\zeta\epsilon_{\nu\varrho\eta} + \left[-\frac{1}{8}\nabla^\mu\nabla_\varrho f_R - \frac{1}{4}R\nabla^\mu\nabla_\varrho f_G\right.\end{aligned}$$

$$+\frac{1}{2}R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\mu}f_G-\frac{1}{2}R^{\mu}_{\varrho}\Box f_G-\frac{1}{2}R^{\mu}_{\alpha\epsilon\varrho}\nabla^{\alpha}\nabla^{\epsilon}f_G\ +R^{\mu}_{\lambda}\nabla^{\lambda}\nabla_{\varrho}f_G\Big]\epsilon^{\varrho\beta}_{\zeta}\epsilon_{\beta\mu\eta}+\Big[\frac{1}{8}\nabla^{\mu}\nabla_{\beta}f_R+\frac{1}{4}\nabla^{\mu}$$
$$\times\nabla_{\beta}f_G-\frac{1}{2}R^{\lambda}_{\beta}\nabla^{\mu}\nabla_{\lambda}f_G+\frac{1}{2}R^{\mu}_{\beta}\Box f_G+\frac{1}{2}R^{\mu}_{\alpha\epsilon\beta}\ \times\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{2}R^{\mu}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G\Big]\epsilon^{\varrho\beta}_{\zeta}\epsilon_{\varrho\mu\eta}-\Big[\frac{1}{4}R\nabla^{\nu}$$
$$\times\nabla_{\varrho}f_G-\frac{1}{2}R^{\lambda}_{\varrho}\nabla^{\nu}\nabla_{\lambda}f_G+\frac{1}{2}R^{\nu}_{\varrho}\Box f_G+\frac{1}{2}R^{\nu}_{\alpha\epsilon\varrho}\ \times\nabla^{\alpha}\nabla^{\epsilon}f_G+\frac{1}{8}R^{\nu}_{\varrho}-\frac{1}{8}f_RR^{\nu}_{\varrho}-\frac{1}{2}R^{\nu}_{\lambda}\nabla^{\lambda}\nabla_{\varrho}f_G$$
$$+\frac{1}{8}\nabla^{\nu}\nabla_{\varrho}f_R\Big]\frac{1}{3}h_{\zeta\eta}\epsilon^{\varrho\beta b}\epsilon_{\nu\beta b}-\Big[\frac{1}{8}\nabla^{\nu}\nabla_{\beta}f_R-\ \frac{1}{4}R\nabla^{\nu}\nabla_{\beta}f_G-\frac{1}{2}R^{\lambda}_{\beta}\nabla_{\lambda}\nabla^{\nu}f_G-\frac{1}{2}R^{\nu}_{\beta}\Box f_G-\frac{1}{2}$$
$$\times R^{\nu}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{8}R^{\nu}_{\beta}+\frac{1}{8}R^{\nu}_{\beta}f_R-\frac{1}{2}R^{\nu}_{\lambda}\nabla_{\beta}\nabla^{\lambda}\ \times f_G\Big]\frac{1}{3}h_{\zeta\eta}\epsilon^{\varrho\beta b}\epsilon_{\nu\varrho b}-\Big[-\frac{1}{8}\nabla_{\varrho}\nabla^{\mu}f_R-\frac{1}{4}R\nabla_{\varrho}\nabla^{\mu}$$
$$\times f_G+\frac{1}{2}R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\mu}f_G-\frac{1}{2}R^{\mu}_{\varrho}\Box f_G-\frac{1}{2}R^{\mu}_{\alpha\epsilon\varrho}\nabla^{\alpha}\ \times\nabla^{\epsilon}f_G+\frac{1}{2}R^{\mu}_{\lambda}\nabla_{\varrho}\nabla^{\lambda}f_G\Big]\frac{1}{3}h_{\zeta\eta}\epsilon^{\varrho\beta a}\epsilon_{\beta\mu a}$$
