The new spin-statistics theorem arising from pseudo-Hermiticity*

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Abstract: We prove a new spin–statistics theorem for a class of quantum field theories with pseudo-Hermitian Hamiltonians. Quantum fields with integer spin exhibit fermionic statistics, whereas those with half-integer spin exhibit bosonic statistics, in contrast to the conventional case. This reversal arises from defining canonical field operators using pseudo-Hermitian conjugation rather than Hermitian conjugation, thereby circumventing the conventional spin–statistics theorem. The free fields retain locality, Lorentz covariance, and unitary evolution. However, interactions can violate unitarity because of the intrinsically non-Hermitian nature of pseudo-Hermitian Hamiltonians. We discuss unitarity issues in this quantum field theory and its potential applications to the dS_4/CFT_3 correspondence.

Keywords: spin–statistics theorem, pseudo-Hermitian Hamiltonian, dS_4/CFT_3 correspondence

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I. INTRODUCTION

Theorems in quantum field theory (QFT) usually have limited applicability. Most theorems apply either to models in lower-dimensional spacetime or to theories with symmetries that have not been observed in nature. The spin-statistics theorem (SST) is an exception, as it correctly describes the connection between spin and statistics for all observed elementary particles [1–6]. In four-dimensional Minkowski spacetime, the SST can be stated simply: *Integer spin fields are bosonic, while half-integer spin fields are fermionic.*

The SST is derived from the principles of locality and Poincaré symmetry in QFT. Consequently, the connection between spin and statistics stated in the preceding paragraph is widely believed to be unique. Here, we demonstrate that, for a class of QFTs with pseudo-Hermitian Hamiltonians (pseudo-Hermitian QFTs), this widespread belief is no longer valid.

A pseudo-Hermitian Hamiltonian is a special class of non-Hermitian Hamiltonian first studied by Pauli [7] and later by Mostafazadeh [8–11]. To define a pseudo-Hermitian Hamiltonian, we must first specify a *linear and Hermitian* operator and use it to define the pseudo-Hermitian operation. Let this operator be η. Given an operator O, the η pseudo-Hermitian conjugation # is defined as

$$O^{\#} \equiv \eta^{-1} O^{\dagger} \eta . \tag{1}$$

The operator η must be linear and Hermitian to ensure that $O^{\#\#} = O$. A Hamiltonian is η-pseudo-Hermitian when

$$H^{\#} = H . \tag{2}$$

Different classes of pseudo-Hermitian QFTs can be constructed by specifying different choices of η [12–19].[1)] The class of pseudo-Hermitian QFT studied in this work was constructed by LeClair *et al.* [20, 21] and Ahluwalia *et al.* [22–24]. By taking η to have a specific form and using it to define field adjoints through pseudo-Hermitian conjugation, they constructed scalar and spinor fields that are fermionic and bosonic, respectively. Owing to this new relationship between spin and statistics, these fields have found applications in the dS_4/CFT_3 correspondence [22–25] [2)].

In this work, we prove a new spin-statistics theorem (NSST) for the class of pseudo-Hermitian QFT constructed by LeClair *et al.* and Ahluwalia *et al.*[3)] The NSST

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1) Y. D. Li and Q. Wang, (2023)

2) C. M. Chang, A. Pathak, and A. Strominger, (2013)

3) From now on, unless otherwise stated, pseudo-Hermitian QFT refers specifically to those constructed by LeClair et al. and Ahluwalia *et al.*

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states that *integer-spin fields are fermionic, whereas half-integer-spin fields are bosonic.* The standard SST is circumvented by exploiting the freedom to construct field adjoints via pseudo-Hermitian conjugation.

In Section 3, we construct the scalar, spinor, and vector fields using the pseudo-Hermitian framework. These fields are shown to be local, to respect Poincaré symmetry, and to satisfy the NSST. The free Lagrangians L_0 for these fields are manifestly pseudo-Hermitian and non-Hermitian. After performing the Legendre transform on L_0, the non-Hermitian contributions cancel, yielding a free Hamiltonian H_0 that is positive definite and Hermitian. The parity and time-reversal transformations of these fields are the same as those of their Hermitian counterparts. However, the charge-conjugation transformation differs. In contrast to the particle states described by Hermitian QFTs, in which the particle and antiparticle have even charge-conjugation phases, the particle and antiparticle associated with pseudo-Hermitian QFT have *odd* charge-conjugation phases.

The free Lagrangian densities for pseudo-Hermitian QFTs are manifestly Poincaré-invariant and pseudo-Hermitian. Therefore, their Poincaré generators satisfy the Poincaré algebra and are pseudo-Hermitian. In Section IV, we show that η commutes with all these generators, so they are simultaneously Hermitian and pseudo-Hermitian. In contrast, the full Hamiltonian H and the Belinfante-Rosenfeld tensor are generally non-Hermitian. Therefore, the interacting theories may violate unitarity.

Using the scattering theory developed in [26], we examine the unitarity of the S-matrix. In doing so, we have been unable to find a fully unitary pseudo-Hermitian QFT. If these fields are non-unitary, they cannot describe elementary particles. One might therefore be inclined to rule out such constructs as unphysical. In our view, however, this negative conclusion is premature. Tentative results in the literature suggest that pseudo-Hermitian scalar and spinor fields are relevant for realizing the dS_4/CFT_3 correspondence [25].[1)] Nevertheless, there is a crucial difference between their constructions and ours, which we explain in Section V.

The paper is organized as follows. In Section II, we review the standard SST. Subsequently, we construct the free pseudo-Hermitian QFT that satisfies the NSST. In Section III, we construct the scalar, spinor, and vector fields that satisfy the NSST. In Section IV, we present the Lagrangian formalism for pseudo-Hermitian QFT. We show that the generators of the Poincaré group obtained from the free Lagrangian density satisfy the Poincaré algebra and are simultaneously Hermitian and pseudo-Hermitian. In Section V, we discuss issues related to unitarity in pseudo-Hermitian QFTs and their potential applications to the dS_4/CFT_3 correspondence. We conclude in Section VI.

II. SPIN-STATISTICS THEOREM

We begin by reviewing the basic structure of Hermitian QFT and the standard SST. We then use results from Hermitian QFT to construct a pseudo-Hermitian QFT that obeys the NSST.

Let $\lambda(x)$ denote a free field and $\bar{\lambda}(x)$ its adjoint in a Hermitian QFT. We expand the fields as

$$\lambda(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma}\left[\mathrm{e}^{-\mathrm{i}p\cdot x}f(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + \mathrm{e}^{\mathrm{i}p\cdot x}g(\mathbf{p},\sigma)a^{c\dagger}(\mathbf{p},\sigma)\right], \tag{3}$$

$$\bar{\lambda}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{f}(\mathbf{p},\sigma)a^{\dagger}(\mathbf{p},\sigma) + \mathrm{e}^{-\mathrm{i}p\cdot x}\bar{g}(\mathbf{p},\sigma)a^{c}(\mathbf{p},\sigma)\right], \tag{4}$$

where $f, g, \bar{f}, \bar{g}$ are coefficient functions, $a^{\dagger}(\mathbf{p},\sigma)$ creates particles, and $a^{c\dagger}(\mathbf{p},\sigma)$ creates antiparticles.

Let $|0\rangle$ denote the free vacuum state. The single-particle and antiparticle states, with appropriate normalizations, are defined as

$$|\mathbf{p},\sigma,a\rangle \equiv \sqrt{2E_{\mathbf{p}}}a^{\dagger}(\mathbf{p},\sigma)|0\rangle, \tag{5}$$

$$|\mathbf{p},\sigma,a^c\rangle \equiv \sqrt{2E_{\mathbf{p}}}a^{c\dagger}(\mathbf{p},\sigma)|0\rangle. \tag{6}$$

These states furnish a unitary representation of the Poincaré group. For massive particles, $\sigma = -j,\cdots,j$, whereas for massless particles, $\sigma = \pm j$, where $j = 0, \frac{1}{2}, 1\cdots$ is the spin of the particle. The creation and annihilation operators must satisfy canonical commutation or anticommutation relations

$$\left[a(\mathbf{p}',\sigma'),a^{\dagger}(\mathbf{p},\sigma)\right]_{\pm} = \left[a^{c}(\mathbf{p}',\sigma'),a^{c\dagger}(\mathbf{p},\sigma)\right]_{\pm} = (2\pi)^3\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta_{\sigma'\sigma}, \tag{7}$$

where the signs $-$ and $+$ denote a commutator and an anticommutator, respectively:

$$[a,b]_{\pm} \equiv ab \pm ba. \tag{8}$$

1) C. M. Chang, A. Pathak, and A. Strominger, (2013)

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Particles and their associated fields obey bosonic (fermionic) statistics when the creation and annihilation operators satisfy commutation (anticommutation) relations.

In standard Hermitian QFT, the connection between spin and statistics is determined by the requirements of Poincaré covariance, Hermiticity, and locality. We now examine these requirements. Poincaré covariance implies that $\lambda(x)$ and $\bar{\lambda}(x)$ transform as

$$\mathrm{U}(\Lambda,\alpha)\lambda(x)\mathrm{U}^{-1}(\Lambda,\alpha)=\mathcal{D}^{-1}(\Lambda)\lambda(\Lambda x+\alpha), \tag{9}$$

$$\mathrm{U}(\Lambda,\alpha)\bar{\lambda}(x)\mathrm{U}^{-1}(\Lambda,\alpha)=\bar{\lambda}(\Lambda x+\alpha)\mathcal{D}(\Lambda), \tag{10}$$

where $\mathrm{U}(\Lambda,\alpha)$ is the unitary representation of Poincaré transformations on the Hilbert space and $\mathcal{D}(\Lambda)$ is the finite-dimensional representation of the Lorentz group. Therefore, $\bar{\lambda}(x)\lambda(x)$ transforms as a scalar operator.

In addition to Poincaré covariance for the field and its adjoint, the full Hamiltonian H constructed from $\lambda(x)$ and $\bar{\lambda}(x)$ must be Hermitian. In perturbative QFT, where $H=H_0+V$ and H_0 and V are the free and interacting parts, respectively, each term must be Hermitian: $H_0^\dagger=H_0$ and $V^\dagger=V$. In a Hermitian QFT, these requirements are met by constructing Hermitian Lagrangian densities $\mathcal{L}^\dagger=\mathcal{L}$. The Hamiltonian obtained by performing a Legendre transform on the Lagrangian $L=\int \mathrm{d}^3x\,\mathcal{L}$ is then automatically Hermitian. As with the full Hamiltonian, the Lagrangian density can be split into free and interacting parts, $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\mathrm{int}}$, where $\mathcal{L}_0^\dagger=\mathcal{L}_0$ and $\mathcal{L}_{\mathrm{int}}^\dagger=\mathcal{L}_{\mathrm{int}}$.