$$-\Big[\frac{1}{8}\nabla_{\beta}\nabla^{\mu}f_R+\frac{1}{4}R\nabla_{\beta}\nabla^{\mu}f_G-\frac{1}{2}R^{\lambda}_{\beta}\nabla_{\lambda}\nabla^{\mu}f_G\ +\frac{1}{2}R^{\mu}_{\beta}\Box f_G+\frac{1}{2}R^{\mu}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{2}R^{\mu}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G\Big]\times\frac{1}{3}h_{\zeta\eta}\epsilon^{\varrho\beta a}\epsilon_{\varrho\mu a},$$

$$\mathcal{S}^{(B)}_{\zeta\eta}=\Big[\frac{1}{2}\nabla^{\mu}\nabla_{\beta}f_R+\frac{1}{2}R^{\mu}_{\beta}+R\nabla_{\beta}\nabla^{\mu}f_G-2R^{\lambda}_{\beta}\ \times\nabla_{\lambda}\nabla^{\mu}f_G+2R^{\mu}_{\beta}\Box f_G+2R^{\mu}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-2$$
$$\times R^{\mu}_{\lambda}\nabla_{\beta}\nabla^{\lambda}f_G-\frac{1}{2}f_RR^{\mu}_{\beta}\Big]h_{\zeta\eta}u^{\beta}u_{\mu}-\frac{1}{3}\Big[-\frac{1}{2}\ \times\Box f_R+2R^{\alpha\epsilon}\nabla_{\alpha}\nabla_{\epsilon}f_G+\frac{1}{4}Rf_R-\frac{1}{2}Rf_R-\frac{1}{2}$$
$$\times Gf_G+\frac{1}{2}f-\frac{1}{2}R\Big]h_{\zeta\eta}+\Big[R\nabla_c\nabla_f f_G-2R_{c\lambda}\ \times\nabla_f\nabla^{\lambda}f_G-2R^{\lambda}_{f}\nabla_{\lambda}\nabla_c f_G+R_{cf}\Box f_G+2R_{c\mu\nu f}$$
$$\times\nabla^{\mu}\nabla^{\nu}f_G-\frac{1}{2}R_{cf}f_R+2R^{\alpha}_{c}\nabla_{\alpha}\nabla_f f_G\Big]h^{c}_{\zeta}h^{f}_{\eta}+\Big[\frac{1}{2}\nabla_{\zeta}\nabla_{\beta}f_R-2\nabla_{\zeta}\nabla_{\lambda}R^{\lambda}_{\beta}f_G-2R_{\zeta\beta}\Box f_G-2$$
$$\times R_{\zeta\mu\nu\beta}\nabla^{\mu}\nabla^{\nu}f_G+2R_{\zeta\lambda}\nabla^{\lambda}\nabla_{\beta}f_G-\frac{1}{2}R_{\zeta\beta}\Big]u_{\eta}u^{\beta}+\Big[\frac{1}{2}\nabla^{\mu}\nabla_{\eta}f_R+R\nabla^{\mu}\nabla_{\eta}f_G-2R^{\lambda}_{\eta}\nabla^{\mu}\nabla_{\lambda}f_G$$
$$+2R^{\mu}_{\eta}\Box f_G+2R^{\mu}_{\alpha\beta\eta}\nabla^{\mu}\nabla^{\nu}f_G-\frac{1}{2}R^{\mu}_{\eta}\Big]u_{\mu}u_{\zeta}+\Big[-\frac{1}{2}\nabla^{\mu}\nabla_{\beta}f_R-\frac{1}{2}R^{\mu}_{\beta}-R^{\mu}\nabla_{\beta}f_G+2R^{\lambda}_{\beta}\nabla^{\mu}$$
$$\times\nabla_{\lambda}f_G-2R^{\mu}_{\beta}\Box f_G-2R^{\mu}_{\alpha\lambda\beta}\nabla^{\alpha}\nabla^{\lambda}f_G+2R^{\mu}_{\lambda}\nabla^{\lambda}\ \times\nabla_{\beta}f_G+\frac{1}{2}f_RR^{\mu}_{\beta}\Big]g_{\zeta\eta}u_{\mu}u^{\beta}+\frac{1}{2}R_{\zeta\eta}+R\nabla_{\zeta}\nabla_{\eta}$$
$$\times f_G-2R_{\zeta\lambda}\nabla_{\eta}\nabla^{\lambda}f_G-2R^{\lambda}_{\eta}\nabla_{\lambda}\nabla_{\zeta}f_G+2R_{\zeta\eta}\times\Box f_G+2R_{\zeta\mu\nu\eta}\nabla^{\mu}\nabla^{\nu}f_G+2R^{\alpha}_{\zeta}\nabla_{\alpha}\nabla_{\eta}f_G+\frac{1}{2}\times u_{\zeta}u_{\eta}f_R.$$

$$\mathcal{S}^{(C)}_{\zeta\eta}=\Big[-\frac{1}{2}Ru^{\beta}\nabla^{\nu}\nabla_{\beta}f_G+u^{\beta}R^{\nu}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G-u^{\beta}R^{\nu}_{\beta}\ \times\Box f_G-\frac{1}{4}u^{\beta}R^{\nu}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{4}u^{\beta}R^{\nu}_{\beta}\Big]\epsilon_{\zeta\eta\nu}$$
$$+\Big[\frac{1}{4}u^{\beta}\nabla_{\beta}\nabla^{\alpha}f_R+\frac{1}{2}u^{\beta}R\nabla_{\beta}\nabla^{\alpha}f_G-u^{\beta}R^{\lambda}_{\beta}\nabla_{\lambda}\ \times\nabla^{\alpha}f_G+u^{\beta}R^{\alpha}_{\beta}\Box f_G+u^{\beta}R^{\alpha}_{\nu\varrho\beta}\nabla^{\nu}\nabla^{\varrho}f_G+\frac{1}{4}u^{\beta}$$
$$\times R^{\alpha}_{\beta}-\frac{1}{4}u^{\beta}R^{\alpha}_{\beta}f_R+u^{\beta}R^{\lambda}_{\beta}\nabla^{\alpha}\nabla_{\lambda}f_G-u^{\beta}R^{\alpha}_{\lambda}\ \nabla^{\lambda}\nabla_{\beta}f_G\Big]\epsilon_{\eta\alpha\zeta}.$$
$$\mathbb{F}=\Big[\frac{1}{4}R\nabla^{\nu}\nabla_{\varrho}f_G-\frac{1}{2}R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\nu}f_G+\frac{1}{2}R^{\nu}_{\varrho}\Box f_G\ +\frac{1}{2}R^{\nu}_{\alpha\epsilon\varrho}\nabla^{\alpha}\nabla^{\epsilon}f_G+\frac{1}{8}R^{\nu}_{\varrho}-\frac{1}{8}f_RR^{\nu}_{\varrho}-\frac{1}{2}R^{\nu}_{\lambda}\nabla^{\lambda}$$
$$\times\nabla_{\varrho}f_G+\frac{1}{8}\nabla^{\nu}\nabla_{\varrho}f_R\Big]\epsilon^{\varrho\beta c}\epsilon_{\nu\beta c}+\Big[\frac{-1}{8}\nabla_{\beta}\nabla^{\nu}\ \times f_R-\frac{1}{4}R\nabla^{\nu}\nabla_{\beta}f_G-\frac{1}{2}R^{\lambda}_{\beta}\nabla_{\lambda}\nabla^{\nu}f_G-\frac{1}{2}R^{\nu}_{\beta}\Box$$
$$\times f_G-\frac{1}{2}R^{\nu}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{8}R^{\nu}_{\beta}+\frac{1}{8}R^{\nu}_{\beta}f_R-\frac{1}{2}\ \times R^{\nu}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G\Big]\epsilon^{\varrho\beta c}\epsilon_{\nu\varrho c}+\Big[-\frac{1}{8}\nabla^{\eta}\nabla_{\varrho}f_R-\frac{1}{4}$$
$$\times R\nabla^{\eta}\nabla_{\varrho}f_G+\frac{1}{2}R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\eta}f_G-\frac{1}{2}R^{\eta}_{\varrho}\Box f_G-\frac{1}{2}\ \times R^{\eta}_{\alpha\epsilon\varrho}\nabla^{\alpha}\nabla^{\epsilon}f_G+\frac{1}{2}R^{\eta}_{\lambda}\nabla^{\lambda}\nabla_{\varrho}f_G\Big]\epsilon^{\varrho\beta c}\epsilon_{\beta\eta c}$$

$$+\left[\frac{1}{8}\nabla^{\eta}\nabla_{\beta}f_R+\frac{1}{4}R\nabla^{\eta}\nabla_{\beta}f_G-\frac{1}{2}R^{\lambda}_{\beta}\nabla^{\eta}\nabla_{\lambda}f_G+\frac{1}{2}R^{\eta}_{\beta}\Box f_G+\frac{1}{2}R^{\eta}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{2}R^{\eta}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G\right]$$