The free Lagrangian density $\mathcal{L}_0$ consists of the kinetic and mass terms. The kinetic term contains derivative operators, while the mass term is proportional to $\bar{\lambda}(x)\lambda(x)$. Both the kinetic and mass terms must be Hermitian; therefore, we require

$$\left[\bar{\lambda}(x)\lambda(x)\right]^\dagger=\bar{\lambda}(x)\lambda(x). \tag{11}$$

For example, if $\lambda(x)$ is a complex scalar field, its adjoint is given by $\bar{\lambda}(x)=\lambda^\dagger(x)$. If $\lambda(x)$ is a Dirac field, then $\bar{\lambda}(x)=\lambda^\dagger(x)\gamma^0$. After substituting Eqs. (3)–(4) into $\bar{\lambda}(x)\lambda(x)$, imposing Hermiticity requires the coefficient functions to satisfy

$$\left[\bar{f}(\mathbf{p}',\sigma')g(\mathbf{p},\sigma)\right]^\dagger=\bar{g}(\mathbf{p},\sigma)f(\mathbf{p}',\sigma'). \tag{12}$$

Thus far, we have imposed Poincaré covariance and Hermiticity on the quantum fields. We must now impose locality. Accordingly, we require the field $\lambda(t,\mathbf{x})$ to commute or anticommute with its adjoint $\bar{\lambda}(t,\mathbf{y})$ at equal times. Using Eq. (7), we obtain

$$\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_\pm=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{1}{2E_{\mathbf{p}}}M_\pm(\mathbf{p}), \tag{13}$$

where

$$M_\pm(\mathbf{p})\equiv\sum_\sigma\left[f(\mathbf{p},\sigma)\bar{f}(\mathbf{p},\sigma)\pm g(-\mathbf{p},\sigma)\bar{g}(-\mathbf{p},\sigma)\right]. \tag{14}$$

The SST states that integer-spin fields are bosonic. The equal-time (spacelike) commutator between $\lambda(t,\mathbf{x})$ and $\bar{\lambda}(t,\mathbf{y})$ is given by

$$\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_-=0,\ M_-(\mathbf{p})=0,\ j=0,1,\cdots. \tag{15}$$

Fields with half-integer spin are fermionic; therefore, $\lambda(t,\mathbf{x})$ and $\bar{\lambda}(t,\mathbf{y})$ anticommute at equal times

$$\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_+=0,\ M_+(\mathbf{p})=0,\ j=\frac{1}{2},\frac{3}{2},\cdots. \tag{16}$$

In $3+1$-dimensional Minkowski spacetime, the SST for Hermitian QFTs is a natural consequence of Poincaré covariance, Hermiticity, and locality. Consequently, the standard SST is widely believed to be unique. However, the works of LeClair *et al.* [20, 21] and Ahluwalia *et al.* [22, 24] have shown otherwise. They have demonstrated that, by defining the appropriate field adjoint using pseudo-Hermitian conjugation, one can construct fermionic scalar and bosonic spinor fields while preserving Poincaré covariance and locality. On the other hand, the full Hamiltonians are no longer Hermitian but are instead pseudo-Hermitian. The important question is whether unitarity can be preserved in pseudo-Hermitian QFTs. We briefly discuss the issue of unitarity in Section V. Detailed analyses of interacting pseudo-Hermitian QFTs have appeared in the literature.[1)]

To construct a general class of pseudo-Hermitian QFTs obeying the NSST, we first introduce the fields $\chi(x)$ and $\bar{\chi}(x)$:

$$\chi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\sigma\left[\mathrm{e}^{-\mathrm{i}p\cdot x}f(\mathbf{p},\sigma)b(\mathbf{p},\sigma)+\mathrm{e}^{\mathrm{i}p\cdot x}g(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{17}$$

1) R. Leng, C. Y. Lee, and S. Zhou, (2025)

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$$\bar{\chi}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{f}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma)+\mathrm{e}^{-\mathrm{i}p\cdot x}\bar{g}(\mathbf{p},\sigma)b^{c}(\mathbf{p},\sigma)\right]. \tag{18}$$

They are identical to $\lambda(x)$ and $\bar{\lambda}(x)$ except that a,a^c are replaced by b,b^c, which also satisfy the canonical relation

$$\left[b(\mathbf{p}',\sigma'),b^{\dagger}(\mathbf{p},\sigma)\right]_{\pm}=\left[b^{c}(\mathbf{p}',\sigma'),b^{c\dagger}(\mathbf{p},\sigma)\right]_{\pm}=(2\pi)^3\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{p}'-\mathbf{p}). \tag{19}$$

Thus, $\chi(x)$ and $\bar{\chi}(x)$ are Poincaré covariant

$$\mathrm{U}(\Lambda,\alpha)\chi(x)\mathrm{U}^{-1}(\Lambda,\alpha)=\mathcal{D}^{-1}(\Lambda)\chi(\Lambda x+\alpha), \tag{20}$$

$$\mathrm{U}(\Lambda,\alpha)\bar{\chi}(x)\mathrm{U}^{-1}(\Lambda,\alpha)=\bar{\chi}(\Lambda x+\alpha)\mathcal{D}(\Lambda). \tag{21}$$

If $\bar{\chi}(x)$ is the field adjoint of $\chi(x)$, then they describe the same physics as $\lambda(x)$ and $\bar{\lambda}(x)$. The NSST arises from the freedom to define a new field adjoint for $\chi(x)$ using pseudo-Hermitian conjugation:[1)]

$$\begin{aligned}\vec{\chi}(x)&\equiv\eta^{-1}\bar{\chi}(x)\eta\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{f}(\mathbf{p},\sigma)\eta^{-1}b^{\dagger}(\mathbf{p},\sigma)\eta+\mathrm{e}^{-\mathrm{i}p\cdot x}\bar{g}(\mathbf{p},\sigma)\eta^{-1}b^{c}(\mathbf{p},\sigma)\eta\right],\end{aligned} \tag{22}$$

where η is a linear Hermitian operator. The Poincaré transformation of $\vec{\chi}(x)$ is

$$\mathrm{U}(\Lambda,\alpha)\vec{\chi}(x)\mathrm{U}^{-1}(\Lambda,\alpha)=\eta^{-1}_{\Lambda,\alpha}\vec{\chi}(\Lambda x+\alpha)\eta_{\Lambda,\alpha}\mathcal{D}(\Lambda), \tag{23}$$

where $\eta_{\Lambda,\alpha}\equiv U(\Lambda,\alpha)\eta U^{-1}(\Lambda,\alpha)$. To preserve Poincaré covariance, η must commute with $\mathrm{U}(\Lambda,\alpha)$. To determine η, we need to establish how it acts on the operators $b^{\dagger}$ and b^c. Since $b^{\dagger}$ and b^c furnish a unitary irreducible representation of the Poincaré group, the Poincaré transformations of $b^{\dagger}$ and b^c are

$$\begin{aligned}&\mathrm{U}(\Lambda,\alpha)b^{\dagger}(\mathbf{p},\sigma)\mathrm{U}^{-1}(\Lambda,\alpha)\\&=\mathrm{e}^{-\mathrm{i}p\cdot\alpha}\sum_{\sigma'}D_{\sigma'\sigma}(W(\Lambda,p))b^{\dagger}(\Lambda\mathbf{p},\sigma'),\end{aligned} \tag{24}$$

$$\begin{aligned}&\mathrm{U}(\Lambda,\alpha)b^{c}(\mathbf{p},\sigma)\mathrm{U}^{-1}(\Lambda,\alpha)\\&=\mathrm{e}^{\mathrm{i}p\cdot\alpha}\sum_{\sigma'}D^{*}_{\sigma'\sigma}(W(\Lambda,p))b^{c}(\Lambda\mathbf{p},\sigma'),\end{aligned} \tag{25}$$

where $D(W)$ is the unitary irreducible representation of the little group $W(\Lambda,p)$ [27, Sec. II]. The simplest nontrivial solution for η is for it to be an internal symmetry operator that leaves the $\mathbf{p}$ and σ labels of $b^{\dagger}$ and b^c invariant. Therefore,

$$\eta^{-1}b^{\dagger}(\mathbf{p},\sigma)\eta=\mathrm{e}^{-\mathrm{i}\theta}b^{\dagger}(\mathbf{p},\sigma), \tag{26}$$

$$\eta^{-1}b^{c}(\mathbf{p},\sigma)\eta=\mathrm{e}^{\mathrm{i}\theta^c}b^{c}(\mathbf{p},\sigma), \tag{27}$$

where θ and θ^c are constant real phases to be determined. Since only the relative phase between $b^{\dagger}(\mathbf{p},\sigma)$ and $b^c(\mathbf{p},\sigma)$ is meaningful, we may set $\theta=0$ and take $\vec{\chi}(x)$ to be

$$\vec{\chi}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{f}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma)+\mathrm{e}^{\mathrm{i}\theta^c}\mathrm{e}^{-\mathrm{i}p\cdot x}\bar{g}(\mathbf{p},\sigma)b^{c}(\mathbf{p},\sigma)\right]. \tag{28}$$

The commutator (anticommutator) of $\chi(t,\mathbf{x})$ and $\vec{\chi}(t,\mathbf{y})$ is given by

$$\left[\chi(t,\mathbf{x}),\vec{\chi}(t,\mathbf{y})\right]_{\pm}=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}N_{\pm}(\mathbf{p},\theta^c), \tag{29}$$

where

$$N_{\pm}(\mathbf{p};\theta^c)\equiv\sum_{\sigma}\left[f(\mathbf{p},\sigma)\bar{f}(\mathbf{p},\sigma)\pm\mathrm{e}^{\mathrm{i}\theta^c}g(-\mathbf{p},\sigma)\bar{g}(-\mathbf{p},\sigma)\right]. \tag{30}$$

Comparing Eqs. (29)−(30) with Eqs. (15)−(16) shows that the spin sum $N_{\pm}(\mathbf{p},\theta^c)$ admits two solutions for which χ and $\vec{\chi}$ are local:

$$N_{\pm}(\mathbf{p};n\pi)=M_{\pm}(\mathbf{p})=0,\ \mathrm{e}^{\mathrm{i}\theta^c}=\mathrm{e}^{\mathrm{i}n\pi}=1, \tag{31}$$

$$N_{\pm}(\mathbf{p};n'\pi)=M_{\mp}(\mathbf{p})=0,\ \mathrm{e}^{\mathrm{i}\theta^c}=\mathrm{e}^{\mathrm{i}n'\pi}=-1. \tag{32}$$

These two solutions, particularly the latter, are the main results of this work. We examine them in turn.