$$\times\epsilon^{\varrho\beta c}\epsilon_{\varrho\eta c},\mathbb{H}=\left[-\frac{1}{2}\nabla_{\varsigma}\nabla_{\beta}f_R-\nabla_{\varsigma}\nabla_{\beta}f_GR+2\nabla_{\varsigma}\nabla_{\lambda}f_GR^{\lambda}_{\beta}\right.$$
$$\left.-2R_{\varsigma\beta}\Box f_G-2R_{\varsigma\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G+2R_{\varsigma\lambda}\nabla^{\lambda}\nabla_{\beta}f_G-\frac{1}{2}R_{\varsigma\beta}+\frac{1}{2}R_{\varsigma\beta}f_R\right]u^{\varsigma}u^{\beta}+\left[-\frac{1}{2}\nabla^{\eta}\nabla_{\varrho}f_R-R\right.$$
$$\left.\times\nabla^{\eta}\nabla_{\varrho}f_G+2R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\eta}f_G-2R^{\eta}_{\varrho}\Box f_G-2R^{\eta}_{\alpha\epsilon\varrho}\times\nabla^{\alpha}\nabla^{\epsilon}f_G-\frac{1}{2}R^{\eta}_{\varrho}+\frac{1}{2}R^{\eta}_{\varrho}f_R\right]u_{\eta}u^{\varrho}+4\left[\frac{1}{2}\nabla^{\eta}\right.$$
$$\left.\times\nabla_{\beta}f_R+\frac{1}{2}R^{\eta}_{\beta}+R^{\eta}\nabla_{\beta}f_G-2R^{\lambda}_{\beta}\nabla_{\lambda}\nabla^{\eta}f_G+2R^{\eta}_{\beta}\Box f_G+2R^{\eta}_{\alpha\epsilon\beta}\nabla^{\alpha}\nabla^{\epsilon}f_G-2R^{\eta}_{\lambda}\nabla^{\lambda}\nabla_{\beta}f_G\right.$$
$$\left.-\frac{1}{2}R^{\eta}_{\beta}f_R\right]u_{\eta}u^{\beta}+3R\Box f_G+f_RR+\frac{1}{2}f_GG+\Box f_R-\frac{5}{4}f+\frac{5}{2}R+R\nabla_{\varsigma}\nabla^{\varsigma}f_G-2R^{\varrho}_{\lambda}\nabla_{\varrho}\nabla^{\lambda}$$
$$\times f_G-2R^{\lambda}_{\varrho}\nabla_{\lambda}\nabla^{\varrho}f_G+2R^{\varrho}_{\alpha\epsilon\varrho}\nabla^{\alpha}\nabla^{\epsilon}f_G.$$

References

[1] S. Sahoo, T. Malakar, and B. K. Roy, Int. J. Dyn. Control. **13**, 30 (2025)
[2] X. Calbet and R. Ruiz, Phys. Rev. E **63**, 066116 (2001)
[3] A. Chaudhari, P. K. Sahu, and S. L. Lee, J. chem. phys. **120**, 170 (2004)
[4] J. Sañudo and R. Ruiz, Phy. Lett. A **372**, 5283 (2008)
[5] L. Herrera, Phys. Rev. D **97**, 044010 (2018)
[6] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **98**, 104059 (2018)
[7] L. Herrera, Int. J. Mod. Phys. D **20**, 1689 (2011)
[8] D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979)
[9] L. Herrera *et al.*, Phys. Lett. A **237**, 113 (1998)
[10] R. G. Catalán, J. Garay, and R. Ruiz, Phys. Rev. E **66**, 011102 (2002)
[11] A. Rehman *et al.*, Eur. Phys. J. C **85**, 949 (2025)