1) Our construction is closely related to the field adjoint defined by Sablevice and Millington (S&M) [16]. Given a field $\psi(x)$, S&M defined its adjoint as $\bar{\psi}(x)\equiv\eta^{-1}\psi(x^{\eta})\eta\pi$, where π is a finite-dimensional Hermitian matrix and x^{η} denotes the point obtained by applying the transformation η to x. For example, if η is parity, then $x^{\eta}=(t,-\mathbf{x})$. If $\psi=(\psi_1,\psi_2,\cdots,\psi_n)$ is a multiplet field consisting of ψ_i $(1\le i\le n)$, then the dimension of π is $n\times n$ (see [16] for more details).

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Trivial solution (31). This solution is trivial because $\vec{\chi}(x)=\bar{\chi}(x)$. The theory therefore reduces to standard Hermitian QFT. The commutator (anticommutator) of $\chi(t,\mathbf{x})$ and $\vec{\chi}(t,\mathbf{y})$ coincides with the commutator (anticommutator) of $\lambda(t,\mathbf{x})$ and $\bar{\lambda}(t,\mathbf{y})$

$$\left[\chi(t,\mathbf{x}),\vec{\chi}(t,\mathbf{y})\right]_{-}=\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_{-}=0,\ j=0,1,\cdots, \tag{33}$$

$$\left[\chi(t,\mathbf{x}),\vec{\chi}(t,\mathbf{y})\right]_{+}=\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_{+}=0,\ j=\frac{1}{2},\frac{3}{2},\cdots. \tag{34}$$

Nontrivial solution (32) When $\mathrm{e}^{\mathrm{i}\theta^c}=-1$, η commutes with $b_\sigma(\mathbf{p})$ and anticommutes with $b^{c\dagger}(\mathbf{p})$,

$$\eta^{-1}b_\sigma(\mathbf{p})\eta=b_\sigma(\mathbf{p}),\eta^{-1}b^{c\dagger}_\sigma(\mathbf{p})=-b^{c\dagger}_\sigma(\mathbf{p}). \tag{35}$$

The solution for η is given in [28]

$$\eta=\exp\left[\mathrm{i}\pi\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_\sigma b^{c\dagger}(\mathbf{p},\sigma)b(\mathbf{p},\sigma)\right]. \tag{36}$$

Therefore, the *commutator (anticommutator) of* $\chi(t,\mathbf{x})$ *and* $\vec{\chi}(t,\mathbf{y})$ *equals the anticommutator (commutator) of* $\lambda(t,\mathbf{x})$ *and* $\bar{\lambda}(t,\mathbf{y})$:

$$\left[\chi(t,\mathbf{x}),\vec{\chi}(t,\mathbf{y})\right]_{+}=\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_{-}=0,\ j=0,1,\cdots, \tag{37}$$

$$\left[\chi(t,\mathbf{x}),\vec{\chi}(t,\mathbf{y})\right]_{-}=\left[\lambda(t,\mathbf{x}),\bar{\lambda}(t,\mathbf{y})\right]_{+}=0,\ j=\frac{1}{2},\frac{3}{2}\cdots. \tag{38}$$

Equations (37)–(38) express the NSST: *integer-spin fields are fermionic, and half-integer-spin fields are bosonic*. The fields $\chi(x)$ and $\vec{\chi}(x)$ associated with Eqs. (36)–(38) have the following expansions

$$r\chi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\sigma\left[\mathrm{e}^{-\mathrm{i}p\cdot x}f(\mathbf{p},\sigma)b(\mathbf{p},\sigma)+\mathrm{e}^{\mathrm{i}p\cdot x}g(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{39}$$

$$]\vec{\chi}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\sigma\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{f}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma)-\mathrm{e}^{-\mathrm{i}p\cdot x}\bar{g}(\mathbf{p},\sigma)b^{c}(\mathbf{p},\sigma)\right]. \tag{40}$$

III. PSEUDO-HERMITIAN QUANTUM FIELDS

In this section, we construct pseudo-Hermitian scalar, spinor, and vector fields using the formalism presented in the previous section. We derive their equations of motion, formulate their Lagrangian densities, and determine the corresponding discrete symmetries, namely charge conjugation C, parity P, and time reversal T.

We show that these fields obey the NSST and satisfy the canonical commutators and anticommutators. The free Lagrangian density $\mathcal{L}_0$ and the corresponding Lagrangian $L_0=\int\mathrm{d}^3x\mathcal{L}_0$ are η pseudo-Hermitian. Consequently, the free Hamiltonian H_0, when expressed in terms of the canonical variables, is manifestly pseudo-Hermitian. Notably, when H_0 is expanded in terms of the annihilation and creation operators, the non-Hermitian contributions cancel.

$$H_0=\int\mathrm{d}^3p\,E_{\mathbf{p}}\sum_\sigma\left[b^{\dagger}_\sigma(\mathbf{p})b_\sigma(\mathbf{p})\pm b^{c}_\sigma(\mathbf{p})b^{c\dagger}_\sigma(\mathbf{p})\right]. \tag{41}$$

Since η commutes with $b_\sigma(\mathbf{p})$ and anticommutes with $b^c_\sigma(\mathbf{p})$, it commutes with H_0. Therefore, H_0 is both Hermitian and η-pseudo-Hermitian

$$H_0^{\#}=\eta^{-1}H_0^{\dagger}\eta=H_0. \tag{42}$$

In the next section, we prove that, for a general QFT with field and adjoint $\chi(x)$ and $\vec{\chi}(x)$, respectively, the free Hamiltonian H_0 and three-momentum $\mathbf{P}_0$ are always Hermitian and positive-definite.

A. Scalar field

The pseudo-Hermitian scalar field is an anticommuting complex scalar field introduced by LeClair *et al.* [20, 21]. This construction is known as symplectic fermion theory. The fields $\phi(x)$ and $\vec{\phi}(x)$ have the expansions

$$\phi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[\mathrm{e}^{-\mathrm{i}p\cdot x}b(\mathbf{p})+\mathrm{e}^{\mathrm{i}p\cdot x}b^{c\dagger}(\mathbf{p})\right], \tag{43}$$

$$\vec{\phi}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[\mathrm{e}^{\mathrm{i}p\cdot x}b^{\dagger}(\mathbf{p})-\mathrm{e}^{-\mathrm{i}p\cdot x}b^{c}(\mathbf{p})\right]. \tag{44}$$

The minus sign in Eq. (44) does not affect the kinematics; therefore, both $\phi(x)$ and $\vec{\phi}(x)$ satisfy the Klein–Gordon

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equation.

$$(\partial^\mu\partial_\mu + m^2)\phi = (\partial^\mu\partial_\mu + m^2)\bar{\phi} = 0, \tag{45}$$

and have the Lagrangian density

$$\mathcal{L}_\phi = \partial^\mu\bar{\phi}\partial_\mu\phi - m^2\bar{\phi}\phi. \tag{46}$$

The fields carry spin $j=0$ and are therefore fermionic. From Eq. (46), the conjugate momenta corresponding to $\phi(x)$ and $\bar{\phi}(x)$ are[1)]

$$\pi_\phi = \frac{\partial\mathcal{L}_\phi}{\partial\dot{\phi}} = \dot{\bar{\phi}}, \bar{\pi}_\phi = \frac{\partial\mathcal{L}_\phi}{\partial\dot{\bar{\phi}}} = -\dot{\phi}. \tag{47}$$

Direct calculations show that the symplectic fermions satisfy the canonical equal-time anticommutation relations

$$\left[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})\right]_+ = \left[\pi(t,\mathbf{x}),\pi(t,\mathbf{y})\right]_+ = 0, \tag{48}$$

$$\left[\phi(t,\mathbf{x}),\pi_\phi(t,\mathbf{y})\right]_+ = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{49}$$

and

$$\left[\bar{\phi}(t,\mathbf{x}),\bar{\phi}(t,\mathbf{y})\right]_+ = \left[\bar{\pi}_\phi(t,\mathbf{x}),\bar{\pi}_\phi(t,\mathbf{y})\right]_+ = 0, \tag{50}$$

$$\left[\bar{\phi}(t,\mathbf{x}),\bar{\pi}_\phi(t,\mathbf{y})\right]_+ = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{51}$$

The remaining anticommutators vanish identically

$$\left[\phi(t,\mathbf{x}),\bar{\phi}(t,\mathbf{y})\right]_+ = \left[\pi_\phi(t,\mathbf{x}),\bar{\pi}_\phi(t,\mathbf{y})\right]_+ = 0, \tag{52}$$

$$\left[\phi(t,\mathbf{x}),\bar{\pi}_\phi(t,\mathbf{y})\right]_+ = \left[\bar{\phi}(t,\mathbf{x}),\pi_\phi(t,\mathbf{y})\right]_+ = 0. \tag{53}$$

The free Hamiltonian H_0 is obtained via the Legendre transform of $L_\phi = \int \mathrm{d}^3x\,\mathcal{L}_\phi$:

$$\begin{aligned} H_0(\phi) &= \int \mathrm{d}^3x\,(\pi_\phi\dot{\phi} + \bar{\pi}_\phi\dot{\bar{\phi}}) - L_\phi \\ &= \int \mathrm{d}^3x\,(\dot{\bar{\phi}}\dot{\phi} - \dot{\phi}\dot{\bar{\phi}}) - L_\phi \\ &= 2\int \mathrm{d}^3x\,\dot{\bar{\phi}}\dot{\phi} - L_\phi. \end{aligned} \tag{54}$$

The free Hamiltonian, expressed in terms of the canonical fields, is manifestly η-pseudo-Hermitian. Substituting the mode expansions for $\bar{\phi}$ and ϕ into $2\int \mathrm{d}^3x\,\dot{\bar{\phi}}\dot{\phi}$ and L_ϕ, we obtain

$$\begin{aligned} 2\int \mathrm{d}^3x\,\dot{\bar{\phi}}\dot{\phi} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\Big[&b^\dagger(\mathbf{p})b^c(\mathbf{p}) - \mathrm{e}^{2\mathrm{i}E_{\mathbf{p}}t}b^\dagger(\mathbf{p})b^{c\dagger}(-\mathbf{p}) \\ &+ \mathrm{e}^{-2\mathrm{i}E_{\mathbf{p}}t}b(-\mathbf{p})b^c(\mathbf{p}) - b(\mathbf{p})b^{c\dagger}(\mathbf{p})\Big], \end{aligned} \tag{55}$$

and

$$L_\phi = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\Big[-\mathrm{e}^{2\mathrm{i}E_{\mathbf{p}}t}b^\dagger(\mathbf{p})b^{c\dagger}(-\mathbf{p}) + \mathrm{e}^{-2\mathrm{i}E_{\mathbf{p}}t}b(-\mathbf{p})b^c(\mathbf{p})\Big]. \tag{56}$$

The time-dependent contributions in Eqs. (55)−(56) render $2\int \mathrm{d}^3x\,\dot{\bar{\phi}}\dot{\phi}$ and L_ϕ non-Hermitian. Substitution of Eqs. (55)−(56) into Eq. (54) shows that the non-Hermitian contributions cancel, yielding the correct expression

$$H_0(\phi) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\left[b^\dagger(\mathbf{p})b(\mathbf{p}) - b^c(\mathbf{p})b^{c\dagger}(\mathbf{p})\right]. \tag{57}$$

The discrete symmetries of symplectic fermions can be analyzed in the same way as for scalar bosons. The C, P, and T transformations of b and $b^{c\dagger}$ are

$$Cb(\mathbf{p})C^{-1} = \eta_C^* b^c(\mathbf{p}), \tag{58}$$

$$Pb(\mathbf{p})P^{-1} = \eta_P^* b(-\mathbf{p}), \tag{59}$$

$$Tb(\mathbf{p})T^{-1} = \eta_T^* b(-\mathbf{p}), \tag{60}$$

and

$$Cb^{c\dagger}(\mathbf{p})C^{-1} = \eta_C^c b^\dagger(\mathbf{p}), \tag{61}$$

$$Pb^{c\dagger}(\mathbf{p})P^{-1} = \eta_P^c b^{c\dagger}(-\mathbf{p}), \tag{62}$$

1) Note that $\bar{\pi}(x)$ is not obtained from $\pi(x)$ by applying η-conjugation. The minus sign in $\bar{\pi}(x)$ arises from the anticommutation of the fields $\bar{\phi}(x)$ and $\phi(x)$.

CPC Accepted

$$T b^{c\dagger}(\mathbf{p}) T^{-1} = \eta_T^c b^{c\dagger}(-\mathbf{p}). \tag{63}$$

Symplectic fermions transform under parity and time reversal in the same way as scalar bosons. Choosing the intrinsic parity and time-reversal phases to be even

$$\eta_P^* = \eta_P^c, \eta_T^* = \eta_T^c, \tag{64}$$

we obtain

$$P\phi(x)P^{-1} = \eta_P^*\phi(\mathscr{P}x), \tag{65}$$

$$T\phi(x)T^{-1} = \eta_T^*\phi(\mathscr{T}x), \tag{66}$$

where

$$\mathscr{P}x = (t, -\mathbf{x}), \quad \mathscr{T}x = (-t, \mathbf{x}). \tag{67}$$

By contrast, charge-conjugation symmetry differs because the field adjoint for ϕ is $\bar{\phi}$ rather than ϕ^*. Charge conjugation is a symmetry when C maps ϕ to $\bar{\phi}$. This can be achieved by choosing the charge-conjugation phase to be odd

$$\eta_C^* = -\eta_C^c, \tag{68}$$

we obtain

$$C\phi(x)C^{-1} = -\eta_C^*\bar{\phi}(x), \tag{69}$$

and

$$C\mathcal{L}_\phi(x)C^{-1} = \mathcal{L}_\phi(x). \tag{70}$$

B. Massive spinor field

The massive pseudo-Hermitian spinor field in the $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ representation was first proposed by Ahluwalia [22] and later studied in [24]. Let $\psi(x)$ and $\bar{\psi}(x)$ denote the massive spinor field and its adjoint, respectively. Their expansions are given by

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm\frac{1}{2}}\left[\mathrm{e}^{-\mathrm{i}p\cdot x}u(\mathbf{p},\sigma)b(\mathbf{p},\sigma) + \mathrm{e}^{\mathrm{i}p\cdot x}v(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{71}$$

$$]\bar{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm\frac{1}{2}}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\bar{u}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma) - \mathrm{e}^{-\mathrm{i}p\cdot x}\bar{v}(\mathbf{p},\sigma)b^{c}(\mathbf{p},\sigma)\right], \tag{72}$$

where u and v are the Dirac spinors, and $\bar{u} = u^\dagger\gamma^0$ and $\bar{v} = v^\dagger\gamma^0$. At rest, $\mathbf{p} = 0$, they are given in Ref. [27] as follows:

$$u(\mathbf{0},\frac{1}{2}) = \sqrt{m}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad u(\mathbf{0},-\frac{1}{2}) = \sqrt{m}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \tag{73}$$

$$v(\mathbf{0},\frac{1}{2}) = \sqrt{m}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, \quad v(\mathbf{0},-\frac{1}{2}) = \sqrt{m}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}. \tag{74}$$

For momentum $\mathbf{p}$, the spinors satisfy the Dirac equation.

$$(\gamma^\mu p_\mu - m)u(\mathbf{p},\sigma) = (\gamma^\mu p_\mu + m)v(\mathbf{p},\sigma) = 0. \tag{75}$$

Thus, $\psi(x)$ and $\bar{\psi}(x)$ satisfy the Dirac equation

$$\mathrm{i}\gamma^\mu\partial_\mu\psi(x) - m\psi(x) = -\mathrm{i}\partial_\mu\bar{\psi}(x)\gamma^\mu - m\bar{\psi}(x) = 0, \tag{76}$$

and have the Lagrangian density

$$\mathcal{L}_\psi = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi. \tag{77}$$

The spinor fields have spin $j = \frac{1}{2}$ and are therefore bosonic. The equal-time commutator of $\psi(t,\mathbf{x})$ and $\bar{\psi}(t,\mathbf{y})$ is given by

$$\left[\psi(t,\mathbf{x}),\bar{\psi}(t,\mathbf{y})\right]_- = \gamma^0\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{78}$$

The conjugate momentum associated with $\psi(x)$ is $\pi_\psi(x) = \mathrm{i}\bar{\psi}(x)\gamma^0$. The field $\psi(x)$ and its conjugate momentum satisfy the canonical commutation relations

$$\left[\psi(t,\mathbf{x}),\psi(t,\mathbf{y})\right]_- = \left[\pi_\psi(t,\mathbf{x}),\pi_\psi(t,\mathbf{y})\right]_- = 0, \tag{79}$$

$$\left[\psi(t,\mathbf{x}),\pi(t,\mathbf{y})\right]_- = \mathrm{i}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{80}$$

CPC Accepted

The free Hamiltonian H_0 for spin-half bosons is given by

$$\begin{aligned} H_0(\psi) &= \int \mathrm{d}^3x \pi_\psi \dot{\psi} - L_\psi \\ &= \mathrm{i}\int \mathrm{d}^3x \bar{\psi}(x)\gamma^0\dot{\psi}(x) - L_\psi, \end{aligned} \tag{81}$$

where $L_\psi = \int \mathrm{d}^3x \mathcal{L}_\psi$. As in the case of symplectic fermions, both terms in $H_0(\psi)$ are η-pseudo-Hermitian. When expanded in terms of annihilation and creation operators, both terms are non-Hermitian. However, these non-Hermitian contributions cancel when the expressions for $\mathrm{i}\int \mathrm{d}^3x \bar{\psi}\gamma^0\psi$ and L_ψ are substituted into $H_0(\psi)$, yielding

$$\begin{aligned} H_0(\psi) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=\pm1/2} \big[& b^\dagger(\mathbf{p},\sigma)b(\mathbf{p},\sigma) \\ & + b^c(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\big]. \end{aligned} \tag{82}$$

The C, P, and T transformations of $b(\mathbf{p},\sigma)$ and $b^{c\dagger}(\mathbf{p},\sigma)$ are the same as those of the creation and annihilation operators for Dirac fermions

$$Cb(\mathbf{p},\sigma)C^{-1} = \eta_C^* b^c(\mathbf{p},\sigma), \tag{83}$$

$$Pb(\mathbf{p},\sigma)P^{-1} = \eta_P^* b(-\mathbf{p},\sigma), \tag{84}$$

$$Tb(\mathbf{p},\sigma)T^{-1} = \eta_T^*(-1)^{1/2-\sigma} b(-\mathbf{p},-\sigma), \tag{85}$$

and

$$Cb^{c\dagger}(\mathbf{p},\sigma)C^{-1} = \eta_C^c b^\dagger(\mathbf{p},\sigma), \tag{86}$$

$$Pb^{c\dagger}(\mathbf{p},\sigma)P^{-1} = \eta_P^c b^{c\dagger}(-\mathbf{p},\sigma), \tag{87}$$

$$Tb^{c\dagger}(\mathbf{p},\sigma)T^{-1} = \eta_T^c(-1)^{1/2-\sigma} b^{c\dagger}(-\mathbf{p},-\sigma). \tag{88}$$

Spin-half bosons have the same parity and time-reversal transformation properties as Dirac fermions. When the intrinsic parity and time-reversal phases are chosen to be odd and even, respectively,

$$\eta_P^* = -\eta_P^c, \eta_T^* = \eta_T^c, \tag{89}$$

we obtain

$$P\psi(x)P^{-1} = \eta_P^*\gamma^0\psi(\mathscr{P}x), \tag{90}$$

$$T\psi(x)T^{-1} = \eta_T^*\gamma^1\gamma^3\psi(\mathscr{T}x). \tag{91}$$

The charge-conjugation symmetry of spin-half bosons must be analyzed more carefully because, in this case, $\bar{\psi}(x) \neq \bar{\psi}(x)$. Applying C to $\psi(x)$ yields

$$\begin{aligned} C\psi(x)C^{-1} =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=\pm1/2} \Big[\mathrm{e}^{-\mathrm{i}p\cdot x}u(\mathbf{p},\sigma)\eta_C^* b^c(\mathbf{p},\sigma) \\ & + \mathrm{e}^{\mathrm{i}p\cdot x}v(\mathbf{p},\sigma)\eta_C^c b^\dagger(\mathbf{p},\sigma)\Big] \\ =& -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \\ & \times \sum_{\sigma=\pm1/2} \mathrm{i}\gamma^2\Big[\mathrm{e}^{-\mathrm{i}p\cdot x}\eta_C^* v^*(\mathbf{p},\sigma)b^c(\mathbf{p},\sigma) \\ & + \mathrm{e}^{\mathrm{i}p\cdot x}\eta_C^c u^*(\mathbf{p},\sigma)b^\dagger(\mathbf{p},\sigma)\Big], \end{aligned} \tag{92}$$

where we have used the identities

$$u(\mathbf{p},\sigma) = -\mathrm{i}\gamma^2 v^*(\mathbf{p},\sigma),\ v(\mathbf{p},\sigma) = -\mathrm{i}\gamma^2 u^*(\mathbf{p},\sigma). \tag{93}$$

For charge conjugation to be conserved, the Lagrangian density must be invariant under the charge-conjugation transformation. Therefore, we cannot choose an even intrinsic phase.[1)] Instead, the intrinsic phase must be odd

$$\eta_C^* = -\eta_C^c. \tag{94}$$

We then obtain

$$C\psi(x)C^{-1} = -\mathrm{i}\eta_C^*\gamma^2\gamma^0\bar{\psi}^T(x), \tag{95}$$

$$C\bar{\psi}(x)C^{-1} = \mathrm{i}\eta_C\psi^T(x)\gamma^0\gamma^2, \tag{96}$$

and $C\bar{\psi}(x)\psi(x)C^{-1} = \bar{\psi}(x)\psi(x)$. This is the appropriate charge-conjugation symmetry for the spin-half boson, since $C\mathcal{L}_\psi(x)C^{-1} = \mathcal{L}_\psi(x)$.