[12] M. Z. Gul *et al.*, Eur. Phys. J. C **85**, 595 (2025)
[13] S. K. Maurya *et al.*, Eur. Phys. J. C **82**, 100 (2022)
[14] A. Rehman *et al.*, Nucl. Phys. B **1026**, 117434 (2026)
[15] S. K. Maurya *et al.*, Eur. Phys. J. C **83**, 348 (2023)
[16] A. Rehman *et al.*, Gen. Relativ. Gravit. **58**, 15 (2026)
[17] M. G. B. De Avellar *et al.*, Phys. Lett. A **378**, 3481 (2014)
[18] S. K. Maurya *et al.*, Eur. Phys. J. C **83**, 317 (2023)
[19] S. K. J. Pacif, and B. Mishra, Ast. Space Science **360**, 48 (2015)
[20] M. Z. Bhatti *et al.*, Int. J. Geom. Methods Mod. Phys. **23**, 2650001 (2026)
[21] S. K. Maurya *et al.*, JCAP **09**, 048 (2024)
[22] G. Mustafa and S. K. Maurya, Eur. Phys. J. C **84**, 686 (2024)
[23] S. K. Maurya *et al.*, Eur. Phys. J. C **82**, 1006 (2022)
[24] H. Chaudhary *et al.*, J. High Energy Astrophys. **45**, 340 (2025)
[25] S. K. Maurya *et al.*, Universe **11**, 33 (2025)
[26] S. Ş. Bayin, Phys. Rev. D **26**, 1262 (1982)
[27] M. M. M. Nasir *et al.*, Eur. Phys. J. C **85**, 189 (2025)
[28] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **103**, 024037 (2021)
[29] M. Sharif, and I. I. Butt, Eur. Phys. Jour. C **78**, 1 (2018)
[30] L. Herrera, and N. O. Santos, Phys. Rep. **286**, 53 (1997)
[31] L. Herrera *et al.*, Phys. Rev. D **79**, 064025 (2009)
[32] E. Elizalde *et al.*, Class. Quantum Grav. **27**, 095007 (2010)
[33] M. Z. Bhatti, Z. Yousaf, and S. Khan, Int. J. Mod. Phys. D **30**, 2150097 (2021)
[34] S. Nojiri, and S. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 16 (2007)
[35] G. Abbas, and H. Nazar, Eur. Phys. J. C **78**, 1 (2018)
[36] K. Bamba *et al.*, Astrophys. Space Sci. **342**, 155 (2012)
[37] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, J. Cosmol. Astropart. Phys. **2013**, 040 (2013)
[38] M. Z. Gul and M. Sharif, Universe **7**, 154 (2021)
[39] G. J. Olmo, D. Rubiera-Garcia, and A. Wojnar, Phys. Rep. **876**, 75 (2020)
[40] M. Z. Gul *et al.*, Eur. Phys. J. C **84**, 775 (2024)
[41] M. Z. Bhatti, Z. Yousaf, and M. Yousaf, Gen. Relativ. Gravit. **56**, 3 (2024)
[42] S. Hanif, and M. Z. Bhatti, Chin. J. Phys. **87**, 1 (2024)
[43] M. Sharif and M. Z. Gul, Phys. Scr. **98**, 035030 (2023)
[44] G. Naz *et al.*, Phys. Dark Universe **46**, 101635 (2024)
[45] I. Hashim *et al.*, Int. J. Geom. Methods Mod. Phys. **22**, 2540049 (2025)
[46] S. Rani *et al.*, Phys. Dark Universe **47**, 101754 (2025)
[47] M. Z. Gul *et al.*, Fortschr. der Phys. **74**, 70076 (2026)