1) If $\eta_C^* = \eta_C^c$, then $C[\bar{\lambda}(x)\lambda(x)]C^{-1} = [\bar{\lambda}(x)\lambda(x)]^\dagger$. This is not the correct symmetry for the spinor field because the Lagrangian density is not invariant under charge conjugation: $C\mathcal{L}_\lambda C^{-1} = \mathcal{L}_\lambda^\dagger$.

CPC Accepted

C. Vector field

We now construct the fermionic vector field in the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation. This representation is reducible, as it decomposes into a direct sum of spin-zero and spin-one irreducible representations. The spin-zero component corresponds to a spin-zero vector field. Here, we focus on the spin-one vector field. Let $v^{\mu}(x)$ be a complex vector field, and let $\bar{v}^{\mu}(x)$ denote its adjoint, with the expansions

$$rv^{\mu}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm1,0}\left[\mathrm{e}^{-\mathrm{i}p\cdot x}\mathrm{e}^{\mu}(\mathbf{p},\sigma)b(\mathbf{p},\sigma)+\mathrm{e}^{\mathrm{i}p\cdot x}\mathrm{e}^{\mu*}(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{97}$$

$$r\bar{v}^{\mu}(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm1,0}\left[\mathrm{e}^{\mathrm{i}p\cdot x}\mathrm{e}^{\mu*}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma)-\mathrm{e}^{-\mathrm{i}p\cdot x}\mathrm{e}^{\mu}(\mathbf{p},\sigma)b^{c}(\mathbf{p},\sigma)\right]. \tag{98}$$

In the rest frame, $\mathbf{p}=\mathbf{0}$, the coefficients are given by [27]

$$e^0(\mathbf{0},0)=0,\ e^0(\mathbf{0},\pm1)=0, \tag{99}$$

$$e^1(\mathbf{0},0)=0,\ e^1(\mathbf{0},\pm1)=\mp\frac{1}{\sqrt{2}}, \tag{100}$$

$$e^2(\mathbf{0},0)=0,\ e^2(\mathbf{0},\pm1)=-\frac{\mathrm{i}}{\sqrt{2}}, \tag{101}$$

$$e^3(\mathbf{0},0)=1,\ e^3(\mathbf{0},\pm1)=0. \tag{102}$$

The coefficients for momentum $\mathbf{p}$ are given by

$$\mathrm{e}^{\mu}(\mathbf{p},\sigma)=L^{\mu}{}_{\nu}(\mathbf{p})\mathrm{e}^{\nu}(\mathbf{0},\sigma), \tag{103}$$

where $L(\mathbf{p})$ denotes the Lorentz boost. Since the momentum p^{μ} lies on the mass shell, $p^2=m^2$, the coefficients satisfy the Klein–Gordon equation. They also satisfy $p_{\mu}\mathrm{e}^{\mu}(\mathbf{p},\sigma)=0$. Therefore, the vector fields satisfy the Klein–Gordon equation

$$(\partial^{\mu}\partial_{\mu}+m^2)v^{\mu}(x)=(\partial^{\mu}\partial_{\mu}+m^2)\bar{v}^{\mu}(x)=0, \tag{104}$$

and the identity

$$\partial_{\mu}v^{\mu}(x)=\partial_{\mu}\bar{v}^{\mu}(x)=0. \tag{105}$$

The Lagrangian density of the vector field is given by

$$\mathcal{L}_v=-\frac{1}{2}\bar{f}^{\mu\nu}f_{\mu\nu}+m^2\bar{v}^{\mu}v_{\mu}, \tag{106}$$

where $f^{\mu\nu}(x)$ and $\bar{f}^{\mu\nu}(x)$ denote the respective field strength tensors corresponding to $v^{\mu}(x)$ and $\bar{v}^{\mu}(x)$,

$$\bar{f}^{\mu\nu}=\partial^{\mu}\bar{v}^{\nu}-\partial^{\nu}\bar{v}^{\mu},\ f_{\mu\nu}=\partial_{\mu}v_{\nu}-\partial_{\nu}v_{\mu}. \tag{107}$$

The equations of motion and the conjugate momenta are given by

$$\partial_{\mu}f^{\mu\nu}+m^2v^{\nu}=\partial_{\mu}\bar{f}^{\mu\nu}+m^2\bar{v}^{\nu}=0, \tag{108}$$

and

$$\pi_{\mu}=\frac{\partial\mathcal{L}_v}{\partial\dot{v}^{\mu}}=\bar{f}_{\mu0}, \tag{109}$$

$$\bar{\pi}_{\mu}=\frac{\partial\mathcal{L}_v}{\partial\dot{\bar{v}}^{\mu}}=-f_{\mu0}. \tag{110}$$

The vector fields have spin $j=1$ and are therefore fermionic. To compute the canonical anticommutators, we must first identify the independent components of the vector fields and their conjugate momenta. From eq. (110), we find $\pi_0(x)=\bar{\pi}_0(x)=0$. The nonzero independent components of the conjugate momenta are

$$\pi_i=\bar{f}_{i0}=\partial_i\bar{v}_0-\dot{\bar{v}}_i, \tag{111}$$

$$\bar{\pi}_i=-f_{i0}=-\partial_iv_0+\dot{v}_i. \tag{112}$$

Using the $\nu=0$ component of the equations of motion (108), we find that $v^0(x)$ and $\bar{v}^0(x)$ can be expressed in terms of the divergence of the conjugate momenta

$$v^0(x)=\frac{1}{m^2}g^{ij}\partial_i\bar{\pi}_j(x),\ \bar{v}^0(x)=-\frac{1}{m^2}g^{ij}\partial_i\pi_j(x). \tag{113}$$

CPC Accepted

Thus, the independent components of the vector fields are $v^i, \bar{v}^i$, and those of the conjugate momenta are $\pi_j, \bar{\pi}_j$. From Eq. (112), π_j and $\bar{\pi}_j$ are given by

$$\pi_j(x) = \mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm1,0}\Big[\mathrm{e}^{\mathrm{i}p\cdot x}f^*_{j0}(\mathbf{p},\sigma)b^\dagger(\mathbf{p},\sigma) + \mathrm{e}^{-\mathrm{i}p\cdot x}f_{j0}(\mathbf{p},\sigma)b^c(\mathbf{p},\sigma)\Big], \tag{114}$$

$$\bar{\pi}_j(x) = \mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=\pm1,0}\Big[\mathrm{e}^{-\mathrm{i}p\cdot x}f_{j0}(\mathbf{p},\sigma)b(\mathbf{p},\sigma) - \mathrm{e}^{\mathrm{i}p\cdot x}f^*_{j0}(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\Big], \tag{115}$$

where

$$f_{j0}(\mathbf{p},\sigma) = p_j e_0(\mathbf{p},\sigma) - p_0 e_j(\mathbf{p},\sigma),\ p_0 = E_{\mathbf{p}}. \tag{116}$$

The canonical equal-time anticommutation relations between the vector fields and their conjugate momenta are given by

$$\left[v^i(t,\mathbf{x}), v^j(t,\mathbf{y})\right]_+ = 0, \tag{117}$$

$$\left[\pi_i(t,\mathbf{x}), \pi_j(t,\mathbf{y})\right]_+ = 0, \tag{118}$$

$$\left[v^i(t,\mathbf{x}), \pi_j(t,\mathbf{y})\right]_+ = \mathrm{i}\delta^i_j\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{119}$$

and

$$\left[\bar{v}^i(t,\mathbf{x}), \bar{v}^j(t,\mathbf{y})\right]_+ = 0, \tag{120}$$

$$\left[\bar{\pi}_i(t,\mathbf{x}), \bar{\pi}_j(t,\mathbf{y})\right]_+ = 0, \tag{121}$$

$$\left[\bar{v}^i(t,\mathbf{x}), \bar{\pi}_j(t,\mathbf{y})\right]_+ = \mathrm{i}\delta^i_j\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{122}$$

The remaining anticommutators vanish identically

$$\left[v^i(t,\mathbf{x}), \bar{v}^j(t,\mathbf{y})\right]_+ = \left[\pi_i(t,\mathbf{x}), \bar{\pi}_j(t,\mathbf{y})\right]_+ = 0, \tag{123}$$

$$\left[v^i(t,\mathbf{x}), \bar{\pi}_j(t,\mathbf{y})\right]_+ = \left[\bar{v}^i(t,\mathbf{x}), \pi_j(t,\mathbf{y})\right]_+ = 0. \tag{124}$$

The free Hamiltonian for the vector fermion is given by

$$\begin{aligned} H_0(v) &= \int \mathrm{d}^3x(\pi_\mu\dot{v}^\mu + \bar{\pi}_\mu\dot{\bar{v}}^\mu) - L_v \\ &= \int \mathrm{d}^3x(\bar{f}_{i0}\dot{v}^i - f_{i0}\dot{\bar{v}}^i) - L_v \\ &= \int \mathrm{d}^3x(\bar{f}_{i0}\dot{v}^i + \dot{\bar{v}}^i f_{i0}) - L_v, \end{aligned} \tag{125}$$

where $L_v = \int \mathrm{d}^3x\mathcal{L}_v$. As in our previous analysis of the free Hamiltonians for scalar fermions and spin-half bosons, the terms contributing to $H_0(v)$ are non-Hermitian. However, these non-Hermitian contributions cancel in $H_0(v)$:

$$H_0(v) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}\sum_{\sigma=\pm1,0}\left[b^\dagger(\mathbf{p},\sigma)b(\mathbf{p},\sigma) - b^c(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right] \tag{126}$$

The discrete symmetry transformations of $b(\mathbf{p},\sigma)$ and $b^{c\dagger}(\mathbf{p},\sigma)$ are

$$Cb(\mathbf{p},\sigma)C^{-1} = \eta^*_C b^c(\mathbf{p},\sigma), \tag{127}$$

$$Pb(\mathbf{p},\sigma)P^{-1} = \eta^*_P b(-\mathbf{p},\sigma), \tag{128}$$

$$Tb(\mathbf{p},\sigma)T^{-1} = \eta^*_T(-1)^{1-\sigma}b(-\mathbf{p},-\sigma), \tag{129}$$

and

$$Cb^{c\dagger}(\mathbf{p},\sigma)C^{-1} = \eta^c_C b^\dagger(\mathbf{p},\sigma), \tag{130}$$

$$Pb^{c\dagger}(\mathbf{p},\sigma)P^{-1} = \eta^c_P b^{c\dagger}(-\mathbf{p},\sigma), \tag{131}$$

$$Tb^{c\dagger}(\mathbf{p},\sigma)T^{-1} = \eta^c_T(-1)^{1-\sigma}b^{c\dagger}(-\mathbf{p},-\sigma). \tag{132}$$

CPC Accepted

For parity and time-reversal transformations, we choose the intrinsic phases to be even

$$\eta_P^* = \eta_P^c, \eta_T^* = \eta_T^c, \tag{133}$$

to obtain

$$Pv^\mu(x)P^{-1} = -\eta_P^* \mathscr{P}^\mu{}_\nu v^\nu(\mathscr{P}x), \tag{134}$$

$$Tv^\mu(x)T^{-1} = -\eta_T^* \mathscr{T}^\mu{}_\nu v^\nu(\mathscr{T}x). \tag{135}$$

Here, charge conjugation maps $v^\mu(x)$ to $\bar{v}^\mu(x)$ rather than to $v^{\mu\dagger}(x)$. Thus, by choosing the intrinsic charge-conjugation phases to be odd

$$\eta_C^* = -\eta_C^c, \tag{136}$$

we obtain

$$Cv^\mu(x)C^{-1} = \eta_C^* \bar{v}^\mu(x). \tag{137}$$

IV. POINCARÉ SYMMETRY & BELINFANTE-ROSENFELD TENSOR

The free quantum fields that satisfy the NSST constructed in the previous section preserve Poincaré symmetry. Consequently, their Lagrangian densities are invariant under a set of global Lorentz transformations, from which the generators of the Poincaré group can be derived via Noether's theorem. In this section, we show that, for a generic theory with field $\chi(x)$ and adjoint $\bar{\chi}(x)$, the free Hamiltonian H_0 and three-momentum $\mathbf{P}_0$ are both Hermitian and pseudo-Hermitian. Using the fact that the free generators satisfy the Poincaré algebra, we then prove that all free generators are Hermitian. Therefore, the free theory is unitary.

Although the free theory is unitary, its interactions are, in general, non-Hermitian. To illustrate this point, we compute the Belinfante–Rosenfeld tensor $\Theta_\phi^{\mu\nu}$ for symplectic fermions and show that it is non-Hermitian. Consequently, the gravitational interaction term $\int \mathrm{d}^3x \Theta_\phi^{\mu\nu} h_{\mu\nu}$, where $h_{\mu\nu}$ is the graviton field, is non-Hermitian. In Section 5, we discuss the issue of unitarity violation in more detail.

Let $\chi(x)$ and $\bar{\chi}(x)$ be the free canonical field and its pseudo-Hermitian conjugate defined in Eqs. (39), (40). The Lagrangian density depends on $\chi(x)$,$\bar{\chi}(x)$, and their derivatives. The theory respects Poincaré symmetry, so the Lagrangian density is invariant under the global transformations

$$\delta\chi(x) = +\frac{\mathrm{i}}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}\chi(x), \delta\bar{\chi}(x) = -\frac{\mathrm{i}}{2}\bar{\chi}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}(x), \tag{138}$$

where $\mathcal{J}$ denotes a finite-dimensional generator of the Lorentz group, and ω denotes the Lorentz transformation parameter. Their derivatives transform as

$$\delta\left[\partial_\kappa\chi(x)\right] = +\frac{\mathrm{i}}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu}\partial_\kappa\chi(x) + \omega_\kappa{}^\lambda\partial_\lambda\chi(x), \tag{139}$$

$$\delta\left[\partial_\kappa\bar{\chi}(x)\right] = -\frac{\mathrm{i}}{2}\partial_\kappa\bar{\chi}(x)\omega^{\mu\nu}\mathcal{J}_{\mu\nu} + \omega_\kappa{}^\lambda\partial_\lambda\bar{\chi}(x). \tag{140}$$

Using the canonical methods [27, Sec. 7], we derive the Belinfante–Rosenfeld tensor

$$\begin{aligned}\Theta^{\mu\nu} = T^{\mu\nu} - \frac{\mathrm{i}}{2}\partial_\kappa\Bigg[&\frac{\partial\mathcal{L}}{\partial(\partial_\kappa\chi)}\mathcal{J}^{\mu\nu}\chi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\mathcal{J}^{\kappa\nu}\chi \\ &- \frac{\partial\mathcal{L}}{\partial(\partial_\nu\chi)}\mathcal{J}^{\kappa\nu}\chi - \bar{\chi}\mathcal{J}^{\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\kappa\bar{\chi})} \\ &+ \bar{\chi}\mathcal{J}^{\kappa\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\chi})} + \bar{\chi}\mathcal{J}^{\kappa\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\bar{\chi})}\Bigg],\end{aligned} \tag{141}$$

where

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)}\partial^\nu\chi + \partial^\nu\bar{\chi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\chi})}. \tag{142}$$

The Hamiltonian H_0 and the three-momentum $\mathbf{P}_0$ are given by

$$H_0 = \int \mathrm{d}^3x\Theta^{00}, P_0^i = \int \mathrm{d}^3x\Theta^{0i}. \tag{143}$$

A third-rank tensor can be constructed from the Belinfante–Rosenfeld tensor

$$M^{\lambda\mu\nu} \equiv x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu}, \tag{144}$$

to obtain the Lorentz generator

$$J_0^{\mu\nu} = \int \mathrm{d}^3x M^{0\mu\nu}. \tag{145}$$

The Belinfante–Rosenfeld tensor and the generators of the Poincaré group are manifestly pseudo-Hermitian with respect to η. However, whether they are Hermitian remains unclear. Here, we show that the generators of the Poincaré group are Hermitian, whereas the Belinfante–Rosenfeld tensor is non-Hermitian. To prove the Hermiticity of the generators, we consider the relationship

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between pseudo-Hermitian and Hermitian QFTs. For a given irreducible representation of the Poincaré group, furnished by both $\chi,\bar{\chi}$ and $\lambda,\bar{\lambda}$, these two formulations are related by

$$\chi(x) = \lambda(x)|_{a\to b, a^{c\dagger}\to b^{c\dagger}}, \tag{146}$$

$$\bar{\chi}(x) = \bar{\lambda}(x)|_{a^{\dagger}\to b^{\dagger}, a^{c}\to -b^{c}}. \tag{147}$$

Hermitian QFTs are well defined in the sense that their H_0 and $\mathbf{P}_0$ operators are Hermitian

$$H_0(\lambda) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma} \left[a^{\dagger}(\mathbf{p},\sigma)a(\mathbf{p},\sigma) \pm a^{c}(\mathbf{p},\sigma)a^{c\dagger}(\mathbf{p},\sigma)\right], \tag{148}$$

$$\mathbf{P}_0(\lambda) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma} \left[a^{\dagger}(\mathbf{p},\sigma)a(\mathbf{p},\sigma) \pm a^{c}(\mathbf{p},\sigma)a^{c\dagger}(\mathbf{p},\sigma)\right], \tag{149}$$

where the upper and lower signs in Eqs. (148)–(149) correspond to bosonic ($j = 0,1,\cdots$) and fermionic fields ($j = \frac{1}{2},\frac{3}{2},\cdots$), respectively. From Eqs. (146)–(147), it follows that H_0 and $\mathbf{P}_0$ are also Hermitian for pseudo-Hermitian QFTs

$$H_0(\chi) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma} \left[b^{\dagger}(\mathbf{p},\sigma)b(\mathbf{p},\sigma) \mp b^{c}(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{150}$$

$$\mathbf{P}_0(\chi) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma} \left[b^{\dagger}(\mathbf{p},\sigma)b(\mathbf{p},\sigma) \mp b^{c}(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right]. \tag{151}$$

In this case, the upper and lower signs apply to fermionic ($j = 0,1,\cdots$) and bosonic fields ($j = \frac{1}{2},\frac{3}{2},\cdots$), respectively. For the Lorentz generator, we do not need the expression for $J^{\rho\sigma}(\chi)$ in terms of the creation and annihilation operators to prove its Hermiticity. It suffices to know that $H_0(\chi)$ and $\mathbf{P}_0(\chi)$ are Hermitian and that the Poincaré generators for $\chi,\bar{\chi}$ derived from the canonical formalism satisfy the Poincaré algebra [27, Sec. 7]. To show that the Lorentz generator $J_0^{\rho\sigma}(\chi)$ is Hermitian, we consider its commutator with the momentum operator $P_0^{\mu}(\chi)$, where $P_0^0(\chi) = H_0(\chi)$

$$\mathrm{i}[P_0^{\mu}(\chi), J_0^{\rho\sigma}(\chi)] = -g^{\mu\rho}P_0^{\sigma}(\chi) + g^{\mu\sigma}P_0^{\rho}(\chi). \tag{152}$$

Taking the Hermitian conjugate of Eq. (152), we obtain

$$\mathrm{i}[P_0^{\mu}(\chi), J_0^{\rho\sigma\dagger}(\chi)] = -g^{\mu\rho}P_0^{\sigma}(\chi) + g^{\mu\sigma}P_0^{\rho}(\chi). \tag{153}$$

Since $P_0^{\mu}(\chi) = P_0^{\mu\dagger}(\chi)$, the Lorentz generator must also be Hermitian, $J_0^{\rho\sigma}(\chi) = J_0^{\rho\sigma\dagger}(\chi)$.

The Hermiticity of the Poincaré generators indicates that the free pseudo-Hermitian QFT is unitary. However, this does not change the fact that the Belinfante-Rosenfeld tensor is non-Hermitian. To see this, consider the symplectic fermion, whose Belinfante-Rosenfeld tensor is

$$\Theta_{\phi}^{\mu\nu} = -g^{\mu\nu}\left(\partial^{\rho}\bar{\phi}\partial_{\rho}\phi - m^2\bar{\phi}\phi\right) + \partial^{\mu}\bar{\phi}\partial^{\nu}\phi + \partial^{\nu}\bar{\phi}\partial^{\mu}\phi. \tag{154}$$

Each term in Eq. (154) is non-Hermitian. Nevertheless, a direct computation shows that

$$\int \mathrm{d}^3x\,\Theta_{\phi}^{0\mu} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^{\mu} \sum_{\sigma} \left[b^{\dagger}(\mathbf{p},\sigma)b(p,\sigma) \pm b^{c}(\mathbf{p},\sigma)b^{c\dagger}(\mathbf{p},\sigma)\right], \tag{155}$$

whereas

$$\int \mathrm{d}^3x\,\Theta_{\phi}^{ij} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{p^i p^j}{E_{\mathbf{p}}} \left[b^{\dagger}(\mathbf{p})b(\mathbf{p}) + b^{\dagger}(\mathbf{p})b^{c\dagger}(-\mathbf{p}) - b^{c}(-\mathbf{p})b(\mathbf{p}) - b^{c}(\mathbf{p})b^{c\dagger}(\mathbf{p})\right], \tag{156}$$

at $t = 0$ and $i \neq j$ is non-Hermitian.