[48] A. Malik *et al.*, Phys. Lett. B **874**, 140260 (2026)
[49] M. Yousaf, and H. Asad, Phys. Dark Universe **48**, 101841 (2025)
[50] U. Farwa and Z. Yousaf, Chin. J. Phys. **85**, 285 (2023)
[51] S. K. Maurya *et al.*, Mon. Not. R. Astron. Soc. **527**, 5192 (2024)
[52] M. Z. Gul *et al.*, Chin. J. Phys. **88**, 388 (2024)
[53] M. Yousaf *et al.*, Commun. Theor. Phys. **78**, 025407 (2026)
[54] M. Z. Bhatti, Z. Yousaf, and A. Rehman, Fortsch. der Phys. **71**, 2200113 (2023)
[55] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, Phys. Lett. B **742**, 160 (2015)
[56] M. Yousaf *et al.*, Ann. Phys. **489**, 170424 (2026)
[57] S. K. Maurya, Eur. Phys. J. C **80**, 429 (2020)
[58] G. Murtaza *et al.*, J. High Energy Astrophys. **44**, 279 (2024)

[59] M. Yousaf *et al*., Indian J. Phys. **100**, 849 (2026)
[60] M. Yousaf *et al*., Int. J. Geom. Methods Mod. Phys. **2650216**, 849 (2026)
[61] A. Ditta *et al*., Phys. Scri. **99**, 065009 (2024)
[62] M. Yousaf *et al*., Phys. Scr. **99**, 115270 (2024)
[63] S. Capozziello, S. De Bianchi, and E. Battista, Phys. Rev. D **109**, 104060 (2024)
[64] M. Yousaf *et al*., Nucl. Phys. B **1028**, 117501 (2026)
[65] M. Z. Bhatti *et al*., Phys. Dark Universe **29**, 100561 (2020)
[66] S. Maurya and S. Maharaj, Eur. Phys. J. C **77**, 328 (2017)
[67] M. Yousaf, H. Asad, and A. Rehman, Phys. Dark Universe **48**, 101888 (2025)
[68] T. Naz *et al*., Phys. Dark Universe **50**, 102166 (2025)
[69] M. Z. Gul *et al*., New Astron. **109**, 102204 (2024)
[70] M. Z. Bhatti *et al*., Galaxies **10**, 40 (2022)
[71] S. K. J. Pacif and Abdussattar, Eur. Phys. J. Plus **129**, 18 (2014)
[72] P. Sahoo and M. Sivakumar, Astrophys. Space Sci. **357**, 60 (2015)
[73] M. Yousaf, Chin. J. Phys. **95**, 1278 (2025)
[74] A. V. Astashenok, S. Capozziello, and S. D. Odintsov, J. Cosmol. Astropart. Phys. **2015**, 001 (2015)
[75] H. Asad *et al*., Phys. Dark Universe **46**, 101666 (2024)
[76] M. Yousaf *et al*., Nucl. Phys. B **995**, 116328 (2023)
[77] P. K. Sahoo, P. Sahoo, and B. K. Bishi, Int. J. Geom. Methods Mod. Phys. **14**, 1750097 (2017)
[78] M. Yousaf *et al*., Commun. Theor. Phys. **77**, 035403 (2024)
[79] M. Z. Gul and M. Sharif, New Astron. **106**, 102137 (2024)
[80] B. Almutairi *et al*., Int. J. Theor. Phys. **63**, 215 (2024)
[81] G. Fatima, F. Javed, and G. Mustafa, Int. J. Geom. Methods Mod. Phys. **22**, 2450279 (2025)