V. PHYSICAL IMPLICATIONS

Thus far, we have studied the kinematics of pseudo-Hermitian QFTs and found that they obey the NSST. This raises the question: *How should pseudo-Hermitian fields be interpreted?* To answer this question, we must study the interacting theory, which may not be unitary because the full Hamiltonian is non-Hermitian. Here, we discuss unitarity issues and potential realizations of the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence using pseudo-Hermitian QFTs.

A. Issues with unitarity

The particles described by the pseudo-Hermitian QFT furnish unitary irreducible representations of the Poincaré group, suggesting that they may be novel elementary particles beyond the Standard Model. However, as we

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will show, unitarity violation in the interacting theories makes this proposal untenable.

The free pseudo-Hermitian QFT is unitary. However, the full Hamiltonian $H = H_0 + V$ is non-Hermitian but is η-pseudo-Hermitian, $\eta^{-1}H^\dagger\eta = H$. Specifically, from Eq. (36), we see that η commutes with H_0 but does not, in general, commute with V. For an interacting potential constructed from $\vec{\chi}$ and χ of the form $V = \int \mathrm{d}^3x\mathcal{H}$, where $\mathcal{H} = \mathcal{H}(\vec{\chi},\chi)$, we have $\eta^{-1}\mathcal{H}^\dagger\eta = \mathcal{H}$ and hence $\eta^{-1}V^\dagger\eta = V$. Therefore, for states whose time evolution is given by

$$|\alpha,t\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\alpha\rangle, \langle\beta,t| = \langle\beta|\mathrm{e}^{\mathrm{i}H^\dagger t}. \tag{157}$$

Time-translation symmetry cannot be preserved using the Dirac inner product, since $\langle\beta,t|\alpha,t\rangle \neq \langle\beta|\alpha\rangle$. This symmetry can instead be preserved by defining the η-inner product following Mostafazadeh [8]

$$\langle\beta|\alpha\rangle_\eta \equiv \langle\beta|\eta|\alpha\rangle, \tag{158}$$

from which it follows that $\langle\beta,t|\alpha,t\rangle_\eta = \langle\beta|\alpha\rangle_\eta$. Within this framework, the S-matrix satisfies the generalized unitarity relation (see Appendix B)

$$S^\# S = \mathbb{1}, S^\# \equiv \eta^{-1}S^\dagger\eta. \tag{159}$$

However, the η-inner product and the generalized unitarity relation do not admit a standard probabilistic interpretation because the η-inner product is not positive definite (the eigenvalues of η are ±1). In particular, the η-inner product between the in- and out-states is not positive definite

$$\langle\beta_\pm|\alpha_\pm\rangle_\eta = \langle\beta_0|\alpha_0\rangle_\eta = (-1)^{\bar{n}_\beta}\delta(\beta-\alpha), \tag{160}$$

where $\bar{n}_\beta$ is the number of antiparticles in $|\beta_0\rangle$.

Despite this difficulty, the η-inner product can be used to identify a class of unitary interacting theories. An interacting theory is unitary if it satisfies the following criterion. For a process $\alpha \to \beta$, the transition amplitude is $S_{\beta\alpha} = \langle\beta_0|S|\alpha_0\rangle$, where $|\alpha_0\rangle, |\beta_0\rangle$ are the free multiparticle states (see (B.11)). The action of η on the free states is

$$\eta|\alpha_0\rangle = \mathrm{e}^{\mathrm{i}\pi n^c_\alpha}|\alpha_0\rangle, \eta|\beta_0\rangle = \mathrm{e}^{\mathrm{i}\pi n^c_\beta}|\beta_0\rangle, \tag{161}$$

where n^c_α and n^c_β are the total numbers of antiparticles in $|\alpha_0\rangle$ and $|\beta_0\rangle$, respectively. The generalized unitarity relation can then be written as

$$\int \mathrm{d}\beta S^\#_{\gamma\beta}S_{\beta\alpha} = \mathrm{e}^{\mathrm{i}\pi(n^c_\alpha+n^c_\gamma)}\int \mathrm{d}\beta S^\dagger_{\gamma\beta}S_{\beta\alpha} = \delta_\Pi(\gamma-\alpha). \tag{162}$$

Equation (162) implies that the theory is unitary provided that the total number of antiparticles in any physical process is *even*. We have been unable to construct interactions that satisfy this criterion. In [26], it was claimed that the LN model for self-interacting symplectic fermions, $(\sum_i \bar{\phi}_i\phi_i)^2$, is unitary. However, this claim holds only for 2-to-2 scattering at low energies below the pair-production threshold [29] [1)].

To resolve this problem, one direct approach is to formulate a unitary scattering theory using the Hermitian Hamiltonian density

$$\mathcal{H}^{(h)}(\mathbf{x}) \equiv \frac{1}{2}\left[\mathcal{H}(\mathbf{x}) + \mathcal{H}^\dagger(\mathbf{x})\right]. \tag{163}$$

However, this approach fails. To see why, consider the Yukawa interaction between a spin-half boson ψ and a real scalar field φ, given by $\mathcal{H}_Y = g\bar{\psi}\psi\varphi$. Since $\psi(\mathbf{x})$ commutes with $\bar{\psi}(\mathbf{y})$, $\mathcal{H}_Y$ is local:

$$[\mathcal{H}_Y(\mathbf{x}),\mathcal{H}_Y(\mathbf{y})]_- = 0. \tag{164}$$

However, the Hermitian Yukawa interaction density

$$\mathcal{H}^{(h)}_Y(\mathbf{x}) = \frac{1}{2}[\mathcal{H}_Y(\mathbf{x}) + \mathcal{H}^\dagger_Y(\mathbf{x})], \tag{165}$$

is nonlocal, $[\mathcal{H}^{(h)}_Y(\mathbf{x}),\mathcal{H}^{(h)}(\mathbf{y})]_- \neq 0$, since $\psi(\mathbf{x})$ *does not* commute with $\psi^\dagger(\mathbf{y})$ at equal times.

The above argument shows that an interacting pseudo-Hermitian QFT satisfying the NSST with respect to the η-inner product is generally non-unitary. Nevertheless, the theory remains consistent in the sense that it is local and obeys Poincaré symmetry. If theories built from these fields are indeed non-unitary, they cannot describe elementary particles, whose scattering processes require a unitary S-matrix. However, this does not imply that they are unphysical. Well-known non-unitary theories exist, such as the Lee–Yang model [30, 31] and logarithmic CFTs in 1+1 dimensions with negative central charge [32, 33]. These theories are not only consistent but also have important applications in statistical physics. Therefore, the task is to identify areas in which pseudo-Hermitian QFT can be applied. In the following, we argue that it may provide a concrete realization of the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence.

1) We thank one of the referees for making this observation and bringing Ref. [29] to our attention.

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B. The dS_4/CFT_3 correspondence

In the literature, three-dimensional models of self-interacting symplectic fermions and Chern-Simons gauge theory coupled to spin-half bosons have been conjectured to provide concrete realizations of the dS_4/CFT_3 correspondence [25].[1] Here, we examine both models, explain why they are unsatisfactory, and propose alternative constructions based on the pseudo-Hermitian framework developed in this work. Although the pseudo-Hermitian QFTs considered here are constructed in four-dimensional spacetime, they can be straightforwardly extended to three dimensions because the actions retain the same form.

Anninos, Hartman, and Strominger (AHS) conjectured that three-dimensional symplectic fermions are holographically dual to minimal Vasiliev higher-spin gravity in dS_4 [25]. AHS described symplectic fermions using real anticommuting scalar fields χ_i $(i=1,\cdots,N)$. They proposed the following action

$$S_\chi=\frac{1}{2}\int \mathrm{d}^3x\sum_{i,j}\left[\Omega_{ij}(\partial^\mu\chi_i)(\partial_\mu\chi_j)+g(\Omega_{ij}\chi_i\chi_j)^2\right]. \tag{166}$$

The free theory is conjectured to be dual to minimal Vasiliev gravity with Neumann boundary conditions. The interacting critical symplectic fermion theory is conjectured to be dual to minimal Vasiliev gravity with Dirichlet boundary conditions. There are two important consequences for formulating the dS_4/CFT_3 correspondence using χ_i. First, χ_i violates the SST, so the proposed duality is non-unitary. Second, there are conserved currents only of even spin

$$J(\chi)=\sum_i\chi_i\chi_i, s=0, \tag{167}$$

$$J_{\mu_1\cdots\mu_s}(\chi)=\sum_i\chi_i\partial_{(\mu_1\cdots\mu_s)}\chi_i+\cdots, s=2,4\cdots. \tag{168}$$

The correction term in (168) makes the current symmetric and traceless. The correlation functions of these currents are conjectured to be dual to the corresponding bulk correlators of minimal Vasiliev gravity in dS_4.

Chang, Pathak, and Strominger (CPS) conjectured that the level-k U(N) Chern-Simons theory coupled to spin-half bosons is dual to parity-violating non-minimal Vasiliev gravity.[2] The action is

$$S_{A,\xi}=\frac{k}{4\pi}\int \mathrm{tr}\left(AdA+\frac{2}{3}A^3\right)+\int \mathrm{d}^3x\bar{\xi}\gamma^iD_i\xi, \tag{169}$$

where A denotes the Chern–Simons gauge field, and $\xi,\bar{\xi}$ are commuting spinor fields in the fundamental representation of U(N). The spin-zero and higher-spin currents are

$$J(\xi)=\bar{\xi}\xi, s=0, \tag{170}$$

$$J_{\mu_1\cdots\mu_s}(\xi)=i^s\bar{\xi}\gamma_{\mu_1}D_{\mu_2}\cdots D_{\mu_s}\xi, s=1,2\cdots. \tag{171}$$

Here, the higher-spin currents are not conserved. They are almost conserved in the sense that the term that violates current conservation is suppressed by $1/N$. The spin-zero operator and higher-spin currents are conjectured to be dual to a scalar field and higher-spin gauge fields in the bulk, respectively.

In our view, the main issue with these models is not the violation of unitarity. Indeed, unitarity violation is expected to be an essential feature of the dS_4/CFT_3 correspondence.

The central problem is that the real anticommuting scalar fields and the commuting spinor fields with $\bar{\xi}=\xi^\dagger\gamma^0$ *cannot* be consistently quantized. Specifically, these fields do not admit proper mode expansions, and thus their Hamiltonians and energy-momentum tensors cannot be expressed in terms of annihilation and creation operators. Their correlation functions cannot be computed directly by contracting fields using Wick's theorem; instead, they can only be obtained by analytic continuation. In the AHS construction, the correlation functions are obtained by analytically continuing those of the O(N) vector model via $N\to -N$. Similarly, in the CPS construction, the correlation functions are obtained by analytically continuing the results from Chern–Simons theory coupled to Dirac fermions.

Since the O(N) vector model and Chern–Simons theory coupled to Dirac fermions are both unitary and have well-defined correlation functions that are analytic in N, it is permissible to analytically continue N to $-N$. Such an analytic continuation implies that the resulting QFTs must satisfy the opposite spin–statistics relation relative to the original theories. However, it does not specify precisely how these theories should be constructed.

To construct QFTs with opposite spin–statistics relations, we must take into account the pseudo-Hermitian adjoint. In this way, the QFTs can be consistently quantized. Since AHS and CPS did not introduce the pseudo-Hermitian adjoint, their constructions are inconsistent.

1) C. M. Chang, A. Pathak, and A. Strominger, (2013)

2) C. M. Chang, A. Pathak, and A. Strominger, (2013)

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We now present the correct constructions using the pseudo-Hermitian framework.

The symplectic fermions of LeClair *et al.* are described by $\bar{\phi}_i$, ϕ_i $(1 \le i \le N)$. The Euclidean action for N symplectic fermions in three dimensions is

$$S_\phi = \int \mathrm{d}^3x \sum_{i,} \left[(\partial^\mu \bar{\phi}_i)\partial_\mu \phi_i + g(\bar{\phi}_i\phi_i)^2\right]. \tag{172}$$

Like its four-dimensional counterpart, the three-dimensional theory can be consistently quantized. Because the fields are complex, in addition to possessing even-spin currents

$$J_{\mu_1\cdots\mu_s}(\phi) = \sum_i \bar{\phi}_i \partial_{(\mu_1\cdots\mu_s)}\phi_i + \cdots, \; s = 0,2,4\cdots, \tag{173}$$

there are also odd spin currents

$$J_{\mu_1\cdots\mu_s}(\phi) = \mathrm{i}\sum_i \bar{\phi}_i \partial_{(\mu_1\cdots\mu_s)}\phi_i + \cdots, \; s = 1,3,5,\cdots. \tag{174}$$

The correction terms in (173)–(174) are introduced to ensure that the currents are symmetric, traceless, and pseudo-Hermitian. For example, the scalar and spin-one currents are

$$J(\phi) = \bar{\phi}\phi, \; J_\mu(\phi) = \mathrm{i}\bar{\phi}(\partial_\mu\phi) - \mathrm{i}(\partial_\mu\bar{\phi})\phi. \tag{175}$$

We conjecture that the correlation functions of $J_{\mu_1\cdots\mu_s}(\phi)$ for all spins $(s = 0,1,2,\cdots)$ are dual to the corresponding bulk correlators of non-minimal Vasiliev gravity in dS_4 with the Neumann boundary condition. For the critical symplectic-fermion theory with quartic self-interaction $(\sum_i \bar{\phi}_i\phi_i)^2$, we conjecture that the theory is dual to non-minimal Vasiliev gravity in dS_4 with the Dirichlet boundary condition. These conjectures are analogous to those relating complex scalar fields with U(N) symmetry in three dimensions to non-minimal Vasiliev gravity in AdS_4 [34–36].

For the level-k U(N) Chern-Simons theory coupled to the spin-1/2 boson, the correct Euclidean action is given by

$$S_{A,\psi} = \frac{k}{4\pi}\int \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right) + \int \mathrm{d}^3x\,\bar{\psi}\gamma^i D_i\psi, \tag{176}$$

where $\bar{\psi}$ denotes the pseudo-Hermitian adjoint of ψ. The currents are

$$J(\psi) = \bar{\psi}\psi, s = 0, \tag{177}$$

and

$$J_{\mu_1\cdots\mu_s}(\psi) = i^s\bar{\psi}\gamma_{\mu_1}D_{\mu_2\cdots\mu_s}\psi, s = 1,2,\cdots. \tag{178}$$

The spectrum of the currents for $\psi,\bar{\psi}$ is the same as that for $\xi,\bar{\xi}$. Consequently, the correlation functions for $J(\psi)$ and $J_{\mu_1\cdots\mu_s}(\psi)$ are also conjectured to be dual to a scalar field and higher-spin gauge fields in the bulk.

VI. CONCLUSION

The pseudo-Hermitian QFT provides a framework for generalizing conventional unitary QFT. Here, we have extended the formalism developed by LeClair *et al.* [20, 21] and Ahluwalia *et al.* [22, 24] to construct a class of free pseudo-Hermitian QFTs that obey the NSST. Contrary to the consensus that fields with the 'wrong' spin-statistics are non-unitary, free pseudo-Hermitian fields, with the appropriate adjoints, are local, respect Poincaré symmetry, and can be consistently quantized. In this framework, integer-spin fields are fermionic, and half-integer-spin fields are bosonic. The free generators of the Poincaré group are Hermitian, which ensures that the free theory is unitary.

We have not introduced new symmetries or degrees of freedom in our construction. The pseudo-Hermitian QFTs are constructed from the same requirements of locality and Poincaré symmetry as their Hermitian counterparts. The crucial difference is the pseudo-Hermitian field adjoint, which alters the connection between spin and statistics. The pseudo-Hermitian QFTs also differ from their Hermitian counterparts in two other respects. First, the particles and antiparticles described by pseudo-Hermitian QFTs must be distinct $(b \ne b^c)$. This is because the operator η, which is used to define the pseudo-Hermitian conjugation, commutes with b and anticommutes with b^c. This is not possible if $b = b^c$.[1)] The second difference concerns charge-conjugation symmetry. As we have shown in Section III, for the scalar, vector, and spinor fields, the particles and antiparticles have *odd* intrinsic charge-conjugation phases. As a result, we have $C^2 = -1$ for pseudo-Hermitian QFTs instead of $C^2 = 1$ for Hermitian QFTs. This result resembles the method used by Bernard and LeClair to classify non-Hermitian random matrices [37].

Once the correct pseudo-Hermitian field adjoint is defined, the derivation of the NSST within the relativistic QFT framework is straightforward. An interesting question is whether the NSST can be derived for non-re-

1) Another reason is that locality forces neutral quantum fields with $b = b^c$ to satisfy the SST. For example, the real scalar field must be bosonic and cannot be fermionic.

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lativistic (pseudo-Hermitian) QM. Berry and Robins have shown that the Pauli phase $(-1)^{2s}$, where s is the spin of the particle, naturally arises from the geometric phase [38]. Single-valuedness of the wave function then implies the standard connection between spin and statistics. The requirement of unitarity is central to their derivation. It would be instructive to re-examine their derivation by relaxing this requirement.

In this work, we have shown that the free pseudo-Hermitian QFT is unitary, whereas the interacting theory is non-unitary. We outline two promising approaches to addressing non-unitarity. As noted in [29], the renormalization group flow for symplectic fermions exhibits a much richer pattern than that of the bosonic ϕ^4 theory, which may be relevant to the hierarchy problem of the Higgs boson. Moreover, since the LN model is unitary at low energies below the pair-production threshold, symplectic fermions may have applications in condensed matter physics.

Alternatively, pseudo-Hermitian QFTs may have applications to the dS_4/CFT_3 correspondence. In Section V, we conjecture that the symplectic fermions and spin-half bosons are dual to Vasiliev gravity. Our results in this work present an interesting paradigm: the dS_4/CFT_3 correspondence is unitary (non-unitary) when the boundary CFT is free (interacting). We hope to verify our conjectures in future work.

APPENDIX A: NOTATIONS & CONVENTIONS

Throughout this paper, we follow the conventions of Ref. [39]. The four-dimensional Minkowski metric is

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1), \tag{A1}$$

where $\mu,\nu = 0,1,2,3$.

The 4×4 Dirac matrices are expressed in the Weyl representation

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \text{with} \begin{cases} \sigma^\mu &= \left(\mathbb{1}_{2\times2}, \sigma^i\right), \\ \bar{\sigma}^\mu &= \left(\mathbb{1}_{2\times2}, -\sigma^i\right), \end{cases} \tag{A2}$$

where $\mu = 0,1,2,3$, and σ^i $(i=1,2,3)$ denote the three Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{A3}$$

and

$$\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}, i,j = 1\cdots3, \tag{A4}$$

where the Kronecker delta function is

$$\delta^{ij} = \begin{cases} 1 & , i = j, \\ 0 & , i \neq j. \end{cases} \tag{A5}$$

The Levi-Civita tensor ϵ_{ijk}, defined for $i,j,k = 1\cdots3$, is totally antisymmetric, with $\epsilon_{123} = +1$.

The chiral operator is also given by

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2} \end{bmatrix}, \tag{A6}$$

and the chiral projection operators

$$P_{L,R} \equiv \frac{\mathbb{1} \mp \gamma^5}{2}. \tag{A7}$$

APPENDIX B: THE *S*-MATRIX WITH THE η-INNER PRODUCT

A scattering theory for a pseudo-Hermitian Hamiltonian was constructed using the η-inner product in Ref. [26]. Here, we briefly review this theory. Let $|\alpha_+\rangle$ and $|\alpha_-\rangle$ denote the in- and out-states, respectively. They are related to the free states $|\alpha_0\rangle$ by

$$|\alpha_\pm\rangle = \Omega_\pm|\alpha_0\rangle, \Omega_\pm = \lim_{t\to\mp\infty} \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}. \tag{B8}$$

The time evolutions of $|\alpha_0\rangle$ and $|\alpha_\pm\rangle$ are generated by H_0 and H, respectively. Because H is pseudo-Hermitian, time-translation symmetry can be preserved with respect to the η-inner product

$$\langle\beta_\pm,t|\alpha_\pm,t\rangle_\eta = \langle\beta_\pm|\alpha_\pm\rangle_\eta. \tag{B9}$$

The S-matrix for the scattering process $\alpha\to\beta$ is given by the η-inner product.

$$\begin{aligned} S_{\beta\alpha} &= \langle\beta_-|\eta|\alpha_+\rangle \\ &= \langle\beta_0|S|\alpha_0\rangle, \end{aligned} \tag{B10}$$

where

$$S = \Omega_-^\dagger\eta\Omega_+. \tag{B11}$$

Using the pseudo-Hermiticity of the Hamiltonian, we find that the inverse of $\Omega_\pm$ is given by

$$\Omega_\pm^{-1} = \Omega_\pm^\# = \eta^{-1}\Omega_\pm^\dagger\eta. \tag{B12}$$

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A direct calculation then shows that the S-matrix satisfies the generalized unitarity relation

$$S^{\#}S = \mathbb{1}, S^{\#} = \eta^{-1}S^{\dagger}\eta. \tag{B13}$$

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