[82] M. Bhatti, Z. Yousaf, and A. Rehman, Pramana **96**, 224 (2022)
[83] G. Calcagni *et al*., Class. Quantum Grav. **22**, 3977 (2005)
[84] S. Nojiri and S. D. Odintsov, Phys. Rev. D **78**, 046006 (2008)
[85] G. Cognola *et al*., Phys. Rev. D **75**, 086002 (2007)
[86] M. Bhatti, Z. Yousaf, and A. Rehman, Indian J. Phys. **97**, 2227 (2023)
[87] A. Rehman, M. Z. Bhatti, and Z. Yousaf, Fortschr. der Phys. **72**, 2300247 (2024)
[88] S. D. Odintsov, V. K. Oikonomou, and S. Banerjee, Nucl. Phys. **938**, 935 (2019)
[89] M. De Laurentis and A. J. Revelles, Int. J. Geom. Methods Mod. Phys. **11**, 1450082 (2014)
[90] K. Atazadeh, and F. Darabi, Gen. Relativ. Gravit. **46**, 1 (2014)
[91] M. De Laurentis, M. Paolella, and S. Capozziello, Phys. Rev. D **91**, 083531 (2015)
[92] M. Bhatti, Z. Yousaf, and A. Rehman, Int. J. Mod. Phys. D **31**, 2150124 (2022)
[93] J. M. Heinzle *et al*., Class. Quantum Grav. **20**, 4567 (2003)
[94] J. S. Read *et al*., Phys. Rev. D **79**, 124032 (2009)
[95] S. Thirukkanesh and F. Ragel, Pramana **78**, 687 (2012)
[96] M. Gokhroo and A. Mehra, Gen. Relativ. Gravit. **26**, 75 (1994)
[97] L. Herrera, A. Di Prisco, and J. Ibanez, Phys. Rev. D **84**, 107501 (2011)
[98] A. Lobo *et al*., Class. Quantum Grav. **25**, 205018 (2008)
[99] Z. Yousaf, M. Bhatti, and A. Rehman, Chin. J. Phys. **73**, 493 (2021)
[100] A. Rehman, T. Naseer, and B. Dayanandan, Nucl. Phys. B **1013**, 116852 (2025)
[101] A. Rehman *et al*., Nucl. Phys. B 116897 (2025)
[102] Z. Yousaf, M. Bhatti, and T. Naseer, Phys. Dark Universe **28**, 100535 (2020)
[103] K. S. Thorne, Phys. Rev. D **138**, 251 (1965)
[104] R. C. Tolman, Phys. Rev. D **35**, 875 (1930)
[105] W. B. Bonnor, Class. Quantum Grav. **18**, 1381 (2001)
[106] L. Herrera, Int. J. Mod. Phys. D **20**, 2773 (2011)
[107] L. Herrera, A. Di Prisco, and J. Ospino, Gen. Relativ. Gravit. **44**, 2645 (2012)
[108] M. Bhatti, Z. Yousaf, and M. Ilyas, Eur. Phys. J. C **77**, 1 (2017)
[109] L. Herrera, J. Ospino, and A. Di Prisco, Phys. Rev. D **77**, 027502 (2008)
[110] A. Di Prisco *et al*., Int. J. Mod. Phys. D **20**, 2351 (2011)
[111] L. Herrera and W. Barreto, Phys. Rev. D **87**, 087303 (2013)
[112] L. Herrera and W. Barreto, Phys. Rev. D **88**, 084022 (2013)
[113] L. Herrera, E. Fuenmayor, and P. Leon, Phys. Rev. D **93**, 024047 (2016)
[114] E. Bertschinger, Astrophys. J. **295**, 1 (1985)
[115] G. Blumenthal *et al*., Astrophys. J. **388**, 234 (1992